Homework

1. Show that the zero of vector addition is unique.

Proof：If there are two zero 0 and O, since 0 is a zero, we have

$$O=0+O \tag{1}$$

Moreover, since O is also zero, we have

$$0=0+O \tag{2}$$

From (1) and (2), we obtain O=0+O=0.

2. Show the axiom 0x = 0 by using other seven axioms in the definition of linear space.

Proof: For arbitrary x in the set X , we have

$$x+0x=(1+0)x=1x=x. \tag{3}$$

Adding the negative of x on the both side of (3), x+(-x)+ 0x=x+(-x)=0. Thus 0x=0.

3. Suppose that the set X is the set of positive real numbers(i.e. $x>0)$, if the addition and scalar multiplication with the field R of real numbers are defined as follows

$$x+y=xy, \quad cx=x^c ,$$

Show this set under this addition and scalar multiplication is a linear space.

Proof:

First， we prove the closeness of addition and scalar multiplication. For any $x, y, z \in X$ and $c \in R$, then $x>0, y>0, z>0$, Thus we have

$x+y=xy>0$ and $cx=x^c>0$. The closeness is proved.

Next, we prove the following results.

1. $x+y=xy=yx=y+x$
2. $(x+y)+z=xy+z=xyz=x+yz=x+(y+z)$
3. $x+1=x1=x$, so 1 is the zero element.
4. $x+1/x=x(1/x)=1$， so $1/x$ is the negative element of x.

For any $k,h \in R$,we have

5. $k(hx)=kx^h=x^{kh}=(kh)x$
6. $k(x+y)=k(xy)=(xy)^k=x^k y^k=kx+ky$
7. $(k+h)x=x^{k+h}=x^k x^h=kx+hx$

8. $1x = x^1 = x$ Therefore, set X with the field R of real numbers is a linear space.

4. Suppose that the set X is 2-dimensional vector set of real numbers R^2 with the following addition and scalar multiplication with the field R of real numbers

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \oplus \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 + x_1 x_2 \end{bmatrix}$$

$$k \bullet \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} kx_1 \\ ky_1 + \dfrac{k(k-1)}{2} x_1^2 \end{bmatrix}$$

Show this set under this addition and scalar multiplication is a linear space.

证明：首先我们证明该集合关于加法和数乘运算的封闭性。对任意 $\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}, \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}, \begin{bmatrix} x_3 \\ y_3 \end{bmatrix} \in X$ 和

$k \in R$ ，我们可以得到

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \oplus \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 + x_1 x_2 \end{bmatrix} \in X$$

$$k \bullet \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} kx_1 \\ ky_1 + \dfrac{k(k-1)}{2} x_1^2 \end{bmatrix} \in X$$

因而，该集合关于加法和数乘运算是封闭的。接下来，我们证明线性空间的另外 8 条。

1. $\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \oplus \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 + x_1 x_2 \end{bmatrix}, \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} \oplus \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} x_2 + x_1 \\ y_2 + y_1 + x_2 x_1 \end{bmatrix}$

   因而，$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \oplus \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} \oplus \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$，交换律成立。

2. $\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \oplus \left(\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} \oplus \begin{bmatrix} x_3 \\ y_3 \end{bmatrix} \right) = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \oplus \begin{bmatrix} x_2 + x_3 \\ y_2 + y_3 + x_2 x_3 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 + x_3 \\ y_1 + y_2 + y_3 + x_2 x_3 + x_1 x_2 + x_1 x_3 \end{bmatrix}$

$\left(\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \oplus \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} \right) \oplus \begin{bmatrix} x_3 \\ y_3 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 + x_1 x_2 \end{bmatrix} \oplus \begin{bmatrix} x_3 \\ y_3 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 + x_3 \\ y_1 + y_2 + y_3 + x_1 x_2 + x_1 x_3 + x_2 x_3 \end{bmatrix}$

因而，$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \oplus \left(\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} \oplus \begin{bmatrix} x_3 \\ y_3 \end{bmatrix} \right) = \left(\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \oplus \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} \right) \oplus \begin{bmatrix} x_3 \\ y_3 \end{bmatrix}$

