Definition: Rigid body transformation

$$g:\mathbb{R}^3 \mapsto \mathbb{R}^3$$

s.t.

1. Length preserving: $\|g(p)-g(q)\| = \|p-q\|$
2. Orientation preserving: $g_*(v\times\omega) = g_*(v)\times g_*(\omega)$

Definition:

$$SO(3) = \{R\in\mathbb{R}^{3\times3}\,|R^TR = I, \det R = 1\}$$

and

$$SO(n) = \{R\in\mathbb{R}^{n\times n}\,|R^TR = I, \det R = 1\}$$

$(G,\cdot)$ is a group if:

1. $g_1, g_2\in G \Rightarrow g_1\cdot g_2\in G$
2. $\exists!\ e\in G$, s.t. $g\cdot e = e\cdot g = g, \forall g\in G$
3. $\forall g\in G, \exists!\ g^{-1}\in G$, s.t. $g\cdot g^{-1} = g^{-1}\cdot g = e$
4. $g_1\cdot(g_2\cdot g_3) = (g_1\cdot g_2)\cdot g_3$

Proof :

For $a\in\mathbb{R}^3$, let $\hat{a} = \begin{bmatrix} 0 & -a_3 & a_2\\ a_3 & 0 & -a_1\\ -a_2 & a_1 & 0\end{bmatrix}$

Note that $\hat{a}\cdot b = a\times b$

follows from $R\hat{v}R^T = (Rv)^\wedge$

$$\begin{cases}\dot{x}(t) = ax(t)\\ x(0) = x_0\end{cases} \Rightarrow x(t) = e^{at}x_0$$

$\dot{q}(t) = \omega\times q(t) = \hat{\omega}q(t)$

$q(0)$: Initial coordinates

Figure 2.5

$$\Rightarrow q(t) = e^{\hat{\omega}t}q_0 \text{ where } e^{\hat{\omega}t} = I + \hat{\omega}t + \frac{(\hat{\omega}t)^2}{2!} + \frac{(\hat{\omega}t)^3}{3!} + \cdots$$

Rodrigues' formula ($\|\omega\| = 1$):

$$e^{\hat{\omega}\theta} = I + \hat{\omega}\sin\theta + \hat{\omega}^2(1-\cos\theta)$$

Proof :

Let $a\in\mathbb{R}^3$, write

$$a = \omega\theta, \omega = \frac{a}{\|a\|}(\text{or } \|\omega\| = 1), \text{and } \theta = \|a\|$$

$$e^{\hat{\omega}\theta} = I + \hat{\omega}\theta + \frac{(\hat{\omega}\theta)^2}{2!} + \frac{(\hat{\omega}\theta)^3}{3!} + \cdots$$

As

$$\hat{a}^2 = aa^T - \|a\|^2 I, \hat{a}^3 = -\|a\|^2\hat{a}$$

$\rightarrow$ 代入 $\hat{\omega}$, $\hat{\omega}^2 = \omega\cdot\omega^T$

we have:

$$e^{\hat{\omega}\theta} = I + (\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots)\hat{\omega} + (\frac{\theta^2}{2!} - \frac{\theta^4}{4!} + \cdots)\hat{\omega}^2$$
$$= I + \hat{\omega}\sin\theta + \hat{\omega}^2(1-\cos\theta)$$

Rodrigues' formula for $\|\omega\| \neq 1$:

$$e^{\hat{\omega}\theta} = I + \frac{\hat{\omega}}{\|\omega\|}\sin\|\omega\|\theta + \frac{\hat{\omega}^2}{\|\omega\|^2}(1-\cos\|\omega\|\theta)$$

Given $R\in SO(3)$, to show $\exists\omega\in\mathbb{R}^3, \|\omega\| = 1$ and θ s.t. $R = e^{\hat{\omega}\theta}$

