Homework of Linear mapping

1. Let $F: R^2 \to R^2$ be the map defined by $F\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2x \\ 3y \end{bmatrix}$ for any $\begin{bmatrix} x \\ y \end{bmatrix} \in R^2$.

   Describe the image by F of the points lying on the unit circle centered at 0, i.e. $\left\{\begin{bmatrix} x \\ y \end{bmatrix} \in \mathrm{R}^2 \mid \mathrm{x}^2 + y^2 = 1\right\}$.

2. Let $F: R^2 \to R^2$ be the map defined by $F\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} xy \\ y \end{bmatrix}$ for any $\begin{bmatrix} x \\ y \end{bmatrix} \in R^2$.

   Describe the image by F of the line $\left\{\begin{bmatrix} x \\ y \end{bmatrix} \in \mathrm{R}^2 \mid x = 2\right\}$.

3. Let V be a linear space of dimension n, and let $\{v_1, v_2, \cdots, v_n\}$ be a basis for V. Let F be a linear map from V into itself. Show that F is uniquely defined if one knows $F(v_j)$ for $j \in \{1, 2, \cdots, n\}$. Is it also true if F is an arbitrary map from om V into itself?

4. Let V, W be two linear space over the same field, and let $T: V \to W$ be a linear mapping. Show that the following set is a subspace of V.

$$\{x \in V \mid T(x) = 0\}$$

5. Let $\mathrm{T}: \mathrm{R}^{n\times n} \to R^{n\times n}$ be the map defined for any $n \times n$ dimensional matrix $A \in R^{n\times n}$ by

$$T(A) = \frac{1}{2}(A + A^T)$$

   where A^T denotes the transpose of matrix A.

   1) Show that T is a linear mapping.

   2) Show that the kernel of T consists in the linear space of all skew-

symmetric matrices.

3) Show that the range of T consists in the linear space of all symmetric matrices.

4) What is the dimension of the linear space of all symmetric matrices, and the dimension of the vector space of all skew-symmetric matrices?

6. Consider the mapping $F\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x \\ x-y \\ x-z \\ x-y-z \end{bmatrix}$

1) Determine the kernel of F.

2) Determine the range of F.

7. Let $\mathrm{T}: R^3 \to R^3$ be a linear mapping which associated matrix has the form $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ with respect to the canonical basis of R^3 ($e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$). What is the matrix associated with T in the basis generated by the three vectors $v_1 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}, v_3 = \begin{bmatrix} 0 \\ 0 \\ \text{-1} \end{bmatrix}$.