HW4 参考答案及常见问题

第一题：

(a). Since $v_1, \dots, v_m$ spans V, for $v \in V$, $v = a_1 v_1 + \dots + a_m v_m$.

$$T\left(\begin{bmatrix} a_1 \\ \vdots \\ a_m \end{bmatrix}\right) = v. \quad T \text{ is surjective.}$$

(b). Let $T(x) = 0$. $T(x) = x_1 v_1 + \dots + x_m v_m = 0$.

Since $v_1, \dots, v_m$ are linearly independent, $x_1, \dots, x_m = 0$.

$$x = \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}. \quad \text{null } T = \{0\}. \Rightarrow T \text{ is injective.}$$

第二题：

(a).

Proof: (a). Let $c_1 T(v_1) + \dots + c_n T(v_n) = 0$. null T.

$\Rightarrow T(c_1 v_1 + \dots + c_n v_n) = 0$.

Since T is injective, null $T = \{0\}$. $\Rightarrow c_1 v_1 + \dots + c_n v_n = 0$.

Note that $v_1, \dots, v_n$ are linearly independent, we have $c_1 = \dots = c_n = 0$.

$\Rightarrow T(v_1), \dots, T(v_n)$ is linearly independent.

(b).

一定要双向证明$T(U) \subseteq T(V)$ 和 $T(V) \subseteq T(U)$。

(b) Given that $\dim V = \dim \operatorname{null} T + \dim \operatorname{range} T$, let U be any subspace V such that $U \cap \operatorname{null} T = \{0\}$, we want to show that $T(U) = T(V)$.
Clearly, $T(U)$ is already a subspace of $T(V)$, so $T(U) \subseteq T(V)$.
Then we only need to prove that $T(V) \subseteq T(U)$.
Suppose that $w \in T(V)$, $T(v) = w$. Let $u = v + n$, where $n \in \operatorname{null} T$.
then

$$T(u) = T(v+n) = T(v) + T(n) = w + 0 = w.$$

Therefore, w is in $T(U)$, $T(V) \subseteq T(U) \Rightarrow T(V) = T(U)$.
To find a basis for U, we can use the fact that $\dim V = \dim U + \dim \operatorname{null} T$.
Choose a basis $\{v_1, v_2, \cdots, v_n\}$ for V such that $\{T(v_1), \cdots, T(v_n)\}$ is a basis for $T(V)$, and $\{v_{n+1}, \cdots, v_{n+m}\}$ is a basis for null T.
Then the set $\{v_1, v_2, \cdots, v_{n+m}\}$ is a basis for U, since it spans both U and V and is linearly independent.

第三题：

（a）充分性和必要性都需要证明

$(\Rightarrow)$ If there exists a surjective $T \in \mathcal{L}(V, W)$, $\operatorname{range} T = W$.

$$\begin{aligned}\dim V &= \dim \operatorname{null} T + \dim \operatorname{range} T \\ &= \dim \operatorname{null} T + \dim W \\ &\geq \dim W.\end{aligned}$$

$(\Leftarrow)$ If $\dim V \geq \dim W$, $n \geq m$.

Construct T: $\underbrace{T(v_1) = w_1}_{\Delta}, \cdots, \underbrace{T(v_m) = w_m}_{\Delta}, T(v_{m+1}) = 0, \cdots, T(v_n) = 0$.

$\forall w \in W$, with $w = c_1 w_1 + \cdots + c_m w_m$,

$$\begin{aligned}&= c_1 T(v_1) + \cdots + c_m T(v_m) \\ &= T(\underline{c_1 v_1 + \cdots + c_m v_m}) \Rightarrow \operatorname{range} T = W. \\ &\Rightarrow T \text{ is surjective.}\end{aligned}$$

（b）充分性和必要性都需要证明

(b) ($\Rightarrow$) $\operatorname{null} T = U$

V and W are finite-dimensional and $T \in \mathcal{L}(V, W)$

$\Rightarrow \quad \dim V = \dim \operatorname{null} T + \dim \operatorname{range} T$

For $T \in \mathcal{L}(V, W)$, range T is a subspace of V and $\dim \operatorname{rang} T \leq \dim W$

$\Rightarrow \quad \dim \operatorname{null} T = \dim V - \dim \operatorname{range} T \geq \dim V - \dim W$

