HW2 参考答案及常见问题

第一题：

部分同学直接一上来就 obviously 得出证明结论,请注意数学思维的连贯性，以及锻炼好数学语言的书面描述能力

proof: because x_1, x_2, x_3, x_4 spans V.

then $\operatorname{span}(x_1, x_2, x_3, x_4) = \{a_1x_1 + a_2x_2 + a_3x_4 + a_4x_4 : a_1, a_2, a_3, a_4 \in F\}$.

we can get

$\operatorname{span}(x_1-x_2, x_2-x_3, x_3-x_4, x_4)$.

$= \{b_1(x_1-x_2) + b_2(x_2-x_3) + b_3(x_3-x_4) + b_4x_4 : b_1, b_2, b_3, b_4 \in F\}$.

$= \{b_1x_1 + (b_2-b_1)x_2 + (b_3-b_2)x_3 + (b_4-b_3)x_4 : b_1, b_2, b_3, b_4 \in F\}$.

Let $a_1 = b_1$, $a_2 = b_2 - b_1$, $a_3 = b_3 - b_2$ $a_4 = b_4 - b_3$.

so we can get.

$\operatorname{span}(x_1, x_2, x_3, x_4) = \operatorname{span}(x_1-x_2, x_2-x_3, x_3-x_4, x_4) = V$

Therefore x_1-x_2, x_2-x_3, x_3-x_4, x_4 also spans V.

第二题：

Prof:

Since x_1, x_2, x_3, x_4 is linearly independent in V, so

$c_1x_1 + c_2x_2 + c_3x_3 + c_4x_4 = 0$ with $c_1, c_2, c_3, c_4 = 0$

$d_1(x_1 - x_2) + d_2(x_2 - x_3) + d_3(x_3 - x_4) + d_4x_4 = 0$

which means: $d_1x_1 + (d_2 - d_1)x_2 + (d_3 - d_2)x_3 + (d_4 - d_3)x_4 = 0$

$$\Rightarrow \begin{cases} d_1 = c_1 = 0 \\ d_2 - d_1 = c_2 = 0 \\ d_3 - d_2 = c_3 = 0 \quad x_1 - x_2 \\ d_4 - d_3 = c_4 = 0 \end{cases} \Rightarrow d_1 = d_2 = d_3 = d_4 = 0$$

Consequently, $x_1 - x_2, x_2 - x_3, x_3 - x_4, x_4$ is also linearly independent

第三题：

注意扩充基的时候，不要取太复杂的，部分同学取得很复杂，但是最后用我们 MATLAB 验证的时候其实是错的

优先取单位标准基

3. $U = \{(x, 3x, y, 7y) \in R^4, x, y \in R\} = \{x(1,3,0,0) + y(0,0,1,7) \in R^4, x, y \in R\}$.

So a basis of U is $\{(1,3,0,0), (0,0,1,7)\}$, since $\{(1,3,0,0), (0,0,1,7)\}$ is linear independent. and $x, y \in R, R \subset F$, $U = \mathrm{Span}((1,3,0,0), (0,0,1,7))$.

Let $W = \mathrm{Span}((1,0,0,0), (0,0,1,0))$

For any $\vec{k} \in R^4$, $\vec{k} = x(1,3,0,0) + y(0,0,1,7) + a(1,0,0,0) + b(0,0,1,0)$, $x, y, a, b \in R$.

Thus $R^4 = U + W$.

~~And~~ meanwhile, $a_1(1,3,0,0) + a_2(0,0,1,7) + a_3(1,0,0,0) + a_4(0,0,1,0) = 0$ if and only if $a_1 = a_2 = a_3 = a_4 = 0$, $\Longrightarrow$ $(1,3,0,0), (0,0,1,7), (1,0,0,0), (0,0,1,0)$ are linear independent

So $U \cap W = \{0\}$.

Above all, $\exists W$ s.t. $U \oplus W = R^4$.

第四题：

尤其需要注意(b)中有线性相关的向量！

4. (a). (1). $\begin{bmatrix} x+2y \\ 2x-3y \\ -x \end{bmatrix} = x\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + y\begin{bmatrix} 2 \\ -3 \\ 0 \end{bmatrix}$, $x,y \in R$.

$\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 0 \end{bmatrix}$ are linear independent.

a basis of it is $\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 0 \end{bmatrix}$, 2) dimension is 2.