3. $\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \oplus \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} x_1 + 0 \\ y_1 + 0 + 0 \end{bmatrix} = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$，所以 $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ 是该线性空间的零元。

4. $\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \oplus \begin{bmatrix} -x_1 \\ -y_1+x_1^2 \end{bmatrix} = \begin{bmatrix} x_1+(-\mathrm{x}_1) \\ y_1+(-y_1+x_1^2)+x_1(-x_1) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ ，所以

该线性空间中$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$的负元为$\begin{bmatrix} -x_1 \\ -y_1+x_1^2 \end{bmatrix}$。

对任意的数域 R 中的数$k,h\in R$，我们有

$$5.\quad k\left(h\left(\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}\right)\right) = k\left(\begin{bmatrix} hx_1 \\ hy_1+\frac{h(h-1)}{2}x_1^2 \end{bmatrix}\right) = \begin{bmatrix} khx_1 \\ k\left(hy_1+\frac{h(h-1)}{2}x_1^2\right)+\frac{k(k-1)}{2}(hx_1)^2 \end{bmatrix}$$

$$(kh)\left(\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}\right) = \begin{bmatrix} khx_1 \\ khy_1+\frac{kh(kh-1)}{2}x_1^2 \end{bmatrix} = \begin{bmatrix} khx_1 \\ k\left(hy_1+\frac{h(h-1)}{2}x_1^2\right)+\frac{k(k-1)}{2}(hx_1)^2 \end{bmatrix}$$

所以$k\left(h\left(\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}\right)\right) = (kh)\left(\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}\right)$

⑥$k(X+Y)=k[x_1+x_2 \quad y_1+y_2+x_1x_2]^T$

$=[k(x_1+x_2) \quad k(y_1+y_2+x_1x_2)+k(k-1)(x_1+x_2)^2/2]^T$

$=[k(x_1+x_2) \quad k(y_1+y_2)+k^2(x_1+x_2)^2/2-k(x_1^2+x_2^2)/2]^T$

$kX+kY=[kx_1 \quad ky_1+k(k-1)x_1^2/2]^{-1}+[kx_2 \quad ky_2+k(k-1)x_2^2/2]^T$

$=[k(x_1+x_2) \quad k(y_1+y_2)+k^2(x_1+x_2)^2/2-k(x_1^2+x_2^2)/2]^T$

$\therefore k(X+Y)=kX+kY$

$(k+h)X=[(k+h)x_1 \quad (k+h)y_1+(k+h)(k+h-1)x_1^2/2]^T$

$=[(k+h)x_1 \quad (k+h)y_1+(k+h)^2x_1^2/2-(k+h)x_1^2/2]^T$

$kX+hX=[kx_1 \quad ky_1+k(k-1)x_1^2/2]^{-1}+[hx_1 \quad hy_1+h(h-1)x_1^2/2]^T$

$=[(k+h)x_1 \quad (k+h)y_1+(k+h)^2x_1^2/2-(k+h)x_1^2/2]^T$

$\therefore (k+h)X=kX+hX$

⑦$1X=[1x_1 \quad 1y_1+1(1-1)x_1^2/2]^T$

$=[x_1 \quad y_1]^T$

$=X$

Homework

1. Show that R^n and P^{n-1} over the same field R (the field of real numbers) are isomorphic.

Proof: Let $e_1, e_2, \cdots, e_n$ be a basis of R^n and $v_1, v_2, \cdots, v_n$ be a basis of P^{n-1}. Generally,

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, e_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \cdots, e_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

$$v_1 = 1, v_2 = t, \cdots, v_n = t^{n-1} \quad .$$

Then , for any $x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in R^n$ and 了齸 ,we have $x = x_1 e_1 + x_2 e_2 + \cdots + x_n e_n$ and

$p(t) = a_0 v_1 + a_1 v_2 + \cdots + a_{n-1} v_n$.

Define a mapping from R^n to P^{n-1} as following,

$$T(x) = x_1 + x_2 t + \cdots + x_n t^{n-1}$$

Next, we prove that this mapping is onto and one to one.

