Homework

1. Show that the function that takes $((x_1, x_2),(y_1, y_2)) \in \mathrm{R}^2 \times R^2$ to $|x_1 y_1| + |x_2 y_2|$ is not an inner product on R^2.

   Solution: $|x_1 y_1| + |x_2 y_2|$ is not a bilinear function. For example, for fixed (y_1, y_2), $f(x) = |y_1 x_1| + |y_2 x_2|$ is a function of $x = (x_1, x_2)$ but is not a linear function. For any $a = (a_1, a_2)$, $b = (b_1, b_2)$, we can find $f(a+b) \neq f(a) + f(b)$ since

   $f(a+b) = |y_1(a_1 + b_1)| + |y_2(a_2 + b_2)| = |y_1 a_1 + y_1 b_1| + |y_2 a_2 + y_2 b_2|$

   $f(a) + f(b) = |y_1 a_1| + |y_2 a_2| + |y_1 b_1| + |y_2 b_2|$

2. Suppose V is a real inner product space, show that

   (a) Show that the inner product $\langle u+v, u-v \rangle = \|u\|^2 - \|v\|^2$ for every $u, v \in V$.

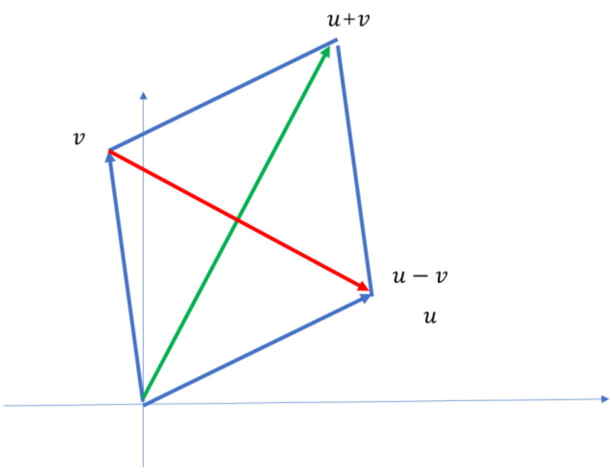
   (b) Show that if $u, v \in V$ have the same norm, then $u+v$ is orthogonal to $u-v$.

   (c) Use part(b) to show that the diagonals of a rhombus are perpendicular to each other.

   Solution: (a) $\langle u+v, u-v \rangle = \langle u, u \rangle + \langle v, u \rangle - \langle u, v \rangle - \langle v, v \rangle = \|u\|^2 - \|v\|^2$

   (b) If $u, v \in V$ have the same norm, we have $\langle u+v, u-v \rangle = \|u\|^2 - \|v\|^2 = 0$, thus $u+v$ is orthogonal to $u-v$.

   (c) For a rhombus, we can set up a coordinate in R^2 as described in the following figure



Thus the conclusion that the diagonals of a rhombus are perpendicular can be obtained by part (b).

3. Suppose $u, v \in V$. Prove that the inner product $\langle u, v \rangle = 0$ if and only if $\|u\| \le \|u + av\|$ for all $a \in R$.

Solution: First , we prove "only if": since $\langle u, v \rangle = 0$,

$$\begin{aligned} \|u+av\|^2 &= \langle u+av, u+av \rangle = \|u\|^2 + a\langle u, v\rangle + a^2\|v\|^2 \\ &= \|u\|^2 + a^2\|v\|^2 \ge \|u\|^2 \end{aligned}$$

Thus, we prove $\|u\| \le \|u + av\|$ by the square root.

Next we prove "if": If , $\|u\| \le \|u + av\|$, we prove that $\langle u, v \rangle = 0$.

From $\|u\| \le \|u + av\|$, we have $\|u\|^2 \le \|u + av\|^2$, thus $a\langle u, v \rangle + a^2\|v\|^2 \ge 0$. This means that quadratic function $f(a) = \|v\|^2 a^2 + \langle u, v \rangle a$ is always lager than zero. Since $\|v\|^2 \ge 0$, the discriminant $\Delta = \langle u, v \rangle^2 \le 0$, thus we have $\langle u, v \rangle = 0$

4. Suppose n is a positive integer. Prove that $\frac{1}{\sqrt{2\pi}}, \frac{\cos x}{\sqrt{\pi}}, \frac{\cos 2x}{\sqrt{\pi}}, \cdots, \frac{\cos nx}{\sqrt{\pi}}, \frac{\sin x}{\sqrt{\pi}}, \frac{\sin 2x}{\sqrt{\pi}}, \cdots, \frac{\sin nx}{\sqrt{\pi}}$ is an orthonormal list of vectors in $C[-\pi, \pi]$, the linear space of continuous real-valued functions on $[-\pi, \pi]$ with inner product $\langle f, g \rangle = \int_{-\pi}^{\pi} f(x)g(x)dx$.