(b). $\begin{bmatrix} x+3y-z \\ 4x+5y+3z \\ 3x+6z \\ -x+7y-9z \end{bmatrix} = x\begin{bmatrix} 1 \\ 4 \\ 3 \\ -1 \end{bmatrix} + y\begin{bmatrix} 3 \\ 5 \\ 0 \\ 7 \end{bmatrix} + z\begin{bmatrix} -1 \\ 3 \\ 6 \\ -9 \end{bmatrix}$, $x,y,z \in R$.

$\because 2\begin{bmatrix} 1 \\ 4 \\ 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \\ 0 \\ 7 \end{bmatrix} + \begin{bmatrix} -1 \\ 3 \\ 6 \\ -9 \end{bmatrix}$

$\therefore$ a basis is $\begin{bmatrix} 3 \\ 5 \\ 0 \\ 7 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \\ 6 \\ -9 \end{bmatrix}$, (2) dimension is 2.

(c)(1) $(x,y,z,w) = (4y-3w, y, z, w) = y(4,1,0,0) + z(0,0,1,0) + w(-3,0,0,1)$ $y,z,w \in R$.

So. a basis is $(4,1,0,0)$, $(0,0,1,0)$, $(-3,0,0,1)$. (2) dimension is 3.

第五题：

有相当一部分人只有结果没有过程，请认真表述解题过程

5. $\because \dim(\mathrm{span}(\vec{x_1},\vec{x_2},\vec{x_3})) = 2 = \dim(\mathrm{span}(\vec{x_1},\vec{x_2})) = 2$.

So, $\vec{x_3}$ can be expressed by $\vec{x_1},\vec{x_2}$ linearly.

$\therefore$ Suppose $\vec{x_3} = a_1\vec{x_1} + a_2\vec{x_2} = \begin{bmatrix} 2a_1+a_2 \\ -a_1-a_2 \\ a_1 \\ 2a_1+a_2 \end{bmatrix} = \begin{bmatrix} 4 \\ -3 \\ 1 \\ a \end{bmatrix}$

Thus $a_1 = 1$, $a_2 = 2$. $a = 6$.

第六题：

1. 有人只给出结论，没有举反例或者给出证明
2. 关于第一小问，有部分同学的反例是错误的，对R^3举例，但是给出

$$S = \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right)$$

6. V is a nonzero finite-dimensional vector space, and the vectors listed belong to V. Mark each statement True or False. Justify each answer (Prove it if True or give an anti-example if False).

a. If $\dim V = p$ and S is a linearly independent set in V, then S is a basis for V.

False. For example, $V = R^2$, $\dim V = 2$, $S = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}$

b. If there exists a linearly independent set $\{v_1, \cdots, v_p\}$ in V, then $\dim V \geq p$.

True.

Proof: Since that set $\{v_1, v_2, \cdots, v_p\}$ is linearly independent, add any vector $v_{p+1} \in V$ to the set

① if set $\{v_1, v_2, \cdots, v_{p+1}\}$ is linearly dependent, implies that ~~set $\{v_1, v_2,$~~ any vector V_{p+1} can be written as a linear combination of set $\{v_1, v_2, \cdots, v_p\}$

Thus set $\{v_1, v_2, \cdots, v_p\}$ is a basis of V

Thus $\dim(V) = p$

② if set $\{v_1, v_2, \cdots, v_{p+1}\}$ is linearly independent, repeat the ~~th~~ same way, add more vectors into the set. Finally, set $\{v_1, v_2, \cdots, v_p, \cdots, v_{p+n}\}$ is linearly ~~in~~ denpendent.

Thus $\dim(V) = p+n-1 \geq p$

To Sum up, $\dim(V) \geq p$

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c. If there exists a set $\{v_1, v_2, \cdots, v_p\}$ that spans V, then $\dim V \le p$.

True.

Proof: Since set $\{v_1, v_2, \cdots, v_p\}$ spans V, thus any vector v in V can be written as: $v = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_p v_p$, ($\alpha_1, \alpha_2, \cdots, \alpha_p$ are scalars)

According to Theorem 2.6.1, the number of any linearly independent vectors is less or equal to n.

Since the basis of V is a set of independent vectors, thus $\dim(V) \le p$

d. If every set of p elements in V fails to span V, then $\dim V > p$.

True.