Onto: For any $p(t) = a_0 v_1 + a_1 v_2 + \cdots + a_{n-1} v_n \in P^{n-1}$, there exists such that $T(a) = p(t)$

One to one: for any $x, y \in R^n$, if $T(x) = T(y)$, that is,

$$x_1 + x_2 t + \cdots + x_n t^{n-1} = y_1 + y_2 t + \cdots + y_n t^{n-1}$$

$$(x_1 - y_1) + (x_2 - y_2)t + \cdots + (x_n - y_n)t^{n-1} = 0$$

Since $v_1 = 1, v_2 = t, \cdots, v_n = t^{n-1}$ are linearly independent, we have $x = y$

Therefore we find the isomorphic mapping between R^n and P^{n-1}.

2. Show that each finite-dimensional linear space X over field K is isomorphic to K^n, n= dim X. Show that this isomorphism is not unique.

Proof:

We assume that the dimension of linear space X is n. There exists a basis $v_1, v_2, \cdots, v_n$.

Then ,for any element $x \in X$, $x = x_1 v_1 + x_2 v_2 + \cdots + x_n v_n$, $x_1, x_2, \cdots x_n \in K$

Define a mapping from X to K^n as following

$$T(x) = (x_1, x_2, \cdots x_n),$$

Next, we show that this mapping is isomorphic.

Onto: For any $(x_1, x_2, \cdots x_n) \in K^n$, there exists $x = x_1 v_1 + x_2 v_2 + \cdots + x_n v_n \in X$ such that $T(x) = (x_1, x_2, \cdots x_n)$.

One to one: for any $x, y \in X$, where $y = y_1 v_1 + y_2 v_2 + \cdots + y_n v_n$, if $T(x) = T(y)$, that is,

$$(x_1, x_2, \cdots x_n) = (y_1, y_2, \cdots, y_n)$$

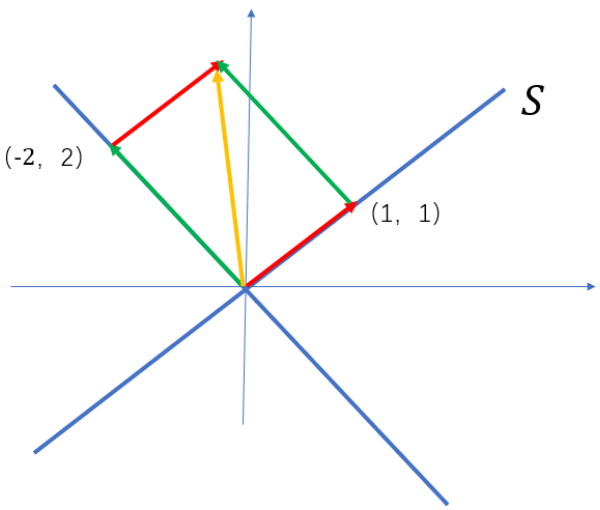
That means $x_i = y_i$, thus $x = y$

Homework

1. Please determine whether the following set is a subspace of R^2. If yes, please show it. If no, please explain the reason.

$$S=\left\{\begin{bmatrix}\omega_1\\ \omega_2\end{bmatrix}\in R^2 \mid \omega_1^2=\omega_2^2\right\}$$

Answer: It is not a subspace. $\omega_1^2=\omega_2^2$ means $\omega_1=\omega_2$ or $\omega_1=-\omega_2$, which are two lines in the R^2 described as the following figure,



(-2,2)+(1,1)=(-1,3), (-1,3) is not in the set *S*. Thus the set S is not closed under the addition. Therefore it is not a subsapace.

2. Please show that the subset $X=\{f(t):R\to R \mid f(t)=f(-t)\}$ of even functions is a subspace of real-valued functions.

Proof: For any $f(t),g(t)\in X$ and $k\in R$, we have

$$(f+g)(t)=f(t)+g(t)=f(-t)+g(-t)=(f+g)(-t)$$

$$(kf)(t)=kf(t)=kf(-t)=(kf)(-t)$$

These show that the set X are closed the addition and scalar multiplication. Thus it is a subspace of real-valued functions.