Let

$$R = \begin{bmatrix} r_{11} & r_{12} & r_{13}\\ r_{21} & r_{22} & r_{23}\\ r_{31} & r_{32} & r_{33}\end{bmatrix}$$

and

$$v_\theta = 1-\cos\theta, c_\theta = \cos\theta, s_\theta = \sin\theta$$

By Rodrigues' formula

$$e^{\hat{\omega}\theta} = \begin{bmatrix} \omega_1^2 v_\theta + c_\theta & \omega_1\omega_2 v_\theta - \omega_3 s_\theta & \omega_1\omega_3 v_\theta + \omega_2 s_\theta\\ \omega_1\omega_2 v_\theta + \omega_3 s_\theta & \omega_2^2 v_\theta + c_\theta & \omega_2\omega_3 v_\theta - \omega_1 s_\theta\\ \omega_1\omega_3 v_\theta - \omega_2 s_\theta & \omega_2\omega_3 v_\theta + \omega_1 s_\theta & \omega_3^2 v_\theta + c_\theta\end{bmatrix}$$

$$\mathrm{tr}(R) = r_{11} + r_{22} + r_{33} = 1 + 2\cos\theta = \sum_{i=1}^3\lambda_i$$

where λ_i is the eigenvalue of $R, i = 1,2,3$

Case 1: $\mathrm{tr}(R) = 3$ or $R = I$, $\theta = 0 \Rightarrow \omega\theta = 0$

Case 2: $-1 < \mathrm{tr}(R) < 3$,

$$\theta = \arccos\frac{\mathrm{tr}(R)-1}{2} \Rightarrow \omega = \frac{1}{2s_\theta}\begin{bmatrix} r_{32}-r_{23}\\ r_{13}-r_{31}\\ r_{21}-r_{12}\end{bmatrix}$$

Case 3: $\mathrm{tr}(R) = -1 \Rightarrow \cos\theta = -1 \Rightarrow \theta = \pm\pi$

$$R_x(\varphi) := e^{\hat{x}\varphi} = \begin{bmatrix} 1 & 0 & 0\\ 0 & \cos\varphi & -\sin\varphi\\ 0 & \sin\varphi & \cos\varphi\end{bmatrix}$$

$$R_y(\theta) := e^{\hat{y}\theta} = \begin{bmatrix} \cos\theta & 0 & \sin\theta\\ 0 & 1 & 0\\ -\sin\theta & 0 & \cos\theta\end{bmatrix}$$

$$R_z(\psi) := e^{\hat{z}\psi} = \begin{bmatrix} \cos\psi & -\sin\psi & 0\\ \sin\psi & \cos\psi & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$R_{ab} = R_z(\psi)R_y(\theta)R_x(\varphi)$$
$$= \begin{bmatrix} C_\theta C_\psi & -S_\varphi S_\theta C_\psi - C_\varphi S_\psi & C_\varphi S_\theta C_\psi - S_\varphi S_\psi\\ C_\theta S_\psi & -S_\varphi S_\theta S_\psi + C_\varphi C_\psi & -C_\varphi S_\theta S_\psi - S_\varphi C_\psi\\ S_\theta & S_\varphi C_\theta & C_\varphi C_\theta\end{bmatrix}$$

(φ) $\beta = \mathrm{atan2}(-r_{31}, \sqrt{r_{32}^2 + r_{33}^2})$

(θ) $\alpha = \mathrm{atan2}(r_{21}/c_\beta, r_{11}/c_\beta)$

(ψ) $\gamma = \mathrm{atan2}(r_{32}/c_\beta, r_{33}/c_\beta)$

$R_z\cdot R_y\cdot R_x$ (先转后乘)

- ZYX Euler angle (continued)

$$R_{ab}(\alpha,\beta,\gamma) = \begin{bmatrix} c_\alpha c_\beta & -s_\alpha c_\gamma + c_\alpha s_\beta s_\gamma & s_\alpha s_\gamma + c_\alpha s_\beta c_\gamma\\ s_\alpha c_\beta & c_\alpha c_\gamma + s_\alpha s_\beta s_\gamma & -c_\alpha s_\gamma + s_\alpha s_\beta c_\gamma\\ -s_\beta & c_\beta s_\gamma & c_\beta c_\gamma\end{bmatrix}$$

Note: When $\beta = \frac{\pi}{2}$, $\cos\beta = 0$, $\alpha + \gamma = \text{const} \Rightarrow$ singularity!

