Problem 1

Show that the function that takes $((x_1, x_2), (y_1, y_2)) \in R^2 \times R^2$ to $|x_1y_1| + |x_2y_2|$ is not an inner product on R^2.

证明上述方程不满足定义中的任意一条就可以，注意证明过程的合理性

这里证明不满足 additivity 和 homogeneity 中任意一条即可，但有同学证明 definiteness 是有问题的

6.3 Definition ***inner product***

An ***inner product*** on V is a function that takes each ordered pair (u, v) of elements of V to a number $\langle u, v\rangle \in \mathbf{F}$ and has the following properties:

positivity
$\langle v, v\rangle \geq 0$ for all $v \in V$;

definiteness
$\langle v, v\rangle = 0$ if and only if $v = 0$;

additivity in first slot
$\langle u + v, w\rangle = \langle u, w\rangle + \langle v, w\rangle$ for all $u, v, w \in V$;

homogeneity in first slot
$\langle \lambda u, v\rangle = \lambda\langle u, v\rangle$ for all $\lambda \in \mathbf{F}$ and all $u, v \in V$;

conjugate symmetry
$\langle u, v\rangle = \overline{\langle v, u\rangle}$ for all $u, v \in V$.

1. If the function is an inner product, we have

$\langle v_1+v_2, u\rangle = \langle v_1, u\rangle + \langle v_2, u\rangle$

Let $v_1 = (x_1, x_2)$, $v_2 = (x_3, x_4)$, $u = (y_1, y_2)$

From definition, $\langle (x_1, x_2)+(x_3, x_4), (y_1, y_2)\rangle = \langle (x_1+x_3, x_2+x_4), (y_1, y_2)\rangle$

$= |x_1y_1+x_3y_1| + |x_2y_2+x_4y_2| \neq |x_1y_1|+|x_2y_2|+|x_3y_1|+|x_4y_2|$

$= \langle (x_1, x_2), (y_1, y_2)\rangle + \langle (x_3, x_4), (y_1, y_2)\rangle$

which is contradict to additivity in first slot

when (y_1, y_2) is fixed, we have that:

$(\lambda(x_1, x_2), (y_1, y_2)) = |\lambda x_1y_1| + |\lambda x_2y_2|$

so when $\lambda < 0$, we have:

$|\lambda x_1y_1| + |\lambda x_2y_2| = -\lambda(|x_1y_1| + |x_2y_2|) \neq \lambda(|x_1y_1| + |x_2y_2|)$

thus: $(\lambda(x_1, x_2), (y_1, y_2)) \neq \lambda((x_1, x_2), (y_1, y_2))$

thus this is not an inner product on R^2

Problem 2

Suppose V is a real inner product space, show that:

a) the inner product $\langle u+v, u-v\rangle = \|u\|^2 - \|v\|^2$ for every $u, v \in V$.

b) if $u, v \in V$ have the same norm, then $u+v$ is orthogonal to $u-v$.

c) use part(b) to show that the diagonals of a rhombus are perpendicular to each other.

a>利用交换性；b>利用 a 的结论；c>利用 b 的结论+菱形邻边相等

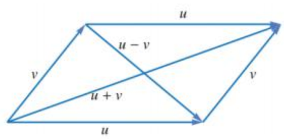
(a) Note that V is a real inner product space, we have $\langle u, v\rangle = \langle v, u\rangle$. Hence

$$\begin{aligned}\langle u+v, u-v\rangle &= \langle u, u\rangle - \langle u, v\rangle + \langle v, u\rangle - \langle v, v\rangle \\ &= \langle u, u\rangle - \langle v, v\rangle = \| u \|^2 - \| v \|^2\end{aligned}$$

(b) By (a).

$$\begin{aligned}\langle u+v, u-v\rangle &= \langle u, u\rangle - \langle u, v\rangle + \langle v, u\rangle - \langle v, v\rangle \\ &= \langle u, u\rangle - \langle v, v\rangle = \| u \|^2 - \| v \|^2 = 0\end{aligned}$$

(c)



Note that $\| u \| = \| v \|$ for a rhombus,

$$\begin{aligned}\langle u+v, u-v\rangle &= \langle u, u\rangle - \langle u, v\rangle + \langle v, u\rangle - \langle v, v\rangle \\ &= \langle u, u\rangle - \langle v, v\rangle = \| u \|^2 - \| v \|^2 = 0\end{aligned}$$

Therefore, the diagonals of a rhombus are perpendicular to each other.

Problem 3

Suppose $u, v \in V$, prove that the inner product $\langle u, v\rangle = 0$ if and only if $\|u\| \leqslant \|u + av\|$ for all $a \in F$.

证明中出现比较多问题：

1. 只证明充分性或者必要性，没有证明另一边或者另一边只写了易证
2. 当 v 作为分母时，未讨论等于 0 的情况
3. 内积忽略复共轭情况，交换顺序没有考虑变共轭

(sufficiency) If $\langle u, v\rangle = 0$, then

$$\| u + av \|^2 = \| u \|^2 + \| av \|^2 \geq \| u \|^2$$

by 6.13.