Proof: Since every set of p elements in V fails to span V, thus exist vector $v_{p+1} \in V$, such that v_{p+1} can not be written as a linear combination of p elements. Thus, suppose p elements are $v_1, v_2, \cdots, v_p$, such that set $\{v_1, v_2, \cdots, v_p, v_{p+1}\}$ is linearaly independent.

~~If~~ Repeat the same way until ~~et~~ set $\{v_1, v_2, \cdots, v_p, v_{p+1}, \cdots, v_{p+n}\}$ can span V.

~~so~~ Set $\{v_1, v_2, \cdots, v_p, v_{p+1}, \cdots, v_{p+n}\}$ is a basis of V.

Thus $\dim V = p+n > p$

e. If there exists a linearly dependent set $\{v_1, \cdots, v_p\}$ in V, then $\dim V \le p$.

False.

For example, $V = R^3$, $\dim(R^3) = 3$, suppose set $\left\{ \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}2\\0\\0\end{pmatrix} \right\}$, which is linearly dependent. Then $\dim V \le 2$, which is a contradition with $\dim(V) = 3$.

第七题：

一定要利用题目条件证明 U 和 V 的基是线性无关的。

可以使用$dim(U+V) = dim(U) + dim(V) - dim(U \cap V)$，但若考试出现类似题目，不建议使用此方法。

下面是推荐的参考解法：

7. To show that $\dim(U+V) = \dim(U) + \dim(V)$, we need to show that there exists a basis for $U+V$ that has $\dim(U)+\dim(V)$ elements.

○ Let $\{u_1, \cdots, u_m\}$ be a basis for U, $\{v_1, \cdots, v_n\}$ be a basis for V.

Suppose $\{u_1, \cdots, u_m, v_1, \cdots, v_n\}$ is a basis for $U+V$.

- First, show this set spans $U+V$.

Let $x \in U+V$. Then $x = u+v$ for some $u \in U$ and $v \in V$.

Since $\{u_1, u_2, \cdots, u_m\}$ is a basis of U, there exist scalars $a_1, a_2, \cdots, a_m$, such that

$$u = a_1 u_1 + a_2 u_2 + \cdots + a_m u_m \qquad (7\text{-}1)$$

Similarly, there exist scalars $b_1, b_2, \cdots, b_n$ such that

$$v = b_1 v_1 + b_2 v_2 + \cdots + b_n v_n. \qquad (7\text{-}2)$$

(7-1) and (7-2) show that $\{u_1, u_2, \cdots, u_m, v_1, v_2, \cdots, v_n\}$ spans $U+V$.

- Next, we need to show that this set is linearly independent.

Suppose that there exist scalars $c_1, c_2, \cdots, c_m, d_1, \cdots d_n$, such that

$$c_1 u_1 + c_2 u_2 + \cdots + c_m u_m + d_1 v_1 + d_2 v_2 + \cdots + d_n v_n = 0$$

Since $U \cap V \{0\}$, we have

$$c_1 u_1 + c_2 u_2 + \cdots + c_m u_m = 0$$

$$d_1 v_1 + d_2 v_2 + \cdots + d_n v_n = 0$$

However, $\{u_1, u_2, \cdots, u_m\}$ and $\{v_1, v_2, \cdots, v_n\}$ are both bases, they're linearly independent, which means

$$c_1 = c_2 = \cdots = c_m = d_1 = d_2 = \cdots = d_n = 0.$$

Therefore, $\{u_1, u_2, \cdots, u_m, v_1, v_2, \cdots, v_n\}$ is a linearly independent set.

According to the 2 points above, we know that this set is a basis for $U+V$ and have $\dim(U)+\dim(V)$ elements. Therefore,

$$\dim(U+V) = \dim(U) + \dim(V)$$

第八题：

问题小结：(1) 部分同学（a)、(b)问题的答案弄反了，需要注意 transition matrix 的定义，注意题目中所问的顺序。

(2) 第四问需要完成两部分的工作，第一步，根据定义计算坐标；第二步，使用前面题目计算的 transition matrix 进行验证，大部分同学只完成了一部分。

