Homework 2

April 14, 2023

1. Suppose $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4$ spans V. Prove that the list

$$\mathbf{x}_1 - \mathbf{x}_2, \mathbf{x}_2 - \mathbf{x}_3, \mathbf{x}_3 - \mathbf{x}_4, \mathbf{x}_4$$

also spans V.

2. Suppose $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4$ is linearly independent in V. Prove that the list

$$\mathbf{x}_1 - \mathbf{x}_2, \mathbf{x}_2 - \mathbf{x}_3, \mathbf{x}_3 - \mathbf{x}_4, \mathbf{x}_4$$

is also linearly independent.

3. Suppose

$$U = \{(x, 3x, y, 7y) \in \mathbb{R}^4 : x, y \in \mathbb{R}\}$$

Find a basis of U and a subspace W of $\mathbb{R}^4$ such that $\mathbb{R}^4 = U \oplus W$.

4. For each subspace in (a)-(d), (1) find a basis, and (2) state the dimension.

(a)

$$\left\{ \begin{bmatrix} x + 2y \\ 2x - 3y \\ -x \end{bmatrix} : x, y \text{ in } \mathbb{R} \right\}$$

(b)

$$\left\{ \begin{bmatrix} x + 3y - z \\ 4x + 5y + 3z \\ 3x + 6z \\ -x + 7y - 9z \end{bmatrix} : x, y, z \text{ in } \mathbb{R} \right\}$$

(c)

$$\{(x, y, z, w) : x - 4y + 3w = 0\}$$

5. Let

$$\mathbf{x}_1 = \begin{bmatrix} 2 \\ -1 \\ 1 \\ 2 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 2 \end{bmatrix}, \quad \mathbf{x}_3 = \begin{bmatrix} 4 \\ -3 \\ 1 \\ a \end{bmatrix}$$

If $\dim(\text{Span}(x_1, x_2, x_3)) = 2$, compute a.

6. V is a nonzero finite-dimensional vector space, and the vectors listed belong to V. Mark each statement True or False. Justify each answer (Prove it if True or give an anti-example if False).

   a. If $\dim V = p$ and S is a linearly independent set in V, then S is a basis for V.

   b. If there exists a linearly independent set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ in V, then $\dim V \geq p$.

   c. If there exists a set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ that spans V, then $\dim V \leq p$.

   d. If every set of p elements in V fails to span V, then dim $V > p$.

   e. If there exists a linearly dependent set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ in V, then $\dim V \leq p$.

7. Show that if U and V are subspaces of $\mathbb{R}^n$ and $U \bigcap V = \{\mathbf{0}\}$, then

$$\dim(U + V) = \dim U + \dim V$$

8. Consider two ordered bases $E = \{v_1, v_2, v_3\}$ and $F = \{w_1, w_2, w_3\}$ for $\mathbb{R}^3$, where

$$v_1 = \begin{bmatrix} 4 \\ 6 \\ 7 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$
$$w_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad w_2 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \quad w_3 = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$$

   (a) Find the transition matrix S_1 from E to F.

   (b) Find the transition matrix S_2 from F to E.

   (c) Verify that $S_1 S_2 = S_2 S_1 = I_3$.

   (d) If $[v]_E = \begin{bmatrix} 2 \\ 3 \\ -4 \end{bmatrix}$, compute $[v]_F$ and use S_1 or S_2 (decide by yourself) to verify your answer.