哈爾濱工業大學
HARBIN INSTITUTE OF TECHNOLOGY

控制系统中的代数基础答疑课

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Part.01

Homeworks1

Problem1

Show that the function that takes $((x_1, x_2), (y_1, y_2)) \in R^2 \times R^2$ to $|x_1 y_1| + |x_2 y_2|$ is not an inner product on R^2.

6.3 Definition ***inner product***

An ***inner product*** on V is a function that takes each ordered pair (u, v) of elements of V to a number $\langle u, v\rangle \in \mathbf{F}$ and has the following properties:

positivity
$\langle v, v\rangle \geq 0$ for all $v \in V$;

definiteness
$\langle v, v\rangle = 0$ if and only if $v = 0$;

additivity in first slot
$\langle u + v, w\rangle = \langle u, w\rangle + \langle v, w\rangle$ for all $u, v, w \in V$;

homogeneity in first slot
$\langle \lambda u, v\rangle = \lambda\langle u, v\rangle$ for all $\lambda \in \mathbf{F}$ and all $u, v \in V$;

conjugate symmetry
$\langle u, v\rangle = \overline{\langle v, u\rangle}$ for all $u, v \in V$.

$$< x+y, z >= |(x_1+y_1)z_1| + |(x_2+y_2)z_2| = |x_1 z_1 + y_1 z_1| + |x_2 z_2 + y_2 z_2|$$

$$< x, z > + < y, z >= |x_1 z_1| + |x_2 z_2| + |y_1 z_1| + |y_2 z_2|$$

The above function does not satisfy the additivity. Thus it is not an inner product. In addition, it also does not satisfy the homogeneity.

Problem2

Suppose V is a real inner product space, show that:

a) the inner product $\langle u+v, u-v\rangle = \|u\|^2 - \|v\|^2$ for every $u, v \in V$.

b) if $u, v \in V$ have the same norm, then $u+v$ is orthogonal to $u-v$.

c) use part(b) to show that the diagonals of a rhombus are perpendicular to each other.

Answer:

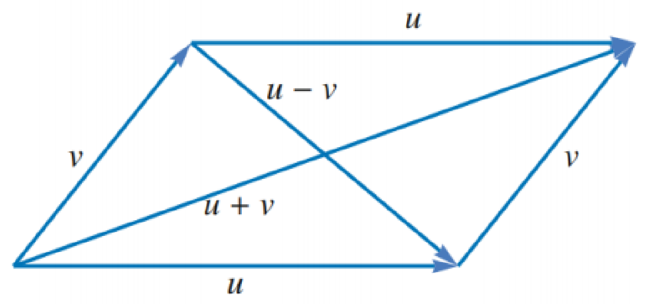
(a) Note that V is a real inner product space, we have $\langle u, v\rangle = \langle v, u\rangle$. Hence

$$\begin{aligned}\langle u+v, u-v\rangle &= \langle u,u\rangle - \langle u,v\rangle + \langle v,u\rangle - \langle v,v\rangle \\ &= \langle u,u\rangle - \langle v,v\rangle = \| u \|^2 - \| v \|^2\end{aligned}$$

(b) By (a).

$$\begin{aligned}\langle u+v, u-v\rangle &= \langle u,u\rangle - \langle u,v\rangle + \langle v,u\rangle - \langle v,v\rangle \\ &= \langle u,u\rangle - \langle v,v\rangle = \| u \|^2 - \| v \|^2 = 0\end{aligned}$$

(c)



Note that $\| u \| = \| v \|$ for a rhombus,

Problem2

Suppose V is a real inner product space, show that:

a) the inner product $\langle u+v, u-v\rangle = \|u\|^2 - \|v\|^2$ for every $u, v \in V$.

b) if $u, v \in V$ have the same norm, then $u+v$ is orthogonal to $u-v$.

c) use part(b) to show that the diagonals of a rhombus are perpendicular to each other.

续上

$$\begin{aligned}\langle u+v, u-v\rangle &= \langle u,u\rangle - \langle u,v\rangle + \langle v,u\rangle - \langle v,v\rangle \\ &= \langle u,u\rangle - \langle v,v\rangle = \| u \|^2 - \| v \|^2 = 0\end{aligned}$$

Therefore, the diagonals of a rhombus are perpendicular to each other.

Problem3

Suppose $u, v \in V$, prove that the inner product $\langle u, v\rangle = 0$ if and only if $\|u\| \leqslant \|u + av\|$ for all $a \in R$.

Proof:

(sufficiency) If $\langle u, v\rangle = 0$, then

$$\| u + av \|^2 = \| u \|^2 + \| av \|^2 \geq \| u \|^2$$

by 6.13.

6.13 Pythagorean Theorem

Suppose u and v are orthogonal vectors in V. Then

$$\|u + v\|^2 = \|u\|^2 + \|v\|^2.$$

(necessity) If $\| u \| \leq \| u + av \|$ for all $a \in \mathbb{F}$, this implies

$$\| u + av \|^2 - \| u \|^2 = |a|^2 \| v \|^2 + a\langle v, u\rangle + \bar{a}\langle u, v\rangle \geq 0.$$

If $v = 0$, then $\langle u, v\rangle = 0$. If $v \neq 0$, plug $a = -\langle u, v\rangle / \| v \|^2$ into the previous equation ,we obtain

$$-\frac{|\langle u, v\rangle|^2}{\| v \|^2} \geq 0.$$

Hence $\langle u, v\rangle = 0$.

Problem4

Suppose $u, v \in V$, prove that $\|au + bv\| = \|bu + av\|$ for all $a, b \in R$ if and only if $\|u\| = \|v\|$.

Proof:

If $\| av + bu \| = \| au + bv \|$ for all $a, b \in \mathbb{R}$, by setting $a = 1$ and $b = 0$, we have $\| u \| = \| v \|$.

Conversely, suppose $\| u \| = \| v \|$. For all $a, b \in \mathbb{R}$, we have

$$\begin{aligned} \| av + bu \|^2 &= \langle av + bu, av + bu \rangle \\ &= a^2 \| u \|^2 + ab(\langle u, v \rangle + \langle v, u \rangle) + b^2 \| v \|^2 \end{aligned}$$

and

$$\begin{aligned} \| au + bv \|^2 &= \langle au + bv, au + bv \rangle \\ &= a^2 \| v \|^2 + ab(\langle u, v \rangle + \langle v, u \rangle) + b^2 \| u \|^2. \end{aligned}$$

Hence if $\| v \| = \| u \|$, we have

$$a^2 \| u \|^2 + b^2 \| v \|^2 = a^2 \| v \|^2 + b^2 \| u \|^2.$$

Therefore $\| av + bu \|^2 = \| au + bv \|^2$, i.e. $\| av + bu \| = \| au + bv \|$.

Problem5

Suppose $u, v \in V$, $\|u\| = \|v\| = 1$ and $\langle u, v\rangle = 1$, prove that $u = v$.

