Homework 4

April 27, 2023

1. Suppose $\mathbf{v}_1, ..., \mathbf{v}_m$ is a list of vectors in V. Define $T \in \mathcal{L}(\mathbb{R}^m, V)$ by

$$T(\mathbf{x}) = x_1\mathbf{v}_1 + ... + x_m\mathbf{v}_m,$$

for $\mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix} \in \mathbb{R}^m$. (Injective or Surjective.)

   (a) What property of T corresponds to $\mathbf{v}_1, ..., \mathbf{v}_m$ spanning V? Why?

   (b) What property of T corresponds to $\mathbf{v}_1, ..., \mathbf{v}_m$ being linearly independent? Why?

2. (a) Suppose $T \in \mathcal{L}(V, W)$ is injective and $\mathbf{v}_1, ..., \mathbf{v}_n$ is linearly independent in V. Prove that $T(\mathbf{v}_1), ...T(\mathbf{v}_n)$ is linearly independent in W.

   (b) Suppose V is finite-dimensional and that $T \in \mathcal{L}(V, W)$. Prove that there exists a subspace U of V such that $U \cap \operatorname{null} T = \{0\}$ and $T(V) = T(U)$. Find a basis.

3. (a) Suppose V and W are both finite-dimensional. Prove that there exists an surjective linear transfomation from V onto W if and only if $\dim V \geq \dim W$.

   (b) Suppose V and W are finite-dimensional and that U is a subspace of V. Prove that there exists $T \in \mathcal{L}(V, W)$ such that $\operatorname{null} T = U$ if and only if $\dim U \geq \dim V - \dim W$

4. (a) Suppose $T \in \mathcal{L}(\mathbb{R}^4, \mathbb{R}^2)$ and $\operatorname{null} T = \{x = [x_1, x_2, x_3, x_4]^{\mathrm{T}} \in \mathbb{R}^4 | x_1 = 2x_2$ and $x_3 = 5x_4\}$. Prove that T is surjective.

   (b) Prove that there does not exist a linear transformation from $\mathbb{R}^5$ to $\mathbb{R}^2$ whose null space equals $\{x = [x_1, x_2, x_3, x_4, x_5]^{\mathrm{T}} \in \mathbb{R}^4 | x_1 = 2x_2$ and $x_3 = x_4 = x_5\}$.

5. Find the standard matrices of the following linear transformations.

(a) $T \in \mathcal{L}(\mathbb{R}^2, \mathbb{R}^3)$ with $T(x) = \begin{bmatrix} 2x_1 - x_2 \\ -x_1 + x_2 \\ x_2 \end{bmatrix}$ for $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

(b) $T \in \mathcal{L}(\mathbb{R}^2, \mathbb{R}^2)$ is a vertical shear transformatin that leaves $\mathbf{e}_2$ unchanged and maps $\mathbf{e}_1$ into $2\mathbf{e}_2 + \mathbf{e}_1$.

(c) $T \in \mathcal{L}(\mathbb{R}^2, \mathbb{R}^2)$ first performs a horizontal shear transformatin that leaves $\mathbf{e}_1$ unchanged and maps $\mathbf{e}_2$ into $-2\mathbf{e}_1 + \mathbf{e}_2$ and then reflects points through the line $x_2 = -x_1$.

6. Let

$$\mathbf{b}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \mathbf{b}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \mathbf{b}_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

and let L be the linear transformation from $\mathbb{R}^2$ into $\mathbb{R}^3$ define by

$$L(\mathbf{x}) = (x_1 - x_2)\mathbf{b}_1 + x_2\mathbf{b}_2 + (x_1 + x_2)\mathbf{b}_3,$$

find the matrix A representing L with respect to the ordered bases $\{\mathbf{e}_1, \mathbf{e}_2\}$ and $\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$.