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1. Solution: $R_{ab} = R_x(\varphi) R_z(\theta)$　　(rotation about fixed axes)

pre multiply

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi & -\sin\varphi \\ 0 & \sin\varphi & \cos\varphi \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \cos\varphi\sin\theta & \cos\varphi\cos\theta & -\sin\varphi \\ \sin\varphi\sin\theta & \sin\varphi\cos\theta & \cos\varphi \end{bmatrix}$$

If $\theta = 30°$ and $\varphi = 45°$, $R_{ab} = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ \frac{\sqrt{2}}{4} & \frac{\sqrt{6}}{4} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{4} & \frac{\sqrt{6}}{4} & \frac{\sqrt{2}}{2} \end{bmatrix}$.

2. Solution: $R_{ab} = R_z(\theta) R_x(\varphi)$　　(rotation about varying axes)

Euler Angle

post multiply

$$= \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi & -\sin\varphi \\ 0 & \sin\varphi & \cos\varphi \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta & -\sin\theta\cos\varphi & \sin\theta\sin\varphi \\ \sin\theta & \cos\theta\cos\varphi & -\cos\theta\sin\varphi \\ 0 & \sin\varphi & \cos\varphi \end{bmatrix}$$

If $\theta = 30°$ and $\varphi = 45°$, $R_{ab} = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4} \\ \frac{1}{2} & \frac{\sqrt{6}}{4} & -\frac{\sqrt{6}}{4} \\ 0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$.

3. Solution:

$$R_{ab} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}}_{\text{Rotation about } X_A} \underbrace{\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\text{Rotation about } Z_A} = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow g = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 & 3 \\ 0 & 0 & 1 & 7 \\ -1 & 0 & 0 & 9 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

4. Solution: The block $\begin{bmatrix} a & 0 & -1 \\ b & 0 & 0 \\ c & -1 & 0 \end{bmatrix}$ is a rotation matrix, thus its columns are mutually orthonormal. Hence $\begin{cases} -c+0+0=0 \\ -a+0+0=0 \end{cases} \Rightarrow a=c=0$

$\det \begin{vmatrix} a & 0 & -1 \\ b & 0 & 0 \\ c & -1 & 0 \end{vmatrix} = 1 = b \Rightarrow b=1$　　$d=0$ follows from definition.

Hence $a=0$, $b=1$, $c=0$, $d=0$.

5. Solution: Consider frame $\{B\}$ as fixed. Then

$$R_{ba} = \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\substack{\text{Rotation about } Z_B \\ (-90^\circ)}} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}}_{\substack{\text{Rotation about } X_B \\ (90^\circ)}} = \begin{bmatrix} 0 & 0 & -1 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \Rightarrow R_{ab} = R_{ba}^T = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix}.$$

can also be obtained by

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}}_{\text{first } X} \underbrace{\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\text{Then } Z}$$

$$\Rightarrow g_{ab} = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 & 3 \\ 0 & 0 & 1 & 7 \\ -1 & 0 & 0 & 9 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} R & P \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} R^T & -R^T P \\ 0 & 1 \end{bmatrix}.$$

6. Solution: $g_{cb} = g_{cu} g_{ua} g_{ab} = g_{cu} g_{ua} g_{ba}^{-1}$

$$= \begin{bmatrix} \sqrt{3}/2 & -1/2 & 0 & -3 \\ \sqrt{3}/4 & 3/4 & -1/2 & -3 \\ 1/4 & \sqrt{3}/4 & \sqrt{3}/2 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{3}/2 & -1/2 & 0 & 1 \\ 1/2 & \sqrt{3}/2 & 0 & -1 \\ 0 & 0 & 1 & 8 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \sqrt{3}/2 & 1/2 & 10-5\sqrt{3} \\ 0 & -1/2 & \sqrt{3}/2 & 5+10\sqrt{3} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 & \frac{11\sqrt{3}}{2}-\frac{5}{2} \\ \frac{3}{4} & \frac{\sqrt{3}}{4} & -\frac{1}{2} & \frac{11\sqrt{3}}{4}-\frac{31}{4} \\ \frac{\sqrt{3}}{4} & \frac{1}{4} & \frac{\sqrt{3}}{2} & \frac{15\sqrt{3}}{4}+\frac{23}{4} \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \sqrt{3}/2 & 1/2 & 10-5\sqrt{3} \\ 0 & -1/2 & \sqrt{3}/2 & 5+10\sqrt{3} \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{3}{4} & -\frac{\sqrt{3}}{4} & 5+\frac{\sqrt{3}}{2} \\ \frac{3}{4} & \frac{5}{8} & -\frac{\sqrt{3}}{8} & \frac{\sqrt{3}}{4}-14 \\ \frac{\sqrt{3}}{4} & -\frac{\sqrt{3}}{8} & \frac{7}{8} & 5\sqrt{3}+\frac{93}{4} \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