Homework

1. Please determine whether the following vectors are linear independent?

(1) $a_1 = (2,-1,0,3),\ a_2 = (1,2,5,-1),\ a_3 = (7,-1,5,8)$

(2) $p_1(x) = 1-x,\ p_2(x) = 5+3x-2x^2,\ p_3(x) = 1+3x-2x^2$

Answer：

（1） We consider the coefficients c_1, c_2, c_3 such that

$$c_1 a_1 + c_2 a_2 + c_3 a_3 = 0 \tag{1}$$

Putting a_1, a_2, a_3 into equation (1), we have

$$\begin{aligned} 2c_1 + c_2 + 7c_3 &= 0 \\ -c_1 + 2c_2 - c_3 &= 0 \\ 5c_2 + 5c_3 &= 0 \\ 3c_1 - c_2 + 8c_3 &= 0 \end{aligned} \qquad \begin{bmatrix} 2 & 1 & 7 \\ -1 & 2 & -1 \\ 0 & 5 & 5 \\ 3 & -1 & 8 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{2}$$

The reduced Row Echelon Form (RREF) of coefficient matrix is

$$\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

That is , $c_1 = -3c_3, c_2 = -c_3$, $\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = c_3 \begin{bmatrix} -3 \\ -1 \\ 1 \end{bmatrix}$. Thus the system (2) of linear equations has nonzero solution

(3) We consider the coefficients c_1, c_2, c_3 such that

$$c_1 p_1(x) + c_2 p_2(x) + c_3 p_3(x) = 0 \tag{3}$$

Putting $p_1(x), p_2(x), p_3(x)$ into equation (3), we have

$$c_1(1-x) + c_2(5+3x-2x^2) + c_3(1+3x-2x^2) = 0$$

Arranging this equation, we obtain

$$(c_1 + 5c_2 + c_3) + (-c_1 + 3c_2 + 3c_3)x + (-2c_2 - 2c_3)x^2 = 0$$

Thus

$$\begin{bmatrix} 1 & 5 & 1 \\ -1 & 3 & 3 \\ 0 & -2 & -2 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The reduced Row Echelon Form (RREF) of coefficient matrix is

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

This means that $c_1 = c_2 = c_3 = 0$. Therefore $p_1(x), p_2(x), p_3(x)$ are linearly independent.

2. Let V be a vector space over R and $\mathrm{n} \in \mathrm{N}$ be an odd number. If the vectors $x_1, x_2, \cdots, x_n \in \mathrm{V}$ are linearly independent, then the same stands also for the vectors $x_1 + x_2, x_2 + x_3, \cdots, x_{n-1} + x_n, x_n + x_1$.

Answer: First， we find the coefficients $c_1, c_2, \cdots, c_n$ such that

$$c_1(x_1 + x_2) + \mathrm{c}_2(x_2 + x_3) + \cdots + c_{n-1}(x_{n-1} + x_n) + \mathrm{c}_n(x_n + x_1) = 0$$

Arranging this equation, we obtain

$$(c_1 + c_n)x_1 + (c_1 + c_2)x_2 + \cdots + (c_{n-1} + c_n)x_n = 0$$

From the linear independence of $x_1, x_2, \cdots, x_n$, we have

$$c_1 + c_n = 0, c_1 + c_2 = 0, \cdots, c_{n-1} + c_n = 0$$

Since n is an odd number, $(c_1 + c_n) - (c_1 + c_2) + (\mathrm{c}_2 + c_3) - \ldots\ldots + (c_{n-1} + c_n) = 2c_n = 0$ .From $c_n = 0$,we can obtain $c_1 = c_2 = \cdots = c_{n-1} = 0$

Homework

1. Let $P^{n\times n}$ be a set of all $n\times n$ matrices,

(1) Please show that the set of all the matrices that are exchangeable with $A\in P^{n\times n}$ is a subspace of $P^{n\times n}$. If this subspace is denoted as $C(A)$. It means $C(A)=\{B\in P^{n\times n}\mid AB=BA, A\in P^{n\times n}\}$.

(2) When $A=\begin{bmatrix}1&0&\cdots&0\\0&1&\cdots&0\\0&0&\ddots&0\\0&0&\cdots&1\end{bmatrix}$, please find $C(A)$.

(3) If $A=\begin{bmatrix}1&0&\cdots&0\\0&2&\cdots&0\\0&0&\ddots&0\\0&0&\cdots&n\end{bmatrix}$, please find the dimension and a basis of $C(A)$.