$\beta = \mathrm{atan2}(-r_{31}, \sqrt{r_{32}^2 + r_{33}^2})$

$\alpha = \mathrm{atan2}(r_{21}/c_\beta, r_{11}/c_\beta)$

$\gamma = \mathrm{atan2}(r_{32}/c_\beta, r_{33}/c_\beta)$

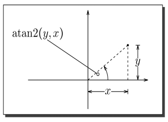


Figure 2.10

$Q = q_0 + q_1 i + q_2 j + q_3 k$
where $i^2 = j^2 = k^2 = -1, i\cdot j = k, j\cdot k = i, k\cdot i = j$

Property 1: Define $Q^* = (q_0, q)^* = (q_0, -q), q_0\in\mathbb{R}, q\in\mathbb{R}^3$
$\|Q\|^2 = QQ^* = q_0^2 + q_1^2 + q_2^2 + q_3^2$

Property 2: $Q = (q_0, q), P = (p_0, p)$
$QP = (q_0p_0 - q\cdot p, q_0p + p_0q + q\times p)$

Property 3: (a) The set of unit quaternions forms a group
(b) If $R = e^{\hat{\omega}\theta}$, then $Q = (\cos\frac{\theta}{2}, \omega\sin\frac{\theta}{2})$
(c) Q acts on $x\in\mathbb{R}^3$ by QXQ^*, where $X = (0, x)$

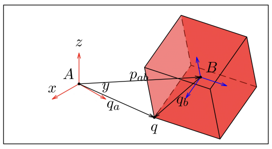
$$R(Q) = I + 2q_0\hat{q} + 2\hat{q}^2$$
$$= \begin{bmatrix} 1-2(q_2^2+q_3^2) & -2q_0q_3 + 2q_1q_2 & 2q_0q_2 + 2q_1q_3\\ 2q_0q_3 + 2q_1q_2 & 1-2(q_1^2+q_3^2) & -2q_0q_1 + 2q_2q_3\\ -2q_0q_2 + 2q_1q_3 & 2q_0q_1 + 2q_2q_3 & 1-2(q_1^2+q_2^2)\end{bmatrix}$$

RPY角：注意下面φ和ψ要反

$$Q = (\cos\frac{\varphi}{2}, x\sin\frac{\varphi}{2})(\cos\frac{\theta}{2}, y\sin\frac{\theta}{2})(\cos\frac{\psi}{2}, z\sin\frac{\psi}{2}) \Rightarrow$$

$$q_0 = \cos\frac{\varphi}{2}\cos\frac{\theta}{2}\cos\frac{\psi}{2} - \sin\frac{\varphi}{2}\sin\frac{\theta}{2}\sin\frac{\psi}{2}$$

$$q = \begin{bmatrix} \cos\frac{\varphi}{2}\sin\frac{\theta}{2}\sin\frac{\psi}{2} + \sin\frac{\varphi}{2}\cos\frac{\theta}{2}\cos\frac{\psi}{2}\\ \cos\frac{\varphi}{2}\sin\frac{\theta}{2}\cos\frac{\psi}{2} - \sin\frac{\varphi}{2}\cos\frac{\theta}{2}\sin\frac{\psi}{2}\\ \cos\frac{\varphi}{2}\cos\frac{\theta}{2}\sin\frac{\psi}{2} + \sin\frac{\varphi}{2}\sin\frac{\theta}{2}\cos\frac{\psi}{2}\end{bmatrix}$$



$$g_{ab} = (p_{ab}, R_{ab}) : \mathbb{R}^3 \mapsto \mathbb{R}^3$$
$$q_b \mapsto q_a = p_{ab} + R_{ab}\cdot q_b$$

