HW5 参考答案及常见问题

第一题：

1. Note: (i) The argmented matrix

$$[b_1\ b_2\ L(u_1)\ L(u_2)\ L(u_3)] \Rightarrow \begin{bmatrix} 1 & 2 & 1 & 3 & 0 \\ -1 & -1 & 2 & 0 & -2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -5 & -3 & 4 \\ 0 & 1 & 3 & 3 & -2 \end{bmatrix}$$

the matrix $A = \begin{bmatrix} -5 & -3 & 4 \\ 3 & 3 & -2 \end{bmatrix}$

$$\Rightarrow L(u_1) = [b_1\ b_2]\begin{bmatrix} -5 \\ 3 \end{bmatrix} \quad L(u_2) = [b_1\ b_2]\begin{bmatrix} -3 \\ 3 \end{bmatrix} \quad L(u_3) = [b_1\ b_2]\begin{bmatrix} 4 \\ -2 \end{bmatrix}$$

(ii) The argmented matrix

$$[b_1\ b_2\ L(u_1)\ L(u_2)\ L(u_3)] = \begin{bmatrix} 1 & 2 & 0 & 4 & 2 \\ -1 & -1 & -1 & -1 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 2 & -2 & -4 \\ 0 & 1 & -1 & 3 & 3 \end{bmatrix}$$

the matrix $A = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 3 \end{bmatrix}$

$$\Rightarrow L(u_1) = [b_1\ b_2]\begin{bmatrix} 2 \\ -1 \end{bmatrix} \quad L(u_2) = [b_1\ b_2]\begin{bmatrix} -2 \\ 3 \end{bmatrix} \quad L(u_3) = [b_1\ b_2]\begin{bmatrix} -4 \\ 3 \end{bmatrix}$$

第二题：

转移矩阵S是从$[1, x, x^2]$ 到 $[2, 4x, 4x^2 - 4]$。很多同学都弄反了。

$$D(1) = 0 = [1, x, x^2] \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$D(x) = 1 = [1, x, x^2] \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$D(x^2) = 2x = [1, x, x^2] \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$$

thus,

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$D(2) = 0 = [2, 4x, 4x^2-4] \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$D(4x) = 4 = [2, 4x, 4x^2-4] \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$$

$$D(4x^2-4) = 8x = [2, 4x, 4x^2-4] \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$$

thus,

$$B = \begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$1 = \frac{1}{2} \cdot 2 + 0 \cdot 4x + 0 \cdot (4x^2 - 4)$$

$$x = 0 \cdot 2 + \frac{1}{4} \cdot 4x + 0 \cdot (4x^2 - 4)$$

$$x^2 = \frac{1}{2} \cdot 2 + 0 \cdot 4x + \frac{1}{4}(4x^2 - 4)$$

thus.

$$S = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & \frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{4} \end{bmatrix}$$

第三题：

3. Suppose V and W are finite-dimensional and $T \in \mathcal{L}(V, W)$. Prove that $\dim \operatorname{range} T = 1$ if and only if there exist a basis of V and a basis of W such that with respect to these bases, all entries of the matrix representation $\mathcal{M}(T)$ equal 1.

Construct.

$\dim V = n$, $\dim W = m$.

Proof. ($\Leftarrow$) Let the basis of V be $\{v_1, \dots, v_n\}$ and the basis of W. $\{w_1, \dots, w_m\}$.

$$[T(v_1) \cdots T(v_n)] = [w_1 \cdots w_m] \begin{bmatrix} 1 & 1 & \cdots & 1 \\ \vdots & \vdots & & \vdots \\ 1 & 1 & \cdots & 1 \end{bmatrix}.$$

$$T(v_1) = \cdots = T(v_n) = w_1 + \cdots + w_m$$

$\forall v \in V$, with $v = c_1 v_1 + \cdots + c_n v_n$.

$$T(v) = c_1 T(v_1) + \cdots + c_n T(v_n) = (c_1 + \cdots + c_n)(w_1 + \cdots + w_m).$$

$\{w_1 + \cdots + w_m\}$ is a basis of range T. $\Rightarrow \dim \operatorname{range} T = 1$.

($\Rightarrow$) If $\dim \operatorname{range} T = 1$, $\dim \operatorname{null} T = n-1$.

