机器人学导论 作业6　　　210320621　吴俊达

1. 解：(1) $\mathcal{I} = \begin{bmatrix} I_{xxi} & I_{xyi} & I_{xzi} \\ I_{xyi} & I_{yyi} & I_{yzi} \\ I_{xzi} & I_{yzi} & I_{zzi} \end{bmatrix}$

其中 $I_{xy_i} = -\int \rho(r)\, xy\, dxdy = -\rho \int xy\, dxdy$，因为积分区间对于 x, y 都是关于原点对称的，xy 分别关于 x, y 是奇函数

因此 $I_{xy_i}=0$。因为 $z_i=0$，则 $I_{xz_i}=I_{yz_i}=0$。

由于 ω_i 仅有 z 分量

$$T_i(\theta,\dot\theta) = \frac{1}{2} V_i^T M_{bi} V_i = \frac{1}{2}\begin{bmatrix} v_i^T & \omega_i^T \end{bmatrix} \begin{bmatrix} mI & 0 \\ 0 & \mathcal{I}_i \end{bmatrix} \begin{bmatrix} v_i \\ \omega_i \end{bmatrix} = \frac{1}{2} m\|v_i\|^2 + \frac{1}{2}\omega_i^T \mathcal{I}_i \omega_i = \frac{1}{2} m\|v_i\|^2 + \frac{1}{2}\omega_i^T I_{zzi}\, \omega_i$$

而 $\omega_1 = \begin{bmatrix} 0 \\ 0 \\ \dot\theta_1 \end{bmatrix}$，$\omega_2 = \begin{bmatrix} 0 \\ 0 \\ \dot\theta_1+\dot\theta_2 \end{bmatrix}$

ω_1：杆1的 body velocity (angular)；ω_2：杆2的 body velocity (angular)

$x_1 = r_1\cos\theta_1 \Rightarrow \dot x_1 = \dot\theta_1 r_1 \sin\theta_1$

$y_1 = r_1\sin\theta_1 \Rightarrow \dot y_1 = \dot\theta_1 r_1 \cos\theta_1$

$\Rightarrow \|v_1\|^2 = \dot\theta_1^2 r_1^2$

$x_2 = l_1\cos\theta_1 + r_2\cos(\theta_1+\theta_2) \Rightarrow \dot x_2 = -\dot\theta_1\times l_1\sin\theta_1 - (\dot\theta_1+\dot\theta_2) r_2 \sin(\theta_1+\theta_2)$

$y_2 = l_1\sin\theta_1 + r_2\sin(\theta_1+\theta_2) \Rightarrow \dot y_2 = \dot\theta_1\times l_1\cos\theta_1 + (\dot\theta_1+\dot\theta_2) r_2\cos(\theta_1+\theta_2)$

$$\Rightarrow \|v_2\|^2 = l_1^2\dot\theta_1^2 + r_2^2(\dot\theta_1+\dot\theta_2)^2 + 2\dot\theta_1(\dot\theta_1+\dot\theta_2)\sin\theta_1\sin(\theta_1+\theta_2) l_1 r_2 + 2\dot\theta_1(\dot\theta_1+\dot\theta_2)\cos\theta_1\cos(\theta_1+\theta_2) l_1 r_2$$

$$= l_1^2\dot\theta_1^2 + r_2^2\dot\theta_1^2 + r_2^2\dot\theta_2^2 + 2r_2\dot\theta_1\dot\theta_2 + 2l_1 r_2\cos\theta_2\,\dot\theta_1^2 + 2l_1r_2\cos\theta_2\,\dot\theta_2\dot\theta_1$$

$$T(\theta,\dot\theta) = \frac{1}{2}\begin{bmatrix}\dot\theta_1^T & \dot\theta_2^T\end{bmatrix} \begin{bmatrix} m_1r_1^2 + m_2[l_1^2+r_2^2] + I_{zz1} + I_{zz2} + m_2 l_1 r_2\cos\theta_2 & m_2 l_1 r_2\cos\theta_2 + I_{zz2} + m_2 r_2^2 \\ m_2 l_1 r_2\cos\theta_2 + I_{zz2} + m_2 r_2^2 & m_2 r_2^2 + I_{zz2} \end{bmatrix} \begin{bmatrix}\dot\theta_1 \\ \dot\theta_2\end{bmatrix}$$

$$\triangleq \frac{1}{2}\begin{bmatrix}\dot\theta_1^T & \dot\theta_2^T\end{bmatrix} \underbrace{\begin{bmatrix} \alpha+2\beta c_2 & \beta c_2+\delta \\ \beta c_2+\delta & \delta \end{bmatrix}}_{M(\theta)} \begin{bmatrix}\dot\theta_1 \\ \dot\theta_2\end{bmatrix}$$

