Homework 1

April 6, 2023

1. Let V be a vector space and let $\mathbf{x}, \mathbf{y} \in V$. Show that

   (a) $\beta\mathbf{0} = \mathbf{0}$ for each scalar β.

   (b) $\mathbf{x} + \mathbf{y} = \mathbf{0}$ implies that $\mathbf{y} = -\mathbf{x}$.

   (c) $(-1)\mathbf{x} = -\mathbf{x}$.

2. Let V be the set of all ordered pairs of real numbers with addition defined by

$$(x_1, x_2) + (y_1, y_2) = (x_1 + y_1, x_2 + y_2)$$

   and scalar multiplication defined by

$$\alpha \circ (x_1, x_2) = (\alpha x_1, x_2)$$

   Scalar multipliacation for this system is defined in an unusual way, and consequently we use the symbol o to avoid confusion with the ordinary scalar multiplication of row vectors. Is V a vector space with these operations? Justify your answer.

3. Let R^+ denote the set of positive real numbers. Define the operation of scalar multiplication, denoted o, by

$$\alpha \circ x = x^\alpha$$

   for each $x \in R^+$ and for any real number α. Define the operation of addition, denoted $\oplus$, by

$$x \oplus y = x \cdot y \quad \text{for all} \quad x, y \in R^+$$

   Thus, for this system, the scalar product of -3 times $\frac{1}{2}$ is given by

$$-3 \circ \frac{1}{2} = \left(\frac{1}{2}\right)^{-3} = 8$$

   and the sum of 2 and 5 is given by

$$2 \oplus 5 = 2 \cdot 5 = 10$$

   Is R^+ a vector space with these operations? Prove your answer.

4. Suppose $\mathbf{u}_1, \ldots, \mathbf{u}_p$ and $\mathbf{v}_1, \ldots, \mathbf{v}_q$ are vectors in a vector space V, and let $H = \text{Span}(\mathbf{u}_1, \ldots, \mathbf{u}_p)$ and $K = \text{Span}(\mathbf{v}_1, \ldots, \mathbf{v}_q)$.

   (a) Show that $H \bigcap K$ is a subspace of V.

   (b) Show that H and K are subspaces of $H + K$.

   (c) Show that $H + K = \text{Span}(\mathbf{u}_1, \ldots, \mathbf{u}_p, \mathbf{v}_1, \ldots, \mathbf{v}_q)$.

5. Let $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k\}$ be a spanning set for a vector space V.

   (a) If we add another vector, $\mathbf{x}_{k+1}$, to the set, will we still have a spanning set? Explain.

   (b) If we delete one of the vectors,say, $\mathbf{x}_k$, from the set, will we still have a spanning set? Explain.