Proof:

Consider $\| u - v \|^2$, we have

$$\| u - v \|^2 = \langle u - v, u - v\rangle = \langle u, u\rangle - \langle u, v\rangle - \langle v, u\rangle + \langle v, v\rangle$$

$$= \| u \|^2 - \langle u, v\rangle - \langle u, v\rangle + \| v \|^2 = 0$$

hence $u - v = 0$ by definiteness. That is $u = v$.

Problem6

Find vectors $u, v \in R^2$ such that u is a scalar multiple of $(1,3)$, v is orthogonal to $(1,3)$, and $(1,2) = u + v$.

Answer:

Let $v = (x, y)$ and $u = z(1,3)$, where $x, y, z \in \mathbb{R}$. Note that v is orthogonal to $(1,3)$, we have

$$(x, y) \cdot (1,3) = x + 3y = 0.$$

It follows that $v = y(-3,1)$. Since $(1,2) = u + v$, we obtain

$$y(-3,1) + z(1,3) = (z - 3y,\ y + 3z) = (1,2).$$

We can solve the equation and get $y = -1/10$ and $z = 7/10$. Hence $u = (7/10, 21/10)$ and $v = (3/10, -1/10)$.

Problem7

Prove that $(x_1 + \cdots + x_n)^2 \leqslant n\left(x_1^2 + \cdots + x_n^2\right)$ for all positive integers n and all real numbers $x_1, \ldots, x_n$.

Proof:

6.15 Cauchy–Schwarz Inequality

Suppose $u, v \in V$. Then

$$|\langle u, v\rangle| \leq \|u\| \, \|v\|.$$

This inequality is an equality if and only if one of u, v is a scalar multiple of the other.

By the Cauchy–Schwarz Inequality, if $x_1, \ldots, x_n, y_1, \ldots, y_n \in \mathbf{R}$, then

$$|x_1 y_1 + \cdots + x_n y_n|^2 \leq (x_1^2 + \cdots + x_n^2)(y_1^2 + \cdots + y_n^2) .$$

Let $y_i = 1$,we can obtain

$$|x_1 + \cdots + x_n|^2 \leq (x_1^2 + \cdots + x_n^2) .$$

Therefore, $(x_1 + \cdots + x_n)^2 \leqslant n(x_1^2 + \cdots + x_n^2)$ for all positive integers n and all real numbers $x_1, \ldots, x_n$.

Problem8

Suppose V is a real inner product space, prove that

$$\langle u, v\rangle = \frac{\|u+v\|^2 - \|u-v\|^2}{4}$$

for all $u, v \in V$.

Proof:

Suppose V is a real inner-product space and $u, v \in V$. Then

$$\begin{aligned}\frac{\| u+v \|^2 - \| u-v \|^2}{4} &= \frac{\langle u+v, u+v\rangle - \langle u-v, u-v\rangle}{4} \\ &= \frac{\| u \|^2 + 2\langle u, v\rangle + \| v \|^2 - (\| u \|^2 - 2\langle u, v\rangle + \| v \|^2)}{4} \\ &= \frac{4\langle u, v\rangle}{4} \\ &= \langle u, v\rangle\end{aligned}$$

as desired.

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Part.02

Homeworks2

Problem1

Suppose $e_1, ..., e_m$ is an orthonormal list of vectors in V. Let $v \in V$. Prove that

$$\|v\|^2 = |\langle v, e_1\rangle|^2 + \cdots + |\langle v, e_m\rangle|^2$$

if and only if $v \in \operatorname{span}(e_1, ..., e_m)$.

- **Solution**

2. Solution: If $v \in \operatorname{span}(e_1, \cdots, e_m)$, then $e_1, \cdots, e_m$ is an orthonormal basis of $\operatorname{span}(e_1, \cdots, e_m)$ by 6.26. By 6.30, it follows that

$$\|v\|^2 = |\langle v, e_1\rangle|^2 + \cdots + |\langle v, e_m\rangle|^2.$$

If $\|v\|^2 = |\langle v, e_1\rangle|^2 + \cdots + |\langle v, e_m\rangle|^2$, we denote

$$\xi = v - (\langle v, e_1\rangle e_1 + \cdots + \langle v, e_m\rangle e_m).$$

It is easily seen that

$$\langle \xi, e_i\rangle = \langle v, e_i\rangle - \langle v, e_i\rangle = 0$$

for $i = 1, \cdots, m$. This implies

$$\langle \xi, e_1\rangle e_1 + \cdots + \langle v, e_m\rangle e_m\rangle = 0.$$

By 6.13, we have

$$\begin{aligned}\|v\|^2 =& \|\xi\|^2 + \|\langle v, e_1\rangle e_1 + \cdots + \langle v, e_m\rangle e_m\|^2 \\ =& \|\xi\|^2 + |\langle v, e_1\rangle|^2 + \cdots + |\langle v, e_m\rangle|^2.\end{aligned}$$

It follows that $\|\xi\|^2 = 0$, hence $\xi = 0$. Thus $v = \langle v, e_1\rangle e_1 + \cdots + \langle v, e_m\rangle e_m$, namely $v \in \operatorname{span}(e_1, \cdots, e_m)$.

Problem2

Suppose n is a positive integer. Prove that

$$\frac{1}{\sqrt{2\pi}}, \frac{\cos x}{\sqrt{\pi}}, \frac{\cos 2x}{\sqrt{\pi}}, \ldots, \frac{\cos nx}{\sqrt{\pi}}, \frac{\sin x}{\sqrt{\pi}}, \frac{\sin 2x}{\sqrt{\pi}}, \ldots, \frac{\sin nx}{\sqrt{\pi}}$$

is an orthonormal list of vectors in $C[-\pi, \pi]$, the vector space of continuous real-valued functions on $[-\pi, \pi]$ with inner product

$$\langle f, g\rangle = \int_{-\pi}^{\pi} f(x)\, g(x)\, dx.$$

sin（α+β）=sinαcosβ+cosαsinβ
sin（α−β）=sinαcosβ−cosαsinβ
cos（α+β）=cosαcosβ−sinαsinβ
cos（α−β）=cosαcosβ+sinαsinβ

● **Solution**

①

COMMENT: This orthonormal list is often used for modeling periodic phenomena such as tides.

