HW3 参考答案及常见问题

第一题：

这题错的人非常多，首先注意基底 v1,v2,v3,w1,w2,w3 都是列向量；其次要注意到底哪个基底是 new 哪个基底是 old，题目要求是 old 到 new，还是 new 到 old，要注意审题；还有相当大一部分人矩阵从上面搬到下面就长得不一样了，计算坐标时简单四则运算错的乱七八糟，还请各位同学注意细节。

① (a) Solution:

$$[v_1\ v_2\ v_3] = [w_1\ w_2\ w_3]\,S$$

$$= [w_1\ w_2\ w_3]\begin{bmatrix} -1 & 0 & 4 \\ 1 & 1 & -2 \\ 1 & 3 & 0 \end{bmatrix}$$

so transition matrix $S = \begin{bmatrix} -1 & 0 & 4 \\ 1 & 1 & -2 \\ 1 & 3 & 0 \end{bmatrix}$

(b) $[v]_E = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$ $\qquad [v]_F = S \cdot [v]_E$

so $[v]_F = \begin{bmatrix} -1 & 0 & 4 \\ 1 & 1 & -2 \\ 1 & 3 & 0 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -6 \\ 5 \\ 5 \end{bmatrix}$

第二题：

2. Solution:

(a) By doing elementary row transformation, we can get the reduced row echelon form of A:

$$A=\begin{bmatrix}1 & -2 & 1 & 1 & 2\\ -1 & 3 & 2 & 0 & -2\\ 0 & 1 & 3 & 1 & 4\\ 1 & 2 & 13 & 5 & 5\end{bmatrix}\rightarrow\begin{bmatrix}1 & 0 & 7 & 3 & 0\\ 0 & 1 & 3 & 1 & 0\\ 0 & 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0 & 0\end{bmatrix}$$

So $\operatorname{rank} A = 3 = \dim C(A)$,

Let $A=[c_1\ c_2\ c_3\ c_4\ c_5]$, then $c_3 = 7c_1+3c_2+0c_5$, $c_4=3c_1+c_2+0c_5$

So $A=\begin{bmatrix}1 & -2 & 2\\ -1 & 3 & -2\\ 0 & 1 & 4\\ 1 & 2 & 5\end{bmatrix}\begin{bmatrix}1 & 0 & 7 & 3 & 0\\ 0 & 1 & 3 & 1 & 0\\ 0 & 0 & 0 & 0 & 1\end{bmatrix}$

A basis of $C(A)$ is $\left\{\begin{bmatrix}1\\-1\\0\\1\end{bmatrix}\begin{bmatrix}-2\\3\\1\\2\end{bmatrix}\begin{bmatrix}2\\-2\\4\\5\end{bmatrix}\right\}$, a basis of $C(A^T)$ is $\left\{\begin{bmatrix}1\\0\\7\\3\\0\end{bmatrix}\begin{bmatrix}0\\1\\3\\1\\0\end{bmatrix}\begin{bmatrix}0\\0\\0\\0\\1\end{bmatrix}\right\}$

Let $Ax=0$, then $\begin{bmatrix} 1 & 0 & 7 & 3 & 0 \\ 0 & 1 & 3 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} x = 0$.

So we have $\begin{cases} x_1 + 7x_3 + 3x_4 = 0 \\ x_2 + 3x_3 + 4x_4 = 0 \\ x_5 = 0 \end{cases}$,

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = x_3 \begin{bmatrix} -7 \\ -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -3 \\ -4 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

a basis of $N(A)$ is $\left\{ \begin{bmatrix} 7 \\ 3 \\ -1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \\ 0 \\ -1 \\ 0 \end{bmatrix} \right\}$, $\dim N(A) = 2$

Do elementary row transformation on A^T:

$$A^T = \begin{bmatrix} 1 & -1 & 0 & 1 \\ -2 & 3 & 1 & 2 \\ 1 & 2 & 3 & 13 \\ 1 & 0 & 1 & 5 \\ 2 & -2 & 4 & 5 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 & 0 & 1 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 3 \end{bmatrix}$$

$$\begin{matrix} 1 & -1 & 0 & 1 \\ 0 & 1 & 1 & 4 \\ 0 & 3 & 3 & 12 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 4 & 3 \end{matrix}$$

Then $A^T x = 0$ equals to $\begin{cases} x_1 - x_2 + x_4 = 0 \\ x_2 + x_3 + 4x_4 = 0 \\ 4x_3 + 3x_4 = 0 \end{cases}$ $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} \frac{17}{3} \\ \frac{13}{3} \\ 1 \\ -\frac{4}{3} \end{bmatrix} \cancel{x_3}$

$x_4 = -\frac{4}{3} x_3$

$x_2 = -x_3 - 4x_4 = -1 + \frac{16}{3}$

$x_1 = x_2 - x_4 = \frac{13}{3} + \frac{4}{3}$

So a basis of $N(A^T)$ is $\left\{ \begin{bmatrix} 17 \\ 13 \\ 3 \\ -4 \end{bmatrix} \right\}$, $\dim N(A^T) = 1$

(b) Part 1: For an $m \times n$ matrix A whose rank is r,

$\dim C(A) = \dim C(A^T) = r$,

$\dim N(A) = n - r$, $\dim N(A^T) = m - r$.

Verification: from (a) we know that $m=4$, $n=5$, $r = \operatorname{rank} A = 3$

$\dim C(A) = \dim C(A^T) = 3 = r$,

$\dim N(A) = 2 = n - r$, $\dim N(A^T) = 1 = m - r$.