Solution: To prove the above functions is an orthonormal list, we need to do the following computations:
First, we prove they are orthogonal each other

$$\left\langle \frac{1}{\sqrt{2\pi}}, \frac{\cos nx}{\sqrt{\pi}} \right\rangle = \int_{-\pi}^{\pi} \frac{\cos nx}{\sqrt{2}\pi} dx = \frac{1}{\sqrt{2}\pi} \frac{1}{n}(2\sin n\pi) = 0$$

$$\left\langle \frac{1}{\sqrt{2\pi}}, \frac{\sin nx}{\sqrt{\pi}} \right\rangle = \int_{-\pi}^{\pi} \frac{\sin nx}{\sqrt{2}\pi} dx = \frac{1}{\sqrt{2}\pi}(-\frac{1}{n})(\mathrm{cosnx})\Big|_{-\pi}^{\pi} = 0$$

$$\left\langle \frac{\cos mx}{\sqrt{\pi}}, \frac{\sin nx}{\sqrt{\pi}} \right\rangle = \int_{-\pi}^{\pi} \frac{\sin nx \cos mx}{\pi} dx = \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{1}{2} [\sin(n+m)x + \sin(n-m)x] dx$$

$$= \frac{1}{2\pi} [(-\frac{1}{n+m}) \cos(n+m)x \Big|_{-\pi}^{\pi} + (-\frac{1}{n-m}) \cos(n-m)x \Big|_{-\pi}^{\pi} = 0$$

Then we prove they are normalized.

$$\left\langle \frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{2\pi}} \right\rangle = \int_{-\pi}^{\pi} \frac{1}{2\pi} dx = 1$$

$$\left\langle \frac{\cos nx}{\sqrt{\pi}}, \frac{\cos nx}{\sqrt{\pi}} \right\rangle = \int_{-\pi}^{\pi} \frac{\cos^2 nx}{\pi} dx = \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{1}{2} [1 + \cos(2nx)] dx$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} [1 + \cos(2nx)] dx = 1 + \int_{-\pi}^{\pi} \cos(2nx) dx = 1$$

$$\left\langle \frac{\sin nx}{\sqrt{\pi}}, \frac{\sin nx}{\sqrt{\pi}} \right\rangle = \int_{-\pi}^{\pi} \frac{\sin^2 nx}{\pi} dx = \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{1}{2} [1 - \cos(2nx)] dx$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} [1 - \cos(2nx)] dx = 1 - \int_{-\pi}^{\pi} \cos(2nx) dx = 1$$

5. On $P^2[x]$, the linear space of polynomial functions of degree ≤ 2, consider the inner product given by $\langle p, q \rangle = \int_0^1 p(x) q(x) dx$. Apply the Gram–Schmidt Procedure to the basis $1, x, x^2$ to produce an orthonormal basis of $P^2[x]$

   Solution:

   Let $u_1 = 1$, $\langle u_1, u_1 \rangle = \int_0^1 1 dx = 1$, thus we obtain the first orthonormal vector $v_1 = 1$

   According to the Gram – Schmidt Procedure, $\langle x, v_1 \rangle = \int_0^1 x dx = \frac{1}{2}$ let $u_2 = x - (x, v_1) v_1 = x - \frac{1}{2}$,

   $\langle u_2, u_2 \rangle = \int_0^1 (x - \frac{1}{2})^2 dx = \int_0^1 (x^2 - x + \frac{1}{4}) dx = (\frac{1}{3} x^3 - \frac{1}{2} x^2 + \frac{1}{4} x) \Big|_0^1 = \frac{1}{12}$

   thus, we obtain the second orthonormal vector $v_2 = \sqrt{12}(x - \frac{1}{2}) = 2\sqrt{3}(x - \frac{1}{2})$

   $$\langle x^2, v_1 \rangle = \int_0^1 x^2 dx = \frac{1}{3}, \quad \langle x^2, v_2 \rangle = 2\sqrt{3} \int_0^1 x^2 (x - \frac{1}{2}) dx$$

   $$= 2\sqrt{3} \int_0^1 (x^3 - \frac{1}{2} x^2) dx = 2\sqrt{3} (\frac{1}{4} x^4 - \frac{1}{6} x^3) \Big|_0^1 = \frac{1}{\sqrt{12}}$$