(3) 需要给出相应的计算步骤，变换步骤，直接给出结论是不提倡的。

8. Solution:

(a)
$$\left(\begin{array}{ccc:ccc} 1 & 1 & 2 & 4 & 0 & 0 \\ 1 & 2 & 3 & 6 & 1 & 1 \\ 1 & 2 & 4 & 7 & 1 & 2 \end{array}\right) \xrightarrow[r_3-r_1]{r_2-r_1} \left(\begin{array}{ccc:ccc} 1 & 1 & 2 & 4 & 0 & 0 \\ 0 & 1 & 1 & 2 & 1 & 1 \\ 0 & 1 & 2 & 3 & 1 & 2 \end{array}\right) \xrightarrow{r_1-r_3} \left(\begin{array}{ccc:ccc} 1 & 0 & 0 & 1 & -1 & -2 \\ 0 & 1 & 1 & 2 & 1 & 1 \\ 0 & 1 & 2 & 3 & 1 & 2 \end{array}\right).$$

$$\xrightarrow{r_3-r_2} \left(\begin{array}{ccc:ccc} 1 & 0 & 0 & 1 & -1 & -2 \\ 0 & 1 & 1 & 2 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{array}\right) \xrightarrow{r_2-r_3} \left(\begin{array}{ccc:ccc} 1 & 0 & 0 & 1 & -1 & -2 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{array}\right).$$

thus, $S_1 = \begin{pmatrix} 1 & -1 & -2 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$

(b)
$$\left(\begin{array}{ccc:ccc} 4 & 0 & 0 & 1 & 1 & 2 \\ 6 & 1 & 1 & 1 & 2 & 3 \\ 7 & 1 & 2 & 1 & 2 & 4 \end{array}\right) \xrightarrow[r_3-r_2]{r_1\times\frac{1}{4}} \left(\begin{array}{ccc:ccc} 1 & 0 & 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \\ 6 & 1 & 1 & 1 & 2 & 3 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \xrightarrow[r_3-r_1]{r_2-6r_1} \left(\begin{array}{ccc:ccc} 1 & 0 & 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \\ 0 & 1 & 1 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 & -\frac{1}{4} & -\frac{1}{4} & \frac{1}{2} \end{array}\right)$$

$$\xrightarrow{r_2-r_3} \left(\begin{array}{ccc:ccc} 1 & 0 & 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \\ 0 & 1 & 0 & -\frac{1}{4} & \frac{3}{4} & -\frac{1}{2} \\ 0 & 0 & 1 & -\frac{1}{4} & -\frac{1}{4} & \frac{1}{2} \end{array}\right)$$

thus, $S_2 = \begin{pmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \\ -\frac{1}{4} & \frac{3}{4} & -\frac{1}{2} \\ -\frac{1}{4} & -\frac{1}{4} & \frac{1}{2} \end{pmatrix}.$

(c)
$$S_1S_2 = \begin{pmatrix} 1 & -1 & -2 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \\ -\frac{1}{4} & \frac{3}{4} & -\frac{1}{2} \\ -\frac{1}{4} & -\frac{1}{4} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$S_2S_1 = \begin{pmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \\ -\frac{1}{4} & \frac{3}{4} & -\frac{1}{2} \\ -\frac{1}{4} & -\frac{1}{4} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} 1 & -1 & -2 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

thus, $S_1S_2 = I_3 = S_2S_1$ $\blacksquare$.

(d)
$$v = \begin{pmatrix} 4 & 0 & 0 \\ 6 & 1 & 1 \\ 7 & 1 & 2 \end{pmatrix}\begin{pmatrix} 2 \\ 3 \\ -4 \end{pmatrix} = \begin{pmatrix} 8 \\ 11 \\ 9 \end{pmatrix} \quad (\text{under base } E)$$

$$v = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 1 & 2 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x+y+2z \\ x+2y+3z \\ x+2y+4z \end{pmatrix} \quad (\text{under base } F)$$

then
$$\begin{cases} x+y+2z = 8 \\ x+2y+3z = 11 \\ x+2y+4z = 9 \end{cases} \Rightarrow \begin{cases} x = 7 \\ y = 5 \\ z = -2 \end{cases} \quad \therefore [v]_F = \begin{pmatrix} 7 \\ 5 \\ -2 \end{pmatrix}.$$

$$[v]_F = S_1 \cdot [v]_E = \begin{pmatrix} 1 & -1 & -2 \\ 1 & 1 & 1 \\ 1 & 0 & 2 \end{pmatrix}\begin{pmatrix} 2 \\ 3 \\ -4 \end{pmatrix} = \begin{pmatrix} 7 \\ 5 \\ -2 \end{pmatrix}$$

It turns out that the answer is correct.