SOLUTION: First we need to show that each element of the list above has norm 1. This follows easily from the following formulas:

$$\sin\frac{\alpha}{2} = \pm\sqrt{\frac{1-\cos\alpha}{2}}$$

$$\cos\frac{\alpha}{2} = \pm\sqrt{\frac{1+\cos\alpha}{2}}$$

$$\int (\sin jt)^2\, dt = \frac{2jt - \sin 2jt}{4j}$$

$$\int (\cos jt)^2\, dt = \frac{2jt + \sin 2jt}{4j}.$$

Next we need to show that any two distinct elements of the list above are orthogonal. This follows easily from the following formulas, valid when $j \neq k$:

②

$$\int (\sin jt)(\sin kt)\, dt = \frac{j\sin(j-k)t + k\sin(j-k)t - j\sin(j+k)t + k\sin(j+k)t}{2(j-k)(j+k)}$$

$$\int (\sin jt)(\cos kt)\, dt = \frac{j\cos(j-k)t + k\cos(j-k)t + j\cos(j+k)t - k\cos(j+k)t}{2(k-j)(j+k)}$$

$$\int (\cos jt)(\cos kt)\, dt = \frac{j\sin(j-k)t + k\sin(j-k)t + j\sin(j+k)t - k\sin(j+k)t}{2(j-k)(j+k)}$$

$$\int (\sin jt)(\cos jt)\, dt = -\frac{\cos 2jt}{2j}$$

Problem3

On $\mathcal{P}_2(\mathbf{R})$, consider the inner product given by

$$\langle p, q\rangle = \int_0^1 p(x)\,q(x)\,dx$$

Apply the Gram–Schmidt Procedure to the basis $1, x, x^2$ to produce an orthonormal basis of $\mathcal{P}_2(\mathbf{R})$.

●**Solution**

① We have the polynomials

$$p_0(x) = 1,\ p_1(x) = x,\ p_2(x) = x^2.$$

For the first vector of the orthonormal basis, we have

$$e_0(x) = p_0(x) = 1$$

because

$$\begin{aligned}\|e_0\|^2 &= \langle e_0, e_0\rangle \\ &= \int_0^1 e_0(x)^2dx \\ &= \int_0^1 1dx \\ &= 1\end{aligned}$$

$\Rightarrow \|e_0\| = 1.$

② For the second polynomial, we first compute

$$\begin{aligned}u_1(x) &= p_1(x) - \langle p_1, e_0\rangle e_0(x) \\ &= x - \int_0^1 p_1(x)e_0(x)dx\cdot 1 \\ &= x - \int_0^1 xdx \\ &= x - \frac{1}{2}\end{aligned}$$

We also compute

$$\begin{aligned}\|u_1\|^2 &= \langle u_1, u_1\rangle \\ &= \int_0^1 u_1(x)^2dx \\ &= \int_0^1 \left(x^2 - x + \frac{1}{4}\right)dx \\ &= \frac{1}{12},\end{aligned}$$

so $\|u_1\| = \frac{\sqrt{3}}{6}$. Therefore,

$$e_1(x) = \frac{u_1(x)}{\|u_1\|} = 2\sqrt{3}\left(x - \frac{1}{2}\right) = 2\sqrt{3}x - \sqrt{3}.$$

③

$$\begin{aligned}u_2(x) &= p_2(x) - \langle p_2, e_0\rangle e_0(x) - \langle p_2, e_1\rangle e_1(x) \\ &= x^2 - \int_0^1 p_2(x)e_0(x)dx\cdot 1 - \int_0^1 p_2(x)e_1(x)dx\cdot\left(2\sqrt{3}x - \sqrt{3}\right) \\ &= x^2 - \int_0^1 x^2dx - 12\left(x - \frac{1}{2}\right)\int_0^1\left(x^3 - \frac{1}{2}x^2\right)dx \\ &= x^2 - \frac{1}{3} - 12\left(x - \frac{1}{2}\right)\cdot\frac{1}{12} \\ &= x^2 - x + \frac{1}{6}.\end{aligned}$$

Now,

$$\begin{aligned}\|u_2\|^2 &= \int_0^1 u_2(x)^2dx \\ &= \int_0^1\left(x^4 - 2x^3 + \frac{4}{3}x^2 - \frac{1}{3}x + \frac{1}{36}\right)dx \\ &= \frac{1}{180},\end{aligned}$$

so $\|u_2\| = \frac{\sqrt{5}}{30}$. Therefore,

$$e_2(x) = \frac{u_2(x)}{\|u_2\|} = 6\sqrt{5}\left(x^2 - x + \frac{1}{6}\right) = 6\sqrt{5}x^2 - 6\sqrt{5}x + \sqrt{5}.$$

Problem3

On $\mathcal{P}_2(\mathbf{R})$, consider the inner product given by

$$\langle p, q\rangle = \int_0^1 p(x)\, q(x)\, dx$$

Apply the Gram–Schmidt Procedure to the basis 1, x, x^2 to produce an orthonormal basis of $\mathcal{P}_2(\mathbf{R})$.

● Solution

④ RESULT

$$e_0(x) = 1$$
$$e_1(x) = 2\sqrt{3}x - \sqrt{3}$$
$$e_2(x) = 6\sqrt{5}x^2 - 6\sqrt{5}x + \sqrt{5}$$

Problem4

For each of the following, use the Gram-Schmidt process find an orthonormal basis for $R(A)$:

$$1. A = \begin{bmatrix} -1 & 3 \\ 1 & 5 \end{bmatrix}$$

$$2. A = \begin{bmatrix} 2 & 5 \\ 1 & 10 \end{bmatrix}$$

where $R(A)$ is the linear space spanned by the columns of A.

●**Solution**

① Using the Gram-Schmidt process,we get

$$r_{11} = \|a_1\| = \sqrt{(-1)^2 + 1^2} = \sqrt{2},$$

$$q_1 = \frac{a_1}{r_{11}} = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix},$$

$$r_{12} = \langle a_2, q_1 \rangle = q_1^T a_2 = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}\begin{bmatrix} 3 \\ 5 \end{bmatrix} = -1/\sqrt{2}\cdot 3 + 1/\sqrt{2}\cdot 5 = \frac{2}{\sqrt{2}} = \sqrt{2},$$

$$p_1 = r_{12} q_1 = \sqrt{2}\begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix},$$

$$r_{22} = \|a_2 - p_1\| = \left\| \begin{bmatrix} 3 \\ 5 \end{bmatrix} - \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\| = \left\| \begin{bmatrix} 4 \\ 4 \end{bmatrix} \right\| = \sqrt{4^2 + 4^2} = \sqrt{32} = 4\sqrt{2},$$

$$q_2 = \frac{a_2 - p_1}{r_{22}} = \frac{\begin{bmatrix} 4 \\ 4 \end{bmatrix}}{4\sqrt{2}} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}.$$

Vectors $\{q_1, q_2\} = \{(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})^T, (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})^T\}$ form an orthonormal basis for $R^2 = R(A)$.

Problem4

For each of the following, use the Gram-Schmidt process find an orthonormal basis for $R(A)$:

$$1. A = \begin{bmatrix} -1 & 3 \\ 1 & 5 \end{bmatrix}$$

$$2. A = \begin{bmatrix} 2 & 5 \\ 1 & 10 \end{bmatrix}$$

where $R(A)$ is the linear space spanned by the columns of A.