   Let $u_3 = x^2 - (x^2, v_1) v_1 - (x^2, v_2) v_2 = x^2 - x + \frac{1}{6}$,

$$\langle u_3,u_3\rangle=\int_0^1(x^2-x+\frac{1}{6})^2dx=\frac{1}{180}$$,thus $v_3=6\sqrt{5}(x^2-x+\frac{1}{6})$

6. Find a polynomial $q\in P^2[x]$ such that $p\left(\frac{1}{2}\right)=\int_0^1 p(x)q(x)dx$ for every $p(x)\in P^2[x]$.

Solution: From problem 5, there is an orthonormal basis v_1,v_2,v_3 for $P^2[x]$, thus $p(x),q(x)$ can be written as $p(x)=p_1v_1+p_2v_2+p_3v_3, q(x)=q_1v_1+q_2v_2+q_3v_3$.

Since $p(x)$ is an arbitrary polynomial, p_1,p_2,p_3 may be any real numbers.

$$\text{leftside}=p(\frac{1}{2})=p_1+p_2[2\sqrt{3}(\frac{1}{2}-\frac{1}{2})]+\text{p}_3[6\sqrt{5}(\frac{1}{4}-\frac{1}{2}+\frac{1}{6})]=\text{p}_1-\frac{\sqrt{5}}{2}p_3$$

$$\text{rightside}=\langle p,q\rangle=p_1q_1+p_2q_2+p_3q_3$$

Considering the arbitrary property of p_1,p_2,p_3, we have $q_1=1,q_2=0,q_3=-\frac{\sqrt{5}}{2}$.

Thus, $q(x)=1-\frac{\sqrt{5}}{2}6\sqrt{5}(x^2-x+\frac{1}{6})=-15x^2+15x-\frac{3}{2}$

7. Suppose U is the subspace of R^4 defined by $U=span\left((1,2,3,-4),(-5,4,3,2)\right)$ Find an orthonormal basis of U and an orthonormal basis of its orthogonal complement $U^\perp$

Solution: $u_1=(1,2,3,-4)$, $\|u_1\|^2=1+2*2+3*3+4*4=30,\|u_1\|=\sqrt{30}$,then

$$v_1=\frac{1}{\sqrt{30}}(1,2,3,-4)$$.

$$u_2=(-5,4,3,2)-\frac{2}{15}(1,2,3,-4)=\frac{1}{15}(-77,56,39,38)$$

$$\|u_2\|^2=\left(\frac{-77}{15}\right)^2+\left(\frac{56}{15}\right)^2+\left(\frac{39}{15}\right)^2+\left(\frac{38}{15}\right)^2$$,$\|u_2\|\approx 7.3$ $v_2=\frac{u_2}{\|u_2\|}$

Let $v=(x_1, x_2, x_3,, x_4)^T\in U^\perp$, then we have

$$x_1+2x_2+3x_3-4x_4=0$$
$$-5x_1+4x_2+3x_3+2x_4=0$$

By using the RREF form, we have

$$x_1 = -\frac{3}{7}x_3 + \frac{10}{7}x_4$$

$$x_2 = -\frac{9}{7}x_3 + \frac{9}{7}x_4$$

This is to say,

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} -\frac{3}{7} \\ -\frac{9}{7} \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} \frac{10}{7} \\ \frac{9}{7} \\ 0 \\ 1 \end{bmatrix}$$

So, $U^{\perp}=\text{span}\left\{ \begin{bmatrix} -\frac{3}{7} \\ -\frac{9}{7} \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{10}{7} \\ \frac{9}{7} \\ 0 \\ 1 \end{bmatrix} \right\}$, we repeat the above process to obtain the orthonormal basis of the orthogonal complement $U^{\perp}$

8. In R^4, let $U = span\left((1,1,0,0),(1,1,1,2)\right)$. Find $u \in U$ such that $\|u-(1,2,3,4)\|_2$ is as small as possible.

Solution: First, we obtain the projection matrix

$$P_U = C(C^TC)^{-1}C^T$$

Where $C = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 1 \\ 0 & 2 \end{bmatrix}$. Then $P_U = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 1 \\ 0 & 2 \end{bmatrix} \frac{1}{10} \begin{bmatrix} 7 & -2 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 2 \end{bmatrix} = \frac{1}{10} \begin{bmatrix} 5 & 5 & 0 & 0 \\ 5 & 5 & 0 & 0 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 4 & 8 \end{bmatrix}$

Then $u = \frac{1}{10} \begin{bmatrix} 5 & 5 & 0 & 0 \\ 5 & 5 & 0 & 0 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 4 & 8 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 3/2 \\ 3/2 \\ 11/5 \\ 22/5 \end{bmatrix}$