7. Solution: (a) $g_{ab} = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix}$, $R = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $p = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}$ $\therefore g_{ab} = \begin{bmatrix} -1 & 0 & 0 & 3 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

(b) $g_{ac} = \begin{bmatrix} R & P \\ 0 & 1 \end{bmatrix}$ $R = \begin{bmatrix} 0 & -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ 0 & \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -1 & 0 & 0 \end{bmatrix}$ $P = \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix}$ $\therefore g_{ac} = \begin{bmatrix} 0 & -\frac{1}{2} & \frac{\sqrt{3}}{2} & 3 \\ 0 & \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ -1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

(c) $g_{bc} = \begin{bmatrix} R & P \\ 0 & 1 \end{bmatrix}$, $R = \begin{bmatrix} 0 & \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ 0 & -\frac{\sqrt{3}}{2} & -\frac{1}{2} \\ -1 & 0 & 0 \end{bmatrix}$ $p = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$ $\therefore g_{bc} = \begin{bmatrix} 0 & \frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ 0 & -\frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ -1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

(d) $g_{ca} = \begin{bmatrix} R & P \\ 0 & 1 \end{bmatrix}$, $R = \begin{bmatrix} 0 & 0 & -1 \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \end{bmatrix}$ $P = \begin{bmatrix} 2 \\ \frac{3}{2} \\ -\frac{3\sqrt{3}}{2} \end{bmatrix}$ $\therefore g_{ca} = \begin{bmatrix} 0 & 0 & -1 & 2 \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} & 0 & \frac{3}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 & -\frac{3\sqrt{3}}{2} \\ 0 & 0 & 0 & 1 \end{bmatrix}$

8、Solution. $g = \begin{bmatrix} e^{\hat{\omega}\theta} & (I - e^{\hat{\omega}\theta})(\omega \times v) - \omega\omega^T\theta v \\ 0 & 1 \end{bmatrix}$

First we find $\hat{\omega}$ and θ.

According to Rodrigue's Formula, $e^{\hat{\omega}\theta} = I + \hat{\omega}\sin\theta + (1-\cos\theta)\hat{\omega}^2$

Let $\omega = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$, $\sin\theta = s\theta$, $1-\cos\theta = v\theta$

$$= \begin{bmatrix} 1-(a_3^2+a_2^2)v\theta & a_1a_2v\theta - a_3 s\theta & a_2 s\theta + a_1a_3 v\theta \\ a_1a_2v\theta + a_3 s\theta & 1-(a_3^2+a_1^2)v\theta & a_2a_3v\theta - a_1 s\theta \\ a_1a_3v\theta - a_2 s\theta & a_2a_3v\theta + a_1 s\theta & 1-(a_1^2+a_2^2)v\theta \end{bmatrix}$$

$tr(e^{\hat{\omega}\theta}) = 0$

$\Rightarrow \theta = \cos^{-1}\left(\frac{0-1}{2}\right) = \frac{2\pi}{3}$, $\theta \in [0, 2\pi)$ $\Rightarrow \begin{cases} 2a_1 s\theta = 0-(-1) = 1 \\ 2a_2 s\theta = 0-(-1) = 1 \\ 2a_3 s\theta = 0-1 = -1 \end{cases}$ Hence $\begin{cases} a_1 = \frac{\sqrt{3}}{3} \\ a_2 = \frac{\sqrt{3}}{3} \\ a_3 = -\frac{\sqrt{3}}{3} \end{cases}$

Next we find ξ. To do this, we first find v.

$$(I - e^{\hat{\omega}\theta})(\omega \times v) + \omega\omega^T\theta v = \left(\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & \frac{\sqrt{3}}{3} & \frac{\sqrt{3}}{3} \\ -\frac{\sqrt{3}}{3} & 0 & -\frac{\sqrt{3}}{3} \\ -\frac{\sqrt{3}}{3} & \frac{\sqrt{3}}{3} & 0 \end{bmatrix} + \frac{2\pi}{3} \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & -\frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \end{bmatrix} \right) v = \begin{bmatrix} 10 \\ 20 \\ 1 \end{bmatrix}$$

解得 $v = \begin{bmatrix} -1.3505 \\ 10.7739 \\ -4.4230 \end{bmatrix}$

则 $\xi = \begin{bmatrix} -1.3505 & 10.7739 & -4.4230 & \frac{\sqrt{3}}{3} & \frac{\sqrt{3}}{3} & -\frac{\sqrt{3}}{3} \end{bmatrix}^T$.

See next page for the answer to Exercise 9.