● **Solution**

②

$r_{11} = \|a_1\| = \sqrt{2^2+1^2} = \sqrt{5}$

$q_1 = \frac{a_1}{r_{11}} = \frac{1}{\sqrt{5}}(2,1)^T = \left(\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right)^T$

$r_{12} = \langle a_2, q_1 \rangle = q_1^T a_2 = \left[\frac{2}{\sqrt{5}} \ \frac{1}{\sqrt{5}}\right]\begin{bmatrix} 5 \\ 10 \end{bmatrix}$

$r_{12} = \frac{10}{\sqrt{5}} + \frac{10}{\sqrt{5}} = 4\sqrt{5}$

$P_1 = r_{12} q_1 = 4\sqrt{5}\left(\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right)^T$

$P_1 = (8,4)^T$

③

$r_{22} = \|a_2 - P_1\| = \|(5,10)^T - (8,4)^T\|$

$= \|(-3,6)^T\|$

$r_{22} = \sqrt{(-3)^2+6^2} = 3\sqrt{5}$

$q_2 = \frac{1}{r_{22}}(a_2 - P_1) = \frac{1}{3\sqrt{5}}(-3,6)^T$

$q_2 = \left(-\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right)^T$

so the orthonormal basis for $R(A)$ will be

$\left\{\left(\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right)^T, \left(\frac{-1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right)^T\right\}$.

Problem5

Given $\mathbf{x}_1 = \frac{1}{2}(1,1,1,-1)^T$ and $\mathbf{x}_2 = \frac{1}{6}(1,1,3,5)^T$, verify that these vectors form an orthonormal set in $\mathbb{R}^4$. Extend this set to an orthonomal basis for $\mathbb{R}^4$ by finding an orthonomal basis for the null space of

$$\begin{bmatrix} 1 & 1 & 1 & -1 \\ 1 & 1 & 3 & 5 \end{bmatrix}$$

[*Hint*: First find a basis for the null space and then use the Gram-Schmidt process.]

- **Solution**

①

$$x_1 = \frac{1}{2}(1,1,1,-1)^T, x_2 = \frac{1}{6}(1,1,3,5)^T$$

$$\begin{aligned}\langle x_1, x_2\rangle &= x_2^T x_1 \\ &= \frac{1}{2}\begin{bmatrix}1 & 1 & 3 & 5\end{bmatrix}\frac{1}{6}\begin{bmatrix}1\\1\\1\\-1\end{bmatrix} \\ &= \frac{1}{12}(1\cdot 1 + 1\cdot 1 + 3\cdot 1 + 5\cdot(-1)) \\ &= \frac{1}{12}\cdot 0 \\ &= 0\end{aligned}$$

②

$$\begin{aligned}\|x_1\| &= \|\frac{1}{2}\begin{bmatrix}1\\1\\1\\-1\end{bmatrix}\| \\ &= \sqrt{(\frac{1}{2})^2(1^2+1^2+1^2+(-1)^2)} \\ &= \sqrt{\frac{1}{4}\cdot 4} \\ &= \sqrt{1} \\ &= 1\end{aligned}$$

$$\begin{aligned}\|x_2\| &= \|\frac{1}{6}\begin{bmatrix}1\\1\\3\\5\end{bmatrix}\| \\ &= \sqrt{(\frac{1}{6})^2(1^2+1^2+3^2+5^2)} \\ &= \sqrt{\frac{1}{36}\cdot 36} \\ &= \sqrt{1} \\ &= 1\end{aligned}$$

Therefore, $\{x_1, x_2\} = \{(1,1,1,-1)^T, (1,1,3,5^T)\}$ form an orthonormal basis for the two-dimensional subspace of R^4.

Problem5

Given $\mathbf{x}_1 = \frac{1}{2}(1,1,1,-1)^T$ and $\mathbf{x}_2 = \frac{1}{6}(1,1,3,5)^T$, verify that these vectors form an orthonormal set in $\mathbb{R}^4$. Extend this set to an orthonomal basis for $\mathbb{R}^4$ by finding an orthonomal basis for the null space of

$$\begin{bmatrix} 1 & 1 & 1 & -1 \\ 1 & 1 & 3 & 5 \end{bmatrix}$$

[*Hint*: First find a basis for the null space and then use the Gram-Schmidt process.]

● **Solution**

③ In order to determine the basis for the column space of matrix A, let's transform it to the reduced row echelon form.

$$\begin{bmatrix} 1 & 1 & 1 & -1 \\ 1 & 1 & 3 & 5 \end{bmatrix} \underset{\sim}{R_2 \to R_2 - R_1} \begin{bmatrix} 1 & 1 & 1 & -1 \\ 0 & 0 & 2 & 6 \end{bmatrix}$$

$$\underset{\sim}{R_2 \to \frac{1}{2} R_2} \begin{bmatrix} 1 & 1 & 1 & -1 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

$$\underset{\sim}{R_1 \to R_1 - R_2} \begin{bmatrix} 1 & 1 & 0 & -4 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Vector $x = (x_1, x_2, x_3, x_4)^T$ belongs to $N(A)$ if

$$\begin{bmatrix} 1 & 1 & 0 & -4 \\ 0 & 0 & 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

We get system of equations

$$x_1 + x_2 - 4x_4 = 0 \implies x_1 = 4x_4 - x_2$$
$$x_3 + 3x_4 = 0 \implies x_3 = -3x_4.$$

If we take $x_2 = \alpha, x_4 = \beta$, we get

$$\boxed{x_1 = 4\beta - \alpha}$$
$$\boxed{x_3 = -3\beta}.$$

Therefore, $N(A)$ consists of all vectors of the form

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 4\beta - \alpha \\ \alpha \\ -3\beta \\ \beta \end{bmatrix} = \alpha \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 4 \\ 0 \\ -3 \\ 1 \end{bmatrix}.$$

Problem5

Given $\mathbf{x}_1 = \frac{1}{2}(1,1,1,-1)^T$ and $\mathbf{x}_2 = \frac{1}{6}(1,1,3,5)^T$, verify that these vectors form an orthonormal set in $\mathbb{R}^4$. Extend this set to an orthonomal basis for $\mathbb{R}^4$ by finding an orthonomal basis for the null space of

$$\begin{bmatrix} 1 & 1 & 1 & -1 \\ 1 & 1 & 3 & 5 \end{bmatrix}$$

[*Hint*: First find a basis for the null space and then use the Gram-Schmidt process.]

● Solution

④

The vectors

$$\{a_1, a_2\} = \{(-1,1,0,0)^T, (4,0,-3,1)^T\}$$

form a basis for the $N(A)$.