Answer:

(1) For any $B,D\in C(A)$ and $k\in R$, we have

$$A(B+D)=AB+AD=BA+DA=(B+D)A$$

$$A(kB)=kAB=kBA=(kB)A$$

Thus the addition and scalar multiplication are closed. The subset is a subspace of $P^{n\times n}$.

(2) When *A=I,* since for any matrix $B\in P^{n\times n}$, we have $BI=B=IB$. Thus $C(A)=P^{n\times n}$.

(3) When $A=\begin{bmatrix}1&0&\cdots&0\\0&2&\cdots&0\\0&0&\ddots&0\\0&0&\cdots&n\end{bmatrix}$, we assume $B=\begin{bmatrix}b_{11}&b_{12}&\cdots&b_{1n}\\b_{21}&b_{22}&\cdots&b_{2n}\\\vdots&\vdots&\ddots&\vdots\\b_{n1}&b_{n2}&\cdots&b_{nn}\end{bmatrix}$. Putting A into $AB=BA$, the left side of equation is

$$\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 2 & \cdots & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & \cdots & n \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ 2b_{21} & 2b_{22} & \cdots & 2b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ nb_{n1} & nb_{n2} & \cdots & nb_{nn} \end{bmatrix}$$

and the right side of equation is

$$\begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{bmatrix} \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 2 & \cdots & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & \cdots & n \end{bmatrix} = \begin{bmatrix} b_{11} & 2b_{12} & \cdots & nb_{1n} \\ b_{21} & 2b_{22} & \cdots & nb_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & 2b_{n2} & \cdots & nb_{nn} \end{bmatrix}$$

By the equality of matrix, we have

$$ib_{ij} = jb_{ij} \text{ when } i \neq j.$$

Thus $b_{ij} = 0$ when $i \neq j$. This shows that all the matrices that are exchangeable with A are diagonal matrices. For the set of diagonal metrices C(A),

$$\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}, \cdots\cdots\cdots \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$ is a basis of C(A). Thus its dimension is n.

2. Please find a basis and the dimension of the solution space of the following system of linear equations

$$\begin{cases} 3x_1 + 2x_2 - 5x_3 + 4x_4 = 0 \\ 3x_1 - x_2 + 3x_3 - 3x_4 = 0 \\ 3x_1 + 5x_2 - 13x_3 + 11x_4 = 0 \end{cases}$$

Answer：

Transforming the system of linear equations into a matrix form, we have

$$\begin{bmatrix} 3 & 2 & -5 & 4 \\ 3 & -1 & 3 & -3 \\ 3 & 5 & -13 & 11 \\ & & & \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \end{bmatrix}$$

Applying the elementary row operations, we can obtain the Reduced Row Echelon Form,

$$\begin{bmatrix} 1 & 0 & 1/9 & -2/9 \\ 0 & 1 & -8/3 & 7/3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = 0$$

This means

$$x_1 = -\frac{1}{9}x_3 + \frac{2}{9}x_4$$
$$x_2 = \frac{8}{3}x_3 - \frac{7}{3}x_4$$
$$x_3 = x_3$$
$$x_4 = x_4$$

Therefore, all the solution can be presented as following

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} -\frac{1}{9} \\ 8/3 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 2/9 \\ -7/3 \\ 0 \\ 1 \end{bmatrix}$$

and a basis of solution space is $\begin{bmatrix} -\frac{1}{9} \\ 8/3 \\ 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 2/9 \\ -7/3 \\ 0 \\ 1 \end{bmatrix}$ and the dimension is 2.

Homework

1. Let $X=\{x|(x_1,x_2)\in R^2\}$,for two fixed c_1,c_2 ,let $f(x)=c_1x_1+c_2x_2$,please show that $f(x)$ is a linear function defined on linear space X . Please show that set $X'=\{f(x)|f(x)=c_1x_1+c_2x_2,c_1\in R,c_2\in R\}$ is a linear space and point out a basis of this space.

Proof: First we prove that $f(x)$ is a linear function.