Let $\{v_1, \dots, v_{n-1}\}$ be a basis of null T, which can be extended to a basis of V, $\{v_1, \dots, v_{n-1}, v_n\}$. $T(v_1) = \cdots = T(v_{n-1}) = 0$. $T(v_n) \triangleq w_1$, which can be extended to a basis of W. $\{w_1, w_2, \dots, w_m\}$.

Construct a basis of V, $\{v_1 + v_n, v_2 + v_n, \dots, v_{n-1} + v_n, v_n\}$ ✓

$$T(v_1 + v_n) = \cdots = T(v_{n-1} + v_n) = T(v_n) = w_1$$

Let $c_1(v_1 + v_n) + \cdots + c_n v_n = 0$. $\Rightarrow c_1 v_1 + \cdots + c_{n-1} v_{n-1} + (c_1 + \cdots + c_n) v_n = 0$.

$v_1, \dots, v_n$ are linearly independent, $c_1 = \cdots = c_{n-1} = c_1 + \cdots + c_n = 0 \Rightarrow c_1 = \cdots = c_n = 0$.

Construct a basis of W: $\{w_1 - w_2 - \cdots - w_m, w_2, \dots, w_m\}$ ✓

Let $a_1(w_1 - w_2 - \cdots - w_m) + a_2 w_2 + \cdots + a_m w_m = 0 \Rightarrow a_1 w_1 + (a_2 - a_1) w_2 + \cdots + (a_m - a_1) w_m = 0$.

$w_1, \dots, w_m$ are linearly independent, $a_1 = a_2 - a_1 = \cdots = a_m - a_1 = 0$

$\Rightarrow a_1 = \cdots = a_m = 0$.

第四题：

4. (i)

$A^TA\hat{x} = A^Tb$ $\quad \begin{bmatrix} 6 & 3 \\ 3 & 6 \end{bmatrix}\hat{x} = \begin{bmatrix} 6 \\ 9 \end{bmatrix}$

$\hat{x} = \begin{bmatrix} 6 & 3 \\ 3 & 6 \end{bmatrix}^{-1}\begin{bmatrix} 6 \\ 9 \end{bmatrix}$ $\quad \hat{x} = \begin{bmatrix} \frac{1}{3} \\ \frac{4}{3} \end{bmatrix}$

(ii)

$A^TA\hat{x} = A^Tb$ $\quad$ a basis for $C(A)$ is $\left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \right\}$

$A' = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$ $\quad (A')^TA' = 6 \quad (A')^Tb = 6$

$6\hat{x} = 6 \quad \hat{x} = 1.$ $\quad \hat{x}' = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$Ax = 0 \quad x = k\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ $\quad k \in R$

$\hat{x} = \hat{x}' + x = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + k\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ $(k \in R)$

第五题：

(a)

(a) Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$. Prove that the following statements are equivalent:
(i) T is invertible;
(ii) T is injective;
(iii) T is surjective.

(i) ⇒ (ii), (ii) ⇒ (iii), (iii) ⇒ (i)

Pf: T is invertible ⟺ T is injective and surjective.

(i) ⇒ (ii), (iii)

(ii) ⇒ (iii) T is injective, null $T = \{0\}$.

$\dim V = \dim \text{null } T + \dim \text{range } T$.

$= \dim \text{range } T$. (range T is a subspace of V).

$\Rightarrow$ range $T = V$.

(iii) ⇒ (ii) T is surjective, range $T = V$.

$\dim V = \dim \text{null } T + \dim \text{range } T$

$= \dim \text{null } T + \dim V$

$\Rightarrow \dim \text{null } T = 0$. $\Rightarrow$ null $T = \{0\}$ $\Rightarrow$ T is injective.

(b)

(b) "$\Rightarrow$" Suppose there exists an invertible operator $T \in \mathcal{L}(V)$ such that $T(u) = S(u)$ for any $u \in U$, we need to proof that S is injective

Since $T \in \mathcal{L}(V)$ and T is invertible, so T is injective.

Assume that for $u_1, u_2 \in U$, $S(u_1) = S(u_2)$, then $T(u_1) = S(u_1) = S(u_2) = T(u_2)$

so $u_1 = u_2$, this proof that S is injective.

"$\Leftarrow$" Suppose that S is injective

① If $\dim U = \dim V$, then $S \in \mathcal{L}(V)$ and S is injective, so S is invertible

let $T = S$, then T satisfies all the items.