Now let's find an orthonormal basis for $N(A)$. Using the Gram-Schmidt process, we get

$$r_{11} = \|a_1\| = \sqrt{(-1)^2+1^2+0^2+0^2} = \sqrt{2},$$

$$q_1 = \frac{a_1}{r_{11}} = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \\ 0 \end{bmatrix},$$

$$r_{12} = \langle a_2, q_1 \rangle = q_1^T a_2 = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} & 0 & 0 \end{bmatrix}\begin{bmatrix} 4 \\ 0 \\ -3 \\ 1 \end{bmatrix} = -1/\sqrt{2}\cdot 4 + 1/\sqrt{2}\cdot 0 + 0\cdot(-3) + 0\cdot 1 = -2\sqrt{2},$$

$$p_1 = r_{12}q_1 = -2\sqrt{2}\begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ 0 \\ 0 \end{bmatrix},$$

$$r_{22} = \|a_2 - p_1\| = \left\|\begin{bmatrix} 4 \\ 0 \\ -3 \\ 1 \end{bmatrix} - \begin{bmatrix} 2 \\ -2 \\ 0 \\ 0 \end{bmatrix}\right\| = \left\|\begin{bmatrix} 2 \\ 2 \\ -3 \\ 1 \end{bmatrix}\right\| = \sqrt{2^2+2^2+(-3)^2+1^2} = \sqrt{18} = 3\sqrt{2},$$

$$q_2 = \frac{a_2 - p_1}{r_{22}} = \frac{\begin{bmatrix} 2 \\ 2 \\ -3 \\ 1 \end{bmatrix}}{3\sqrt{2}} = \begin{bmatrix} 2/3\sqrt{2} \\ 2/3\sqrt{2} \\ -1/\sqrt{2} \\ 1/3\sqrt{2} \end{bmatrix},$$

Problem5

Given $\mathbf{x}_1 = \frac{1}{2}(1,1,1,-1)^T$ and $\mathbf{x}_2 = \frac{1}{6}(1,1,3,5)^T$, verify that these vectors form an orthonormal set in $\mathbb{R}^4$. Extend this set to an orthonomal basis for $\mathbb{R}^4$ by finding an orthonomal basis for the null space of

$$\begin{bmatrix} 1 & 1 & 1 & -1 \\ 1 & 1 & 3 & 5 \end{bmatrix}$$

[*Hint*: First find a basis for the null space and then use the Gram-Schmidt process.]

- **Solution**

⑤

Therefore, vectors $\{q_1, q_2\} = \{(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, 0)^T, (\frac{2}{3\sqrt{2}}, \frac{2}{3\sqrt{2}}, -\frac{1}{\sqrt{2}}, \frac{1}{3\sqrt{2}})^T$ form an orthonormal basis for $N(A)$.

Let's check whether vectors

$\{x_1, x_2, q_1, q_2\}$ form a basis for R^4. In order for this to be true it's enough to prove that these vectors are linearly independent. (This is because $R(A^T)$ and $N(A)$ are orthogonal and the vectors make up bases for $R(A^T)$ and $N(A)$ respectively.)

Let $\alpha, \beta, \gamma, \delta \in R$.

$$\alpha \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix} + \beta \begin{bmatrix} 1 \\ 1 \\ 3 \\ 5 \end{bmatrix} + \gamma \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \\ 0 \end{bmatrix} + \delta \begin{bmatrix} 2/3\sqrt{2} \\ 2/3\sqrt{2} \\ -1/\sqrt{2} \\ 1/3\sqrt{2} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

We get system of equations

$$\alpha + \beta - \frac{1}{\sqrt{2}}\gamma + \frac{2}{3\sqrt{2}}\delta = 0$$
$$\alpha + \beta + \frac{1}{\sqrt{2}}\gamma + \frac{2}{3\sqrt{2}}\delta = 0$$
$$\alpha + 3\beta - \frac{1}{\sqrt{2}}\delta = 0$$
$$-\alpha + 5\beta + \frac{1}{3\sqrt{2}}\delta = 0.$$

If we subtract second equation from the first, and also add third equation to the fourth, we get

$$\sqrt{2}\gamma = 0 \implies \boxed{\gamma = 0}$$
$$8\beta - \frac{2}{3\sqrt{2}}\delta = 0 \implies \boxed{\beta = \frac{1}{12\sqrt{2}}\delta}.$$

Problem5

Given $\mathbf{x}_1 = \frac{1}{2}(1, 1, 1, -1)^T$ and $\mathbf{x}_2 = \frac{1}{6}(1, 1, 3, 5)^T$, verify that these vectors form an orthonormal set in $\mathbb{R}^4$. Extend this set to an orthonomal basis for $\mathbb{R}^4$ by finding an orthonomal basis for the null space of

$$\begin{bmatrix} 1 & 1 & 1 & -1 \\ 1 & 1 & 3 & 5 \end{bmatrix}$$

[*Hint*: First find a basis for the null space and then use the Gram-Schmidt process.]

● **Solution**

⑥

If we substitute $\gamma = 0$ into the original system, we get these equations

$$\alpha + \beta + \frac{2}{3\sqrt{2}}\delta = 0$$
$$\alpha + 3\beta - \frac{1}{\sqrt{2}}\delta = 0$$
$$-\alpha + 5\beta + \frac{1}{3\sqrt{2}}\delta = 0.$$

If we subtract first equation from the second and add the first equation to the third, we get

$$2\beta - \frac{5}{3\sqrt{2}}\delta = 0$$
$$6\beta + \frac{1}{\sqrt{2}} = 0.$$

If we substitute $\beta = -\frac{1}{6\sqrt{2}}\delta$ in the first equation, we get

$$-\frac{1}{3\sqrt{2}}\delta - \frac{5}{3\sqrt{2}}\delta = 0 \implies -\sqrt{2}\delta = 0 \implies \boxed{\delta = 0}.$$

From $\beta = -\frac{1}{6\sqrt{2}}\delta$ follows that

$$\boxed{\beta = 0}.$$

Now, if we substitute $\beta = \gamma = \delta = 0$ in the first equation of the original system, we get

$$\boxed{\alpha = 0}.$$

Since $\alpha = \beta = \gamma = \delta = 0$, it follows that

$$\{x_1, x_2, q_1, q_2\} = \{(1,1,1,-1)^T, (1,1,3,5^T), (-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, 0)^T, (\frac{2}{3\sqrt{2}}, \frac{2}{3\sqrt{2}}, -\frac{1}{\sqrt{2}}, \frac{1}{3\sqrt{2}})^T\}$$

are linearly independent and therefore form an orthonormal basis for R^4.

Problem6

Find a polynomial $q \in \mathcal{P}_2(\mathbf{R})$ such that

$$p\left(\frac{1}{2}\right) = \int_0^1 p(x)\,q(x)\,dx$$

for every $p \in \mathcal{P}_2(\mathbf{R})$.

●Solution

① First off, we would need an orthonormal basis of $P_2(R)$, which is where the inner product of two polynomials in $P_2(R)$ is defined to be the integral from 0 to 1 of the product of the two polynomials. So, an orthonormal basis of $P_2(R)$ was already computed in a previous exercise.

Now, specifically, let $e_1(x) = 1$, $e_2(x) = \sqrt{3}(-1+2x)$, and $e_3(x) = \sqrt{5}(1 - 6x + 6x^2)$. Next, note that (e_1, e_2, e_3) is an orthonormal basis of $P_2(R)$. So, define a linear functional φ on $P_2(R)$ by $\varphi(p) = p(\frac{1}{2})$.