For any $x=(x_1,x_2),y=(y_1,y_2)\in R^2$, $x+y=(x_1+y_1,x_2+y_2),kx=(kx_1,kx_2)$

$$f(x+y)=c_1(x_1+y_1)+c_2(x_2+y_2)=c_1x_1+c_2x_2+c_1y_1+c_2y_2=f(x)+f(y)$$

$$f(kx)=c_1(kx_1)+c_2(kx_2)=k(c_1x_1+c_2x_2)=kf(x)$$

Thus $f(x)$ is a linear function on linear space X .

Second, we prove that X' is a linear space, where the addition and scalar multiplication are the addition and scalar multiplication of real-valued functions.

For any $f(x)=f_1x_1+f_2x_2,g(x)=g_1x_1+g_2x_2\in X',f_1,f_2\in R,g_1,g_2\in R$,

$$f(x)+g(x)=f_1x_1+f_2x_2+g_1x_1+g_2x_2=(f_1+g_1)x_1+(f_2+g_2)x_2\in X'$$

$$kf(x)=kf_1x_1+kf_2x_2=(kf_1)x_1+(kf_2)x_2\in X'$$

Thus the set is closed under addition and scalar multiplication. Next we prove the eight axioms:

1. $f(x)+g(x)=f_1x_1+f_2x_2+g_1x_1+g_2x_2=g_1x_1+g_2x_2+f_1x_1+f_2x_2=g(x)+f(x)$

2. $$\begin{aligned}f(x)+(g(x)+h(x))&=f_1x_1+f_2x_2+(g_1x_1+g_2x_2+h_1x_1+h_2x_2)\\&=(f_1x_1+f_2x_2+g_1x_1+g_2x_2)+h_1x_1+h_2x_2\\&=(f(x)+g(x))+h(x)\end{aligned}$$

3. $f(x)+0(x)=f(x)$, where $0(x)=0x_1+0x_2$ is the zero element.

4. $f(x)+(-f(x))=0$, where $-f(x)=-f_1x_1-f_2x_2$

5. $k(lf(x))=k(lc_1x_1+lc_2x_2)=(kl)(c_1x_1+c_2x_2)=(kl)f(x)$

6. $$\begin{aligned}k(f(x)+g(x))&=k(f_1x_1+f_2x_2+g_1x_2+g_2x_2)=kf_1x_1+kf_2x_2+kg_1x_2+kg_2x_2\\&=k(f_1x_1+f_2x_2)+k(g_1x_2+g_2x_2)=kf(x)+kg(x)\end{aligned}$$

7. $(k+l)f(x)=(k+l)(f_1x_1+f_2x_2)=kf(x)+lf(x)$

8. $1f(x)=f(x)$

2. Let $X=\{h(s)\,|\,h(s)\text{is a continuous function defined on }[0,1]\ ,0\le \mathrm{s}\le 1\}$, for any point s_1 in [0,1], please show that $f(h)=h(s_1)$ is a linear function defined on linear space X .

Proof: For any $h(s),g(s)\in X$,

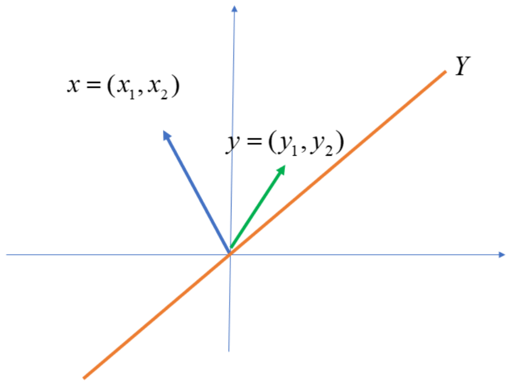
$f(h(s)+g(s))=h(s_1)+g(s_1)=f(h(s))+f(g(s))$

$f(kh(s))=kh(s_1)=kf(h(s))$

Thus $f(h)=h(s_1)$ is a linear function.

Homework of quotient space

Let $X = \{x \mid x = (x_1, x_2) \in R^2\}$. The vector x, y in X can be drawn as a vector in the following figure. Y is a line though the origin as described in the figure. Then Y is a subspace. Please draw the congruence class mod Y of x and y and draw the sum of two congruence classes $\{x\} + \{y\}$.



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Answer：