② If $\dim U < \dim V$, suppose $\{u_1, u_2 \cdots u_m\}$ is a basis of U, and it can be extended to $\{u_1 u_2 \cdots u_m\ u_{m+1} \cdots u_n\}$ which is a basis of V

now we need to prove that $S(u_1)\ S(u_2) \cdots S(u_m)$ are linearly independen

suppose $c_1 S(u_1) + c_2 S(u_2) + \cdots + c_m S(u_m) = 0$

$\Rightarrow S(c_1 u_1 + c_2 u_2 + \cdots + c_m u_m) = 0$

since S is injective, null $S = \{0\}$, so $c_1 = c_2 = \cdots = c_m = 0$

we can extend $\{S(u_1)\ S(u_2) \cdots S(u_m)\ v_{m+1} \cdots v_n\}$ as a basis of V

let $T(u_1) = S(u_1)$ $T(u_2) = S(u_2)$ $\cdots$ $T(u_m) = S(u_m)$ $T(u_{m+1}) = v_{m+1}$ $\cdots$ $T(u_n) = v_n$

now we need to proof that T is invertible.

since $\dim \operatorname{range} T = n = \dim V$, T is surjective

suppose $T(u) = T(a_1 u_1 + a_2 u_2 + \cdots + a_n u_n) = 0$

then $a_1 T(u_1) + a_2 T(u_2) + \cdots + a_n T(u_n) = 0$ and $\{T(u_1)\ T(u_2) \cdots T(u_n)\}$ is a basis of V

so $a_1 = a_2 = \cdots = a_n = 0$, null $T = \{0\}$ so T is injective. So T is invertible.

(c). if part 和 only if part 都需要证明

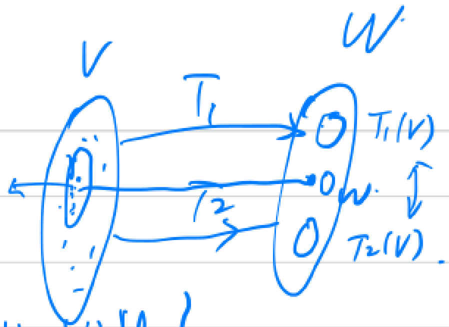
Proof: ($\Leftarrow$). $\forall v \in \text{null}\, T_1$, $T_1(v)=0$. $ST_2(v) = T_1(v) = 0 = S(T_2(v))$

S is invertible, $T_2(v)=0$. $\Rightarrow v \in \text{null}\, T_2 \Rightarrow \text{null}\, T_1 \subset \text{null}\, T_2$.

$\forall v \in \text{null}\, T_2$, $T_2(v) = 0$, $T_1(v) = ST_2(v) = S(0) = 0$.

$\Rightarrow v \in \text{null}\, T_1 \Rightarrow \text{null}\, T_2 \subset \text{null}\, T_1$.

($\Rightarrow$) If $\text{null}\, T_1 = \text{null}\, T_2$, let $\{v_1, \cdots, v_r\}$ be a basis of $\text{null}\, T_1$ ($\text{null}\, T_2$), which can be extended to a basis of V, $\{\underline{v_1, \cdots, v_r}, \underline{v_{r+1}, \cdots, v_n}\}$.



$T_1(\underset{\approx}{v_1}) = \cdots = T_1(\underset{\approx}{v_r}) = 0$. Let $T_1(v_{r+1}) = w_1, \cdots, T_1(v_n) = w_{n-r}$.

$T_2(v_1) = \cdots = T_2(v_r) = 0$. Let $T_2(v_{r+1}) = w_1', \cdots, T_2(v_n) = w_{n-r}'$.

Let $c_1 w_1 + \cdots + c_{n-r} w_{n-r} = 0$. $\Rightarrow T_1(\underline{c_1 v_{r+1} + \cdots + c_{n-r} v_n}) = 0$.

$\Rightarrow c_1 v_{r+1} + \cdots + c_{n-r} v_n \in \text{null}\, T_1 \Rightarrow c_1 v_{r+1} + \cdots + c_{n-r} v_n = 0$.

$\Rightarrow c_1 = \cdots = c_{n-r} = 0$.

$\Rightarrow w_1, \cdots, w_{n-r}$ are linearly independent, which can be extended to a basis of W. $\{w_1, \cdots, w_{n-r}, \cdots, w_m\}$.