$\alpha = m_1r_1^2 + m_2[l_1^2+r_2^2] + I_{zz1} + I_{zz2} = 20.02$

$\beta = m_2 l_1 r_2 = 6$，$\delta = m_2 r_2^2 + I_{zz2} = 4.01$

$L = T$

运动方程：$\frac{d}{dt}\frac{\partial L}{\partial\dot\theta_i} - \frac{\partial L}{\partial\theta_i} = \tau_i \Rightarrow M(\theta)\begin{bmatrix}\ddot\theta_1 \\ \ddot\theta_2\end{bmatrix} + \begin{bmatrix} -\beta s_2\dot\theta_2 & -\beta s_2(\dot\theta_1+\dot\theta_2) \\ \beta s_2\dot\theta_1 & 0 \end{bmatrix}\begin{bmatrix}\dot\theta_1 \\ \dot\theta_2\end{bmatrix} = \begin{bmatrix}\tau_1 \\ \tau_2\end{bmatrix}$

(2) $\Gamma_{111} = \frac{1}{2}\left(\frac{\partial M_{11}}{\partial\theta_1} + \frac{\partial M_{11}}{\partial\theta_1} - \frac{\partial M_{11}}{\partial\theta_1}\right) = \frac{1}{2}\frac{\partial M_{11}}{\partial\theta_1} = 0$

$\Gamma_{211} = \frac{1}{2}\left(\frac{\partial M_{21}}{\partial\theta_1} + \frac{\partial M_{21}}{\partial\theta_1} - \frac{\partial M_{11}}{\partial\theta_2}\right) = -\frac{1}{2}\frac{\partial M_{11}}{\partial\theta_2} = \beta s_2$

$\Gamma_{112} = \frac{1}{2}\left(\frac{\partial M_{11}}{\partial\theta_2} + \frac{\partial M_{12}}{\partial\theta_1} - \frac{\partial M_{21}}{\partial\theta_1}\right) = \frac{1}{2}\frac{\partial M_{11}}{\partial\theta_2} = \frac{1}{2}(-2\beta s_2) = -\beta s_2$

$\Gamma_{212} = \frac{1}{2}\left(\frac{\partial M_{21}}{\partial\theta_2} + \frac{\partial M_{22}}{\partial\theta_1} - \frac{\partial M_{21}}{\partial\theta_2}\right) = 0$

$\Gamma_{121} = \frac{1}{2}\left(\frac{\partial M_{12}}{\partial\theta_1} + \frac{\partial M_{11}}{\partial\theta_2} - \frac{\partial M_{12}}{\partial\theta_1}\right) = \frac{1}{2}\frac{\partial M_{11}}{\partial\theta_2} = -\beta s_2$

$\Gamma_{221} = \frac{1}{2}\left(\frac{\partial M_{22}}{\partial\theta_1} + \frac{\partial M_{21}}{\partial\theta_2} - \frac{\partial M_{12}}{\partial\theta_2}\right) = 0$

$\Gamma_{122} = \frac{1}{2}\left(\frac{\partial M_{12}}{\partial\theta_2} + \frac{\partial M_{12}}{\partial\theta_2} - \frac{\partial M_{22}}{\partial\theta_1}\right) = \frac{\partial M_{12}}{\partial\theta_2} = -\beta s_2$

$\Gamma_{222} = \frac{1}{2}\left(\frac{\partial M_{22}}{\partial\theta_2} + \frac{\partial M_{22}}{\partial\theta_2} - \frac{\partial M_{22}}{\partial\theta_2}\right) = 0$

Γ_{ijk}'s 即为待求的 Christoffel 记号。

则 $C_{ij}(\theta,\dot\theta) = \sum_{k=1}^{n}\Gamma_{ijk}\dot\theta_k \Rightarrow C = \begin{bmatrix} -\beta s_2\dot\theta_2 & -\beta s_2(\dot\theta_1+\dot\theta_2) \\ \beta s_2\dot\theta_1 & 0 \end{bmatrix}$，又 $V=0 \Rightarrow N=0$

动力学方程 $M(\theta)\begin{bmatrix}\ddot\theta_1 \\ \ddot\theta_2\end{bmatrix} + C(\theta,\dot\theta)\begin{bmatrix}\dot\theta_1 \\ \dot\theta_2\end{bmatrix} = \begin{bmatrix}\tau_1 \\ \tau_2\end{bmatrix}$ 与上一问推出的方程相同。

亦可验证 $\dot M - 2C$ 为反对称矩阵。