② So, we would seek $q \in P_2(R)$ such that $\varphi(p) = \langle p, q\rangle$ for every single $p \in P_2(R)$. So, by using a former formula, we would now have $q = \varphi(e_1)e_1 + \varphi(e_2)e_2 + \varphi(e_3)e_3$.

So, when you now evaluate the right side of the given equation, then you would not have $q = -\frac{3}{2} + 15x - 15x^2$.

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Part.03

Homeworks3

Problem 1

Find a polynomial $q \in \mathcal{P}_2(\mathbf{R})$ such that

$$\int_0^1 p(x)(\cos \pi x)\,dx = \int_0^1 p(x)\,q(x)\,dx$$

for every $p \in \mathcal{P}_2(\mathbf{R})$.

Problem 1: (1) step 1. find the orthonormal basis by Gram-S Procedure

参考 Ex 6.44 及 6.43.

$e_1 = 1$

$e_2 = 2\sqrt{3}\,(x - \frac{1}{2})$

$e_3 = 6\sqrt{5}\,(x^2 - x + \frac{1}{6})$

(2) Step 2. let $\varphi(p) = \int_0^1 p(x)(\cos \pi x)\,dx$

(3) Step 3. $q(x) = \underline{\varphi(e_1)e_1}^{0} + \varphi(e_2)e_2 + \underline{\varphi(e_3)e_3}^{0}$

$= \frac{12 - 24x}{\pi^2}$

Problem 2

Let

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 12 \\ 6 \\ 18 \end{bmatrix}$$

(a) Use the Gram–Schmidt process to find an orthonormal basis for the column space of A.
(b) Factor A into a product QR, where Q has an orthonormal set of column vectors and R is upper triangular.
(c) Solve the least squares problem $A\mathbf{x} = \mathbf{b}$.

We are given $A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix}$, and $\mathbf{b} = \begin{bmatrix} 12 \\ 6 \\ 18 \end{bmatrix}$. Part (b), which asks for the QR factorization of A, makes part (a) redundant. We first normalize the first column of A, obtaining $r_{11} = 3$, and $\mathbf{q}_1 = (2/3, 1/3, 2/3)^T$. We then find $r_{12} = \mathbf{q}_1^T \mathbf{a}_2 = 5/3$, and update $\mathbf{a}_2$ to $\mathbf{a}_2 - r_{12}\mathbf{q}_1 = (-1/9, 4/9, -1/9)^T$. Finally, we normalize $\mathbf{a}_2$, obtaining $r_{22} = \sqrt{2}/3$, and $\mathbf{q}_2 = \left(1/3\sqrt{2}\right)(-1, 4, -1)^T = \left(-\sqrt{2}/6, 2\sqrt{2}/3, -\sqrt{2}/6\right)^T$.

The factorization that results is

$$A = QR = \begin{bmatrix} 2/3 & -\sqrt{2}/6 \\ 1/3 & 2\sqrt{2}/3 \\ 2/3 & -\sqrt{2}/6 \end{bmatrix} \begin{bmatrix} 3 & 5/3 \\ 0 & \sqrt{2}/3 \end{bmatrix},$$

Problem2

so the least squares problem in part (c) reduces to solving the triangular system $R\mathbf{x} = Q^T\mathbf{b} = \begin{bmatrix} 22 \\ -\sqrt{2} \end{bmatrix}$. The solution is $\hat{\mathbf{x}} = \begin{bmatrix} 9 \\ -3 \end{bmatrix}$.

Problem 3

Let U be an m-dimensional subspace of $\mathbb{R}^n$ and let V be a k-dimensional subspace of U, where $0 < k < m$.

(a) Show that any orthonormal basis

$$\{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_k\}$$

for V can be expanded to form an orthonormal basis $\{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_k, \mathbf{v}_{k+1}, ..., \mathbf{v}_m\}$ for U.

(b) Show that if $W = \mathrm{Span}\{\mathbf{v}_{k+1}, ..., \mathbf{v}_m\}$, then $U = V \oplus W$.

(a)

Let U be an m-dimensional subspace of $\mathbb{R}^n$ and let V be a k-dimensional subspace of U, wheere $0 < k < m$. Now let $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_k\}$ be an orthonormal basis for V. Then it can be extended to form a basis for U, $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_k, \boldsymbol{u}_{k+1}, \ldots, \boldsymbol{u}_m\}$. Applying the Gram-Schmidt Orthogonalization process on the new base we will get an orthonormal base $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_k\, \boldsymbol{v}_{k+1}, \ldots, \boldsymbol{v}_m\}$ which is an extension of $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_k\}$.

(b)

By exercise 33 and 34 of section 5.5, we see that $W = V^\perp$. Also, we know that $U = V \oplus V^\perp$, hence we have $U = V \oplus W$.

Problem 4

Suppose $v_1, ..., v_m \in V$. Prove that

$$\{v_1, ..., v_m\}^{\perp} = (\operatorname{span}(v_1, ..., v_m))^{\perp}$$

Solution: Suppose $w \in \{v_1, \ldots, v_m\}^{\perp}$. Let $v =\in \operatorname{span}(v_1, \ldots, v_m)$. We have that

$$v = a_1 v_1 + \ldots a_m v_m$$

for some $a_1, \ldots, a_m \in \mathbb{F}$. Moreover

$$\langle v, w\rangle = \langle a_1 v_1 + \ldots a_m v_m, w\rangle = a_1\langle v_1, w\rangle + \cdots + a_m\langle v_m, w\rangle = 0.$$

Thus $w \in (\operatorname{span}(v_1, \ldots, v_m))^{\perp}$ and so $\{v_1, \ldots, v_m\}^{\perp} \subset (\operatorname{span}(v_1, \ldots, v_m))^{\perp}$.

Now suppose $w \in (\operatorname{span}(v_1, \ldots, v_m))^{\perp}$. Since each v_j is in $\operatorname{span}(v_1, \ldots, v_m)$, it follows that w is orthogonal to each v_j. Therefore $w \in \{v_1, \ldots, v_m\}^{\perp}$ and thus $(\operatorname{span}(v_1, \ldots, v_m))^{\perp} \subset \{v_1, \ldots, v_m\}^{\perp}$.

Problem 5

Suppose U is the subspace of $\mathbf{R}^4$ defined by

$$U = \operatorname{span}\left((1,2,3,-4),(-5,4,3,2)\right).$$

Find an orthonormal basis of U and an orthonormal basis of $U^\perp$.

It's easy to check that $(1,2,3,-4),(-5,4,3,2),(1,0,0,0),(0,1,0,0)$ is a basis of $\mathbb{R}^4$. Applying the Gram-Schmidt Procedure yields

$$e_1 = \frac{1}{\sqrt{30}}(1,2,3,-4), \qquad e_2 = \frac{1}{\sqrt{12030}}(-77,56,39,38)$$

$$e_3 = \frac{1}{\sqrt{76190}}(190,117,60,151), \qquad e_4 = \frac{1}{\sqrt{190}}(0,9,-10,-3).$$

Hence e_1, e_2 is an orthonormal basis of U, and e_3, e_4 is an orthonormal basis of $U^\perp$.

