Homework 5

May 8, 2023

1. Let $E = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ and $F = \{\mathbf{b}_1, \mathbf{b}_2\}$, where

$$\mathbf{u}_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \mathbf{u}_2 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \mathbf{u}_2 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$$

and

$$\mathbf{b}_1 = (1, -1)^T, \quad \mathbf{b}_2 = (2, -1)^T$$

For each of the following linear transformations L from $\mathbb{R}^3$ into $\mathbb{R}^2$, find the matrix representing L with respect to the ordered bases E and F:

(i) $L(\mathbf{x}) = \begin{pmatrix} x_1 + x_2 \\ x_1 - x_3 \end{pmatrix}$

(ii) $L(\mathbf{x}) = \begin{pmatrix} 2x_2 \\ -x_1 \end{pmatrix}$

2. Let D be the differentiation operator on $\mathbb{P}_2(\mathbb{R})$. Find the matrix B representing D with respect to $\left[1, x, x^2\right]$, the matrix A representing D with respect to $\left[2, 4x, 4x^2 - 4\right]$, and the nonsingular matrix S such that $B = S^{-1}AS$.

3. Suppose V and W are finite-dimensional and $T \in \mathcal{L}(V, W)$. Prove that $\dim \operatorname{range} T = 1$ if and only if there exist a basis of V and a basis of W such that with respect to these bases, all entries of the matrix representation $\mathcal{M}(T)$ equal 1.

4. For each of the following systems $Ax = b$, find all the least squares solutions:

(i) $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \\ -1 & 1 \end{bmatrix}, b = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$;

(ii) $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \\ -1 & -2 \end{bmatrix}, b = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$.

5. (a) Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$. Prove that the following statements are equivalent:
(i) T is invertible;
(ii) T is injective;
(iii) T is surjective.

(b) Suppose V is finite-dimensional, U is a subspace of V, and $S \in \mathcal{L}(U, V)$. Prove there exists an invertible operator $T \in \mathcal{L}(V)$ such that $T(u) = S(u)$ for every $u \in U$ if and only if S is injective.

(c) Suppose V, W are finite-dimensional and $T_1, T_2 \in \mathcal{L}(V, W)$. Prove that $\operatorname{null} T_1 = \operatorname{null} T_2$ if and only if there exists an invertible operator $S \in \mathcal{L}(W)$ such that $T_1 = ST_2$.

(d) Suppose V, W are finite-dimensional and $T_1, T_2 \in \mathcal{L}(V, W)$. Prove that $\dim \operatorname{null} T_1 = \dim \operatorname{null} T_2$ if and only if there exists an invertible operators $R \in \mathcal{L}(V)$ and $S \in \mathcal{L}(W)$ such that $T_1 = ST_2R$.