Part 2: $N(A) = C(A^T)^{\perp}$, $N(A^T) = C(A)^{\perp}$

Verification: for each basis for $N(A)$ and $C(A^T)$,

$$\begin{bmatrix} 7 \\ 3 \\ -1 \\ 0 \\ 0 \end{bmatrix}^T \begin{bmatrix} 1 \\ 0 \\ 7 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 0 \\ -1 \\ 0 \end{bmatrix}^T \begin{bmatrix} 1 \\ 0 \\ 7 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 7 \\ 3 \\ -1 \\ 0 \\ 0 \end{bmatrix}^T \begin{bmatrix} 0 \\ 1 \\ 3 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 0 \\ -1 \\ 0 \end{bmatrix}^T \begin{bmatrix} 0 \\ 1 \\ 3 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 7 \\ 3 \\ -1 \\ 0 \\ 0 \end{bmatrix}^T \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 0 \\ -1 \\ 0 \end{bmatrix}^T \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = 0$$

So it can be infered that $N(A)$ and $C(A^T)$ are orthogonal.

Similarly, for the bases for $N(A^T)$ and $C(A)$:

$$\begin{bmatrix} 17 \\ 13 \\ 3 \\ -4 \end{bmatrix}^T \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 17 \\ 13 \\ 3 \\ -4 \end{bmatrix}^T \begin{bmatrix} -2 \\ 3 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 17 \\ 13 \\ 3 \\ -4 \end{bmatrix}^T \begin{bmatrix} 2 \\ -2 \\ 4 \\ 5 \end{bmatrix} = 0.$$

So $N(A^T)$ and $C(A)$ are orthogonal, too.

第三题：

Solutions:

(a) From R, we can get

$$\begin{cases} x_1 - x_3 + x_5 = 0 \\ x_2 - 2x_3 = 0 \\ x_4 + 2x_5 = 0 \end{cases} \Rightarrow \text{2 special solutions: } \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 0 \\ -2 \\ 1 \end{bmatrix},$$

which are the basis for $N(A)$

(b) from R, we know $a_5 = a_1 + 2a_4 = \begin{bmatrix} -2 \\ -5 \\ -4 \\ -10 \end{bmatrix}$, $a_3 = -a_1 - 2a_2$

) Suppose $a_2 = \begin{bmatrix} a_{21} \\ a_{22} \\ a_{23} \\ a_{24} \end{bmatrix}$, $a_3 = \begin{bmatrix} a_{31} \\ a_{32} \\ a_{33} \\ a_{34} \end{bmatrix} = \begin{bmatrix} -4-2a_{21} \\ -1-2a_{22} \\ -2-2a_{23} \\ -2a_{24} \end{bmatrix}$

For $Ax_0 = b$,

$$\begin{bmatrix} 4 & a_{21} & -4-2a_{21} & -3 & -2 \\ 1 & a_{22} & -1-2a_{22} & -3 & -5 \\ -2 & a_{23} & -2-2a_{23} & -1 & -4 \\ 0 & a_{24} & -2a_{24} & 5 & -10 \end{bmatrix} \begin{bmatrix} 0 \\ 3 \\ 1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ -3 \\ 0 \\ 1 \end{bmatrix}.$$

we can get $\begin{cases} a_{21} = -1 \\ a_{22} = -5 \\ a_{23} = -3 \\ a_{24} = -4 \end{cases}$ and $\begin{cases} a_{31} = -2 \\ a_{32} = 9 \\ a_{33} = \cancel{4}\ 8 \\ a_{34} = 8 \end{cases}$

$a_2 = \begin{bmatrix} -1 \\ -5 \\ -3 \\ -4 \end{bmatrix}$, $a_3 = \begin{bmatrix} -2 \\ 9 \\ 8 \\ 8 \end{bmatrix}$

第四题：

该题第一小问要从两个方向加以证明

(a) ① suppose $m \in S$, then $m = a_1 x + a_2 y$

suppose $M \in N(A)$, then $AM = 0$

$\therefore (\cancel{x}\ xy^T + yx^T) M = 0$

$\therefore M^T (xy^T + yx^T)^T = 0$

$\therefore M^T (yx^T + xy^T) = 0$

$\therefore M^T (yx^T + xy^T) x = 0$

$\therefore M^T (yx^T x + xy^T x) = M^T (y|x| + 0) = 0$

$\therefore M^T y = 0$

$\therefore M^T (yx^T + xy^T) y = M^T (yx^T y + xy^T y) = M^T (0 + x|y|) = 0$

$\therefore M^T x = 0$

$\therefore M^T \cdot m = a_1 M^T x + a_2 M^T y = 0$

$\therefore N(A) \in S^\perp$

② suppose $N \in S^\perp$, then $N^T \cdot m = 0$

$\therefore a_1 N^T x + a_2 N^T y = 0$

$\therefore N^T x = 0,\ N^T y = 0$

$\therefore N^T (yx^T + xy^T) = N^T y x^T + N^T x y^T = 0$

$\therefore (xy^T + yx^T) N = 0$

$\therefore S^\perp \in N(A)$

$\therefore N(A) = S^\perp$

(b) $\because \dim S = 2,\ S \in R^n$

$\therefore \dim S^\perp = n - 2$

$\because N(A) = S^\perp$,

$\therefore \dim N(A) = n - 2 = n - R(A)$

$\therefore R(A) = 2$