> 6.13 Pythagorean Theorem
>
> Suppose u and v are orthogonal vectors in V. Then
>
> $$\|u + v\|^2 = \|u\|^2 + \|v\|^2.$$

(necessity) If $\| u \| \leq \| u + av \|$ for all $a \in \mathbb{F}$, this implies

$$\| u + av \|^2 - \| u \|^2 = |a|^2 \| v \|^2 + a\langle v, u\rangle + \bar{a}\langle u, v\rangle \geq 0.$$

If $v = 0$, then $\langle u, v\rangle = 0$. If $v \neq 0$, plug $a = -\langle u, v\rangle / \| v \|^2$ into the previous equation ,we obtain

$$-\frac{|\langle u, v\rangle|^2}{\| v \|^2} \geq 0.$$

Hence $\langle u, v\rangle = 0$.

Problem 4

Suppose $u, v \in V$, prove that $\|au + bv\| = \|bu + av\|$ for all $a, b \in R$ if and only if $\|u\| = \|v\|$.

Proof:

only if part

If $\| av + bu \| = \| au + bv \|$ for all $a, b \in \mathbb{R}$, by setting $a = 1$ and $b = 0$, we have $\| u \| = \| v \|$.

if part

Conversely, suppose $\| u \| = \| v \|$. For all $a, b \in \mathbb{R}$, we have

$$\begin{aligned} \| av + bu \|^2 &= \langle av + bu, av + bu\rangle \\ &= a^2 \| u \|^2 + ab(\langle u, v\rangle + \langle v, u\rangle) + b^2 \| v \|^2 \end{aligned}$$

and

$$\begin{aligned} \| au + bv \|^2 &= \langle au + bv, au + bv\rangle \\ &= a^2 \| v \|^2 + ab(\langle u, v\rangle + \langle v, u\rangle) + b^2 \| u \|^2. \end{aligned}$$

Hence if $\| v \| = \| u \|$, we have

$$a^2 \| u \|^2 + b^2 \| v \|^2 = a^2 \| v \|^2 + b^2 \| u \|^2.$$

Therefore $\| av + bu \|^2 = \| au + bv \|^2$, i.e. $\| av + bu \| = \| au + bv \|$.

Problem 5

Suppose $u, v \in V$, $\|u\| = \|v\| = 1$ and $\langle u, v \rangle = 1$, prove that $u = v$.

Proof:

Consider $\| u - v \|^2$, we have

$$\| u - v \|^2 = \langle u - v, u - v \rangle = \langle u, u \rangle - \langle u, v \rangle - \langle v, u \rangle + \langle v, v \rangle$$

$$= \| u \|^2 - \langle u, v \rangle - \langle u, v \rangle + \| v \|^2 = 0$$

hence $u - v = 0$ by definiteness. That is $u = v$.

Problem 6

Find vectors $u, v \in R^2$ such that u is a scalar multiple of $(1, 3)$, v is orthogonal to $(1, 3)$, and $(1, 2) = u + v$.

Answer:

Let $v = (x, y)$ and $u = z(1,3)$, where $x, y, z \in \mathbb{R}$. Note that v is orthogonal to $(1,3)$,we have

$$(x, y) \cdot (1,3) = x + 3y = 0.$$

It follows that $v = y(-3,1)$. Since $(1,2) = u + v$, we obtain

$$y(-3,1) + z(1,3) = (z - 3y,\ y + 3z) = (1,2).$$

We can solve the equation and get $y = -1/10$ and $z = 7/10$. Hence $u = (7/10,21/10)$ and $v = (3/10, -1/10)$.

Problem 7

Prove that $(x_1 + \cdots + x_n)^2 \leqslant n\,(x_1^2 + \cdots + x_n^2)$ for all positive integers n and all real numbers $x_1, \ldots, x_n$.

注意证明时的充分性 all positive integers n 和 all real numbers x_i

有同学按照第一题通过举例可以证明不成立的思路，在这一题里也举例子证明是不可行的

Proof:

6.15 Cauchy–Schwarz Inequality

Suppose $u, v \in V$. Then

$$|\langle u, v \rangle| \leq \|u\|\,\|v\|.$$

This inequality is an equality if and only if one of u, v is a scalar multiple of the other.

By the Cauchy–Schwarz Inequality, if $x_1, \ldots, x_n, y_1, \ldots, y_n \in \mathbf{R}$, then

$$|x_1 y_1 + \cdots + x_n y_n|^2 \leq (x_1^2 + \cdots + x_n^2)(y_1^2 + \cdots + y_n^2) \text{ .}$$

Let $y_i = 1$,we can obtain

$$|x_1 + \cdots + x_n|^2 \leq (x_1^2 + \cdots + x_n^2) \text{ .}$$

Therefore, $(x_1 + \cdots + x_n)^2 \leqslant n(x_1^2 + \cdots + x_n^2)$ for all positive integers n and all real numbers $x_1, \ldots, x_n$.

Problem 8

Suppose V is a real inner product space, prove that

$$\langle u, v \rangle = \frac{\|u+v\|^2 - \|u-v\|^2}{4}$$

for all $u, v \in V$.

Proof:

Suppose V is a real inner-product space and $u, v \in V$. Then

$$\begin{aligned}\frac{\| u+v \|^2 - \| u-v \|^2}{4} &= \frac{\langle u+v, u+v \rangle - \langle u-v, u-v \rangle}{4} \\ &= \frac{\| u \|^2 + 2\langle u, v \rangle + \| v \|^2 - (\| u \|^2 - 2\langle u, v \rangle + \| v \|^2)}{4} \\ &= \frac{4\langle u, v \rangle}{4} \\ &= \langle u, v \rangle\end{aligned}$$

as desired.