Hence $\dim U \geq \dim V - \dim W$

($\Leftarrow$) $\dim U \geq \dim V - \dim W$

Aussme $\dim U = r \quad \dim V = n \quad \dim W = m$

$\Rightarrow r \geq n - m \quad (m \geq n - r)$

Let $\{v_1, v_2 \cdots v_r\}$ is a basis of U, which can be extended to a basis of V $\{v_1, v_2 \cdots v_r, v_{r+1} \cdots v_n\}$

construct

$T(v_1) = T(v_2) = \cdots\cdots = T(v_r) = 0$

$T(v_{r+1}) = w_1, T(v_{r+2}) = w_2, \cdots\cdots, T(v_n) = w_{n-r} \quad (n - r \leq m)$

Hence there exists $T \in \mathcal{L}(V, W)$. when $\dim U \geq \dim V - \dim W$ we have $\operatorname{null} T = U$.

第四题：

4. Solution:

(a) Since $T \in \mathcal{L}(R^4, R^2)$, we have

$$\dim R^4 = \dim \text{null}\, T + \dim T(R^4)$$

Since $\text{null}\, T = \left\{ \begin{bmatrix} 2a \\ a \\ 5b \\ b \end{bmatrix} \right\}$, a basis for null T can be $\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 5 \\ 1 \end{bmatrix} \right\}$.

Therefore, $\dim \text{null}\, T = 2$, $\dim T(R^4) = \dim R^4 - \dim \text{null}\, T = 4 - 2 = 2 = \dim R^2$

This means that range T is a 2-dimensional subspace of R^2. But since the only 2-dimensional subspace of R^2 is R^2 itself, it follows that T is surjective.

(b) Suppose that there exists $T \in \mathcal{L}(R^5, R^2)$ such that

$$\text{null}\, T = \{ x = [x_1\ x_2\ x_3\ x_4\ x_5]^T \in R^5 \mid x_1 = 2x_2,\ x_3 = x_4 = x_5 \}$$

Then null T can be expressed as $\text{Span}\left(\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right)$, it follows that $\dim \text{null}\, T = 2$

We know that $\dim R^5 = 5$. Since range T is a subspace of R^2, we can infer that $\dim \text{range}\, T \leq 2$.

However, this contradicts with the theorem that

$$\dim R^5 = \dim \text{null}\, T + \dim \text{range}\, T,$$

since 2 plus a number that is not larger than 2 cannot be 5.

Therefore, there does not exist a linear transformation like this.

第五题：

5. (a) Since $T(e_1) = \begin{bmatrix}2\\-1\\0\end{bmatrix}$ $T(e_2) = \begin{bmatrix}-1\\1\\1\end{bmatrix}$. The standard matrix $A = \begin{bmatrix}2 & -1\\-1 & 1\\0 & 1\end{bmatrix}$

$[e_1\ e_2]\begin{bmatrix}1\\2\end{bmatrix}$

(b) Since $2e_2+e_1 = \begin{bmatrix}1\\2\end{bmatrix}$, the standard matrix $A = \begin{bmatrix}1 & 0\\2 & 1\end{bmatrix}$.

$[e_1,\ e_2]\begin{bmatrix}-2\\1\end{bmatrix}$

(c) Since $-2e_1+e_2 = \begin{bmatrix}-2\\1\end{bmatrix}$ and the matrix of reflection is $\begin{bmatrix}0 & -1\\-1 & 0\end{bmatrix}$,

the standard matrix $A = \begin{bmatrix}0 & -1\\-1 & 0\end{bmatrix}\begin{bmatrix}1 & -2\\0 & 1\end{bmatrix} = \begin{bmatrix}0 & -1\\-1 & 2\end{bmatrix}$

第六题：

solution: $e_1 = \begin{bmatrix}1\\0\end{bmatrix}$ $L(e_1) = b_1+b_3 = [b_1\ b_2\ b_3]\begin{bmatrix}1\\0\\1\end{bmatrix}$

$e_2 = \begin{bmatrix}0\\1\end{bmatrix}$ $L(e_2) = -b_1+b_2+b_3 = [b_1\ b_2\ b_3]\begin{bmatrix}-1\\1\\1\end{bmatrix}$

$A = \begin{bmatrix}1 & -1\\0 & 1\\1 & 1\end{bmatrix}$