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Part.04

Homeworks4

Problem 1

In $\mathbf{R}^4$, let

$$U = \operatorname{span}((1,1,0,0),(1,1,1,2)).$$

Find $u \in U$ such that $\|u-(1,2,3,4)\|$ is as small as possible.

We will use the normal equations for the formula of an orthogonal projection. Let

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 1 \\ 0 & 2 \end{pmatrix} \text{ and } b = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}.$$

Then u is the orthogonal projection of b onto the subspace spanned by the column of A and it is given by the formula:

$$u = A(A^TA)^{-1}A^Tb.$$

We get that

$$A^Tb = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 3 \\ 14 \end{pmatrix}$$

$$A^TA = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 1 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 2 & 7 \end{pmatrix}$$

Problem1

$$(A^TA)^{-1} = \frac{1}{10}\begin{pmatrix} 7 & -2 \\ -2 & 2 \end{pmatrix}$$

$$(A^TA)^{-1}A^Tb = \frac{1}{10}\begin{pmatrix} 7 & -2 \\ -2 & 2 \end{pmatrix}\begin{pmatrix} 3 \\ 14 \end{pmatrix} = \frac{1}{10}\begin{pmatrix} -7 \\ 22 \end{pmatrix}$$

$$u = A(A^TA)^{-1}A^Tb = \frac{1}{10}\begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 1 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} -7 \\ 22 \end{pmatrix} = \frac{1}{10}\begin{pmatrix} 15 \\ 15 \\ 22 \\ 44 \end{pmatrix}.$$

The answer is

$$u = \frac{1}{10}\begin{pmatrix} 15 \\ 15 \\ 22 \\ 44 \end{pmatrix}.$$

How do we know this is the desired vector? First, clearly $u \in U$, (since it is equal by construction to $-\frac{7}{10}a_1 + \frac{22}{10}a_2$). Second, we can check that the residual polynomial $r = b - u$ is orthogonal to U. Indeed, we have

$$A^T(b-u) = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 2 \end{pmatrix}\left(\begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} - \frac{1}{10}\begin{pmatrix} 15 \\ 15 \\ 22 \\ 44 \end{pmatrix}\right) = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 2 \end{pmatrix}\left(\frac{1}{10}\begin{pmatrix} -5 \\ 5 \\ 8 \\ -4 \end{pmatrix}\right) = 0$$

Problem 2

Find $p \in \mathcal{P}_3(\mathbf{R})$ such that $p(0) = 0, p'(0) = 0$, and

$$\int_0^1 |2 + 3x - p(x)|^2 dx$$

is as small as possible.

Proof. We consider the inner product space $\mathcal{P}_3(\mathbb{R})$ of all polynomials of degree less or equal than 3 with inner product

$$\langle p, q \rangle = \int_0^1 p(x)q(x)dx.$$

We also consider the subspace U of $\mathcal{P}_3(\mathbb{R})$ defined as

$$U = \{p \in \mathcal{P}_3(\mathbb{R}) \, : \, p'(0) = p(0) = 0\},$$

and the polynomial $b(x) = 2+3x$. To solve our problem, we need to construct $u(x)$, the orthogonal projection of $b(x) = 2 + 3x$ onto U.

Problem2

A basis for the subspace U is

$$\left(x^2, x^3\right).$$

We use the Gram-Schmidt algorithm to obtain an orthonormal basis. We get

$$q_1 = \frac{x^2}{||x^2||} = \sqrt{5}x^2,$$

$$q_2 = \frac{x^3 - \langle x^3, \sqrt{5}x^2\rangle \cdot \sqrt{5}x^2}{||x^3 - \langle x^3, \sqrt{5}x^2\rangle \cdot \sqrt{5}x^2||} = \frac{x^3 - \frac{5}{6}x^2}{||x^3 - \frac{5}{6}x^2||} = 6\sqrt{7}\left(x^3 - \frac{5}{6}x^2\right).$$

So an orthonormal basis for U is

$$\left(\sqrt{5}x^2, 6\sqrt{7}\left(x^3 - \frac{5}{6}x^2\right)\right)$$

So we have

$$P_U(2+3x) = \langle 2+3x, \sqrt{5}x^2\rangle \cdot \sqrt{5}x^2 + \langle 2+3x, 6\sqrt{7}\left(x^3 - \frac{5}{6}x^2\right)\rangle \cdot 6\sqrt{7}\left(x^3 - \frac{5}{6}x^2\right)$$

$$= (\frac{17}{12}\sqrt{5})\sqrt{5}x^2 + (-\frac{29}{60}\sqrt{7})6\sqrt{7}\left(x^3 - \frac{5}{6}x^2\right) = -\frac{203}{10}x^3 + 24x^2.$$

We get

$$u(x) = -\frac{203}{10}x^3 + 24x^2$$

Problem2

which is the desired polynomial.

How do we know this is the desired polynomial? First, clearly $u(x) \in U$, so this is good. Second, we can check that the residual polynomial $r(x) = b(x) - u(x)$ is orthogonal to U. Indeed, we have

$$\langle r, x^2 \rangle = \langle -\frac{203}{10}x^3 + 24x^2 - 3x - 2, x^2 \rangle = 0,$$

$$\langle r, x^3 \rangle = \langle -\frac{203}{10}x^3 + 24x^2 - 3x - 2, x^3 \rangle = 0,$$

so $u(x) = -\frac{203}{10}x^3 + 24x^2$ is the good answer.

Problem 3

Suppose $T \in \mathcal{L}(V)$ and U is a subspace of V.

(a) Prove that if $U \subset \text{null } T$, then U is invariant under T.

(b) Prove that if range $T \subset U$, then U is invariant under T.

1. Solution: (a) For any $u \in U$, then $Tu = 0 \in U$ since $U \subset \text{null} T$, hence U is invariant under T.

(b) For any $u \in U$, then $Tu \in \text{range} T \subset U$, hence U is invariant under T.

Problem 4

Suppose $S, T \in \mathcal{L}(V)$ are such that $ST = TS$. Prove that range S is invariant under T.

Solution: For any $u \in \mathrm{range} S$, there exists $v \in V$ such that $Sv = u$, hence

$$Tu = TSv = STv \in \mathrm{range} S.$$

Therefore range S is invariant under T.

Problem 5

Suppose $S, T \in \mathcal{L}(V)$ are such that $ST = TS$. Prove that null S is invariant under T.

Solution 5 For any $v \in \texttt{Nul}(S)$, $S(v) = 0$. Since $ST = TS$, $S(T(v)) = T(S(v)) = T(0) = 0$. Then $T(v) \in \texttt{Nul}(S)$. Then $\texttt{Nul}(S)$ is invariant under T.

Part.05

homework5

Problem1

Define $T \in \mathcal{L}(\mathbf{F}^3)$ by

$$T(z_1, z_2, z_3) = (2z_2, 0, 5z_3)$$

Find all eigenvalues and eigenvectors of T.