Similarly, $w_1', \cdots w_{n-r}'$ are linearly independent, which can be also extended to a basis of W, $\{w_1', \cdots, w_{n-r}', \cdots, w_m'\}$.

Construct S. $S(w_1') = w_1, \cdots, S(w_m') = w_m$. ✓

(d). 以作业内的题目为准，按照不同题目解答，只要过程正确，也算对。

题目: $T_1, T_2 \in \mathcal{L}(V,W)$ need to prove: $\underset{r}{\dim \operatorname{null} T_1 = \dim \operatorname{null} T_2} \iff \exists$ invertible operators. $R \in \mathcal{L}(V)$. $S \in \mathcal{L}(W)$ s.t. $T_1 = ST_2R$.

solution:

$\Leftarrow$ $\exists$ $R\,S$ st. $T_1 = ST_2R$

let $\{v_1 \cdots v_r\}$ be a basis of $\operatorname{null} T_1$. we can extend it to a basis of V.

$\{v_1 \cdots v_r, v_{r+1}, \cdots, v_n\}$. $\Rightarrow \{T_1(v_{r+1}), \cdots, T_1(v_n)\}$ is a basis of $\operatorname{rang} T_1$.

since S & R are invertible. $\{R(v_1), \cdots, R(v_n)\}$ is a basis of V.

$ST_2R(v_1) = T_1(v_1) = 0$ $\cdots$ $ST_2R(v_r) = T_1(v_r) = 0$ $\Rightarrow T_2R(v_1) = \cdots = T_2R(v_r) = 0$

~~$\Rightarrow \{R(v_1), \cdots, R(v_r)\}$ is a basis of null T_2~~.

$\Rightarrow$ $\forall$ $v \in V$ $T_2(v) = a_1T_2R(v_1) + \cdots + a_nT_2R(v_n) = a_{r+1}T_2R(v_{r+1}) + \cdots + a_nT_2R(v_n)$

if $c_{r+1}T_2R(v_{r+1}) + \cdots + c_nT_2R(v_n) = 0$. $\Rightarrow c_{r+1}ST_2R(v_{r+1}) + \cdots + c_nST_2R(v_n) = 0$

$\Rightarrow c_{r+1}T_1(v_{r+1}) + \cdots + c_nT_1(v_n) = 0$ $\Rightarrow$ $c_{r+1} = \cdots = c_n = 0$

we have $\{T_2R(v_{r+1}), \cdots, T_2R(v_n)\}$ is a basis of $\operatorname{rang} T_2$.

$\dim \operatorname{rang} T_2 = n - r$ $\Rightarrow$ $\dim \operatorname{null} T_2 = r = \dim \operatorname{null} T_1$.

$\Rightarrow$ if $\dim \operatorname{null} T_1 = \dim \operatorname{null} T_2 = r$.

let $\{v_1, \cdots, v_r\}$ be a basis of $\operatorname{null} T_1$. and extend it to $\{v_1, \cdots, v_r, v_{r+1}, \cdots, v_n\}$ a basis of V

$\{v_1', \cdots, v_r'\}$ $\cdots$ $\operatorname{null} T_2$. $\{v_1', \cdots, v_r', v_{r+1}', \cdots v_n'\}$ V.

let $R(v_1) = v_1'$ $\cdots$ $R(v_r) = R(v_r')$ $\cdots$ $R(v_n) = v_n'$

$\{T_1(v_{r+1}) \cdots T_1(v_n)\}$ is a basis of $\operatorname{rang} T_1$. $\{T_2(v_{r+1}'), \cdots, T_2(v_n')\}$ is a basis of $\operatorname{rang} T_2$

let $w_1 = T_1(v_{r+1}) \cdots w_{n-r} = T_1(v_n)$ $\underline{w_1' = T_2(v_{r+1}') \cdots w_{n-r}' = T_2(v_n')}$
and extend them to $\{w_1, \cdots, w_r, w_{r+1}, \cdots w_m\}$ be a basis of W the same $\{w_1', \cdots, w_r', w_{r+1}', \cdots w_m'\}$

let $S(w_1') = w_1$

$\vdots$

$S(w_m') = w_m$

with S & R.

$\forall$ $i = 1, \cdots, r$.

$ST_2R(v_i) = 0 = T_1(v_i)$

$\forall$ $i = r+1, \cdots, n$.

$ST_2R(v_i) = ST_2(v_i') = S(w_{i-r}') = w_{i-r} = T_1(v_i)$