- Solution

①

SOLUTION: Suppose λ is an eigenvalue of T. For this particular operator, the eigenvalue-eigenvector equation $T(z_1, z_2, z_3) = \lambda(z_1, z_2, z_3)$ becomes the system of equations

$$\begin{aligned} 2z_2 &= \lambda z_1 \\ 0 &= \lambda z_2 \\ 5z_3 &= \lambda z_3. \end{aligned}$$

If $\lambda \neq 0$, then the second equation implies that $z_2 = 0$, and the first equation then implies that $z_1 = 0$. Because an eigenvalue must have a nonzero eigenvector, there must be a solution to the system above with $z_3 \neq 0$. The third equation then shows that $\lambda = 5$. In other words, 5 is the only nonzero eigenvalue of T. The set of eigenvectors corresponding to the eigenvalue 5 is

$$\{(0, 0, z_3) : z_3 \in \mathbf{F}\}.$$

Problem1

Define $T \in \mathcal{L}(\mathbf{F}^3)$ by

$$T(z_1, z_2, z_3) = (2z_2, 0, 5z_3)$$

Find all eigenvalues and eigenvectors of T.

- **Solution**

②

If $\lambda = 0$, the first and third equations above show that $z_2 = 0$ and $z_3 = 0$. With these values for z_2, z_3, the equations above are satisfied for all values of z_1. Thus 0 is an eigenvalue of T. The set of eigenvectors corresponding to the eigenvalue 0 is

$$\{(z_1, 0, 0) : z_1 \in \mathbf{F}\}.$$

Problem2

Define $T : \mathcal{P}(\mathbf{R}) \to \mathcal{P}(\mathbf{R})$ by $Tp = p'$. Find all eigenvalues and eigenvectors of T.

- **Solution**

Suppose λ is an eigenvalue of T with an eigenvector q, then

$$q' = Tq = \lambda q.$$

Note that in general $\deg p' < \deg p$(because we consider $\deg 0 = -\infty$). If $\lambda \neq 0$, then $\deg \lambda q > \deg q'$. We get a contradiction. If $\lambda = 0$, then $q = c$ for nonzero $c \in \mathbb{R}$. Hence the only eigenvalue of T is zero with nonzero constant polynomials as eigenvectors.

Problem3

Suppose $T \in \mathcal{L}(V)$. Suppose $S \in \mathcal{L}(V)$ is invertible.

(a) Prove that T and $S^{-1}TS$ have the same eigenvalues.

(b) What is the relationship between the eigenvectors of T and the eigenvectors of $S^{-1}TS$?

●Solution

(a) Suppose λ is an eigenvalue of T, then there exists a nonzero vector $v \in V$ such that $Tv = \lambda v$. Hence

$$S^{-1}TS(S^{-1}v) = S^{-1}Tv = S^{-1}(\lambda v) = \lambda S^{-1}v.$$

Note that $S^{-1}v \neq 0$ as S^{-1} is invertible, hence λ is an eigenvalue of $S^{-1}TS$, namely every eigenvalue of T is an eigenvalue of $S^{-1}TS$. Similarly, note that $S(S^{-1}TS)S^{-1} = T$, we have every eigenvalue of $S^{-1}TS$ is an eigenvalue of T. Hence T and $S^{-1}TS$ have the same eigenvalues.

(b) From the process of (a), one can easily deduce that v is an eigenvector of T if and only if $S^{-1}v$ is an eigenvector of $S^{-1}TS$.

Problem4

Find all eigenvalues and eigenvectors of the backward shift operator $T \in \mathcal{L}(\mathbf{F}^\infty)$ defined by

$$T(z_1, z_2, z_3, \ldots) = (z_2, z_3, \ldots).$$

● **Solution**

① SOLUTION: Suppose λ is an eigenvalue of T. For this particular operator, the eigenvalue-eigenvector equation $Tz = \lambda z$ becomes the system of equations

$$\begin{aligned} z_2 &= \lambda z_1 \\ z_3 &= \lambda z_2 \\ z_4 &= \lambda z_3 \\ &\vdots \end{aligned}$$

② From this we see that we can choose z_1 arbitrarily and then solve for the other coordinates:

$$\begin{aligned} z_2 &= \lambda z_1 \\ z_3 &= \lambda z_2 = \lambda^2 z_1 \\ z_4 &= \lambda z_3 = \lambda^3 z_1 \\ &\vdots \end{aligned}$$

Problem4

Find all eigenvalues and eigenvectors of the backward shift operator $T \in \mathcal{L}(\mathbf{F}^\infty)$ defined by

$$T(z_1, z_2, z_3, \dots) = (z_2, z_3, \dots).$$

● Solution

③ Thus each $\lambda \in \mathbf{F}$ is an eigenvalue of T and the set of corresponding eigenvectors is

$$\{(w, \lambda w, \lambda^2 w, \lambda^3 w \dots) : w \in \mathbf{F}\}.$$

Problem5

If A is a matrix with $m \times n$ dimension, please show that $A^T A$ and AA^T have the same nonzero eigenvalues.

●Solution

①

since A is an $m \times n$ matrix with rank k, there exist otrhonormal bases $\mathcal{B}_1 = \{v_1, \ldots, v_n\}$ and $\mathcal{B}_2 = \{u_1, \ldots, u_m\}$ for R^n and R^m, respectively, and scalars $\sigma_1 \geq \cdots \geq \sigma_k > 0$ such that (9) and (10) are satisfied. Now we have, for $i = 1, \ldots, k$,

$$\begin{aligned} A^T A v_i &= A^T(\sigma_i u_i) \\ &= \sigma_i A^T u_i \\ &= \sigma_i \sigma_i v_i \\ &= \sigma_i^2 v_i \end{aligned}$$

Therefore, for $i = 1, \ldots, k$; $\sigma_1^2, \ldots, \sigma_k^2$ are eigenvalues of $A^T A$ corresponding to $v_1, \ldots, v_k \in \mathcal{B}_1$.

$$\begin{aligned} AA^T u_i &= A(\sigma_i v_i) \\ &= \sigma_i A v_i \\ &= \sigma_i \sigma_i u_i \\ &= \sigma_i^2 u_i \end{aligned}$$

Therefore, for $i = 1, \ldots, k$; $\sigma_1^2, \ldots, \sigma_k^2$ are eigenvalues of AA^T corresponding to $u_1, \ldots, u_k \in \mathcal{B}_2$.

Hence, $A^T A$ and AA^T have the same eigenvalues.

②

假设x是$A^T A$的输入特征值λ的特征向量。$A^T A x = \lambda x$.
两边同乘以A,得到$AA^T A x = \lambda A x$, 则有$AA^T(Ax) = \lambda(Ax)$。
所以$A^T A$和AA^T有相同的非零特征值。
同理可得，AB 和BA有相同的非零特征值。

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谢　谢！