$$\bar{q} = \begin{bmatrix} q_1\\ q_2\\ q_3\\ 1\end{bmatrix}\in\mathbb{R}^4$$

$$\bar{v} = \bar{p} - \bar{q} = \begin{bmatrix} p_1\\ p_2\\ p_3\\ 1\end{bmatrix} - \begin{bmatrix} q_1\\ q_2\\ q_3\\ 1\end{bmatrix} = \begin{bmatrix} v_1\\ v_2\\ v_3\\ 0\end{bmatrix}$$

$$q_a = p_{ab} + R_{ab}\cdot q_b$$

$$\begin{bmatrix} q_a\\ 1\end{bmatrix} = \underbrace{\begin{bmatrix} R_{ab} & p_{ab}\\ 0 & 1\end{bmatrix}}_{\bar{g}_{ab}}\begin{bmatrix} q_b\\ 1\end{bmatrix}$$

$$\bar{q}_a = \bar{g}_{ab}\cdot\bar{q}_b$$

$$\bar{g}_{ac} = \bar{g}_{ab}\cdot\bar{g}_{bc} = \begin{bmatrix} R_{ab}R_{bc} & R_{ab}p_{bc} + p_{ab}\\ 0 & 1\end{bmatrix}$$

$$SE(3) = \left\{\begin{bmatrix} R & p\\ 0 & 1\end{bmatrix}\in\mathbb{R}^{4\times4} \middle| p\in\mathbb{R}^3, R\in SO(3)\right\}$$

Property 4: $SE(3)$ forms a group.

Proof :

1. $g_1\cdot g_2\in SE(3)$
2. $e = I_4$
3. $(\bar{g})^{-1} = \begin{bmatrix} R^T & -R^Tp\\ 0 & 1\end{bmatrix}$
4. Associativity: Follows from property of matrix multiplication

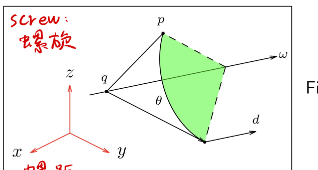
Definition:

$$se(3) = \left\{\begin{bmatrix} \hat{\omega} & v\\ 0 & 0\end{bmatrix}\in\mathbb{R}^{4\times4} \middle| v, \omega\in\mathbb{R}^3\right\}$$

is called the twist space. There exists a 1-1 correspondence between $se(3)$ and $\mathbb{R}^6$, defined by $\wedge : \mathbb{R}^6 \mapsto se(3)$

$$\xi := \begin{bmatrix} v\\ \omega\end{bmatrix} \mapsto \hat{\xi} = \begin{bmatrix} \hat{\omega} & v\\ 0 & 0\end{bmatrix}$$

Property 6: $\exp : se(3) \mapsto SE(3), \hat{\xi}\theta \mapsto e^{\hat{\xi}\theta}$



Pitch: $h = \frac{d}{\theta}(\theta = 0, h = \infty), d = h\cdot\theta$

Axis: $l = \{q + \lambda\omega | \lambda\in\mathbb{R}\}$

幅值 Magnitude: $M = \theta$

$$g = \begin{bmatrix} e^{\hat{\omega}\theta} & (I - e^{\hat{\omega}\theta})q + h\theta\omega\\ 0 & 1\end{bmatrix}$$

螺旋运动变换了

$\xi = (v, \omega)\in\mathbb{R}^6$

1. Pitch: $h = \begin{cases}\frac{\omega^Tv}{\|\omega\|^2}, & \text{if } \omega\neq0\\ \infty, & \text{if } \omega = 0\end{cases}$
2. Axis: $l = \begin{cases}\frac{\omega\times v}{\|\omega\|^2} + \lambda\omega, & \lambda\in\mathbb{R}, \text{if } \omega\neq0\\ 0 + \lambda v & \lambda\in\mathbb{R}, \text{if } \omega = 0\end{cases}$
3. Magnitude: $M = \begin{cases}\|\omega\|, & \text{if } \omega\neq0\\ \|v\|, & \text{if } \omega = 0\end{cases}$

Special cases:

1. $h = \infty$, Pure translation (prismatic joint)
2. $h = 0$, Pure rotation (revolute joint)

□ Velocity of Rotational Motion:

$R_{ab}(t)\in SO(3), t\in(-\varepsilon,\varepsilon),\ q_a(t) = R_{ab}(t)q_b$

$V^a = \frac{\mathrm{d}}{\mathrm{d}t}q_a(t) = \dot{R}_{ab}(t)q_b = \dot{R}_{ab}(t)R_{ab}^T(t)R_{ab}(t)q_b = \dot{R}_{ab}R_{ab}^Tq_a$

$R_{ab}(t)R_{ab}^T(t) = I \Rightarrow \dot{R}_{ab}R_{ab}^T + R_{ab}\dot{R}_{ab}^T = 0, \dot{R}_{ab}R_{ab}^T = -(\dot{R}_{ab}R_{ab}^T)^T$

$$\hat{\omega}_{ab}^s = \dot{R}_{ab}R_{ab}^T, \omega_{ab}\in\mathbb{R}^3 \quad V^a = \hat{\omega}_{ab}^s\cdot q_a = \omega_{ab}^s\times q_a$$

$$\hat{\omega}_{ab}^b = R_{ab}^T\cdot\dot{R}_{ab}, v^b \triangleq R_{ab}^T\cdot v^a = \omega_{ab}^b\times q_b$$

$$\omega_{ab}^b = R_{ab}^T\cdot\omega_{ab}^s \text{ or } \hat{\omega}_{ab}^b = R_{ab}^T\hat{\omega}_{ab}^sR_{ab}$$

$$g_{ab} = \begin{bmatrix} R_{ab}(t) & p_{ab}(t)\\ 0 & 1\end{bmatrix}, q_a(t) = g_{ab}(t)q_b$$

$$\frac{\mathrm{d}}{\mathrm{d}t}q_a(t) = \dot{g}_{ab}(t)q_b = \dot{g}_{ab}\cdot g_{ab}^{-1}\cdot g_{ab}\cdot q_b = \hat{V}_{ab}^s\cdot q_a$$

$$\hat{V}_{ab}^s = \dot{g}_{ab}\cdot g_{ab}^{-1} = \begin{bmatrix} \dot{R}_{ab} & \dot{p}_{ab}\\ 0 & 0\end{bmatrix}\begin{bmatrix} R_{ab}^T & -R_{ab}^Tp_{ab}\\ 0 & 1\end{bmatrix}$$
$$= \begin{bmatrix} \dot{R}_{ab}R_{ab}^T & -\dot{R}_{ab}R_{ab}^Tp_{ab} + \dot{p}_{ab}\\ 0 & 0\end{bmatrix}$$
$$= \begin{bmatrix} \hat{\omega}_{ab}^s & -\omega_{ab}^s\times p_{ab} + \dot{p}_{ab}\\ 0 & 0\end{bmatrix} \triangleq \begin{bmatrix} \hat{\omega}_{ab}^s & v_{ab}^s\\ 0 & 0\end{bmatrix}$$

$$V_{ab}^s = \begin{bmatrix} v_{ab}^s\\ \omega_{ab}^s\end{bmatrix} = \underbrace{\begin{bmatrix} R_{ab} & \hat{p}_{ab}R_{ab}\\ 0 & R_{ab}\end{bmatrix}}_{\mathrm{Ad}_g}V_{ab}^b$$

$$\mathrm{Ad}_g = \begin{bmatrix} R & \hat{p}R\\ 0 & R\end{bmatrix}\in\mathbb{R}^{6\times6}, \text{ for } g = (p, R)$$

$$e^{\hat{\xi}\theta} = \begin{bmatrix} e^{\hat{\omega}\theta} & (I - e^{\hat{\omega}\theta})\hat{\omega}v + \omega\omega^Tv\theta\\ 0 & 1\end{bmatrix}$$

$$\xi = \begin{bmatrix} v\\ \omega\end{bmatrix} = \begin{bmatrix} -\omega\times q\\ \omega\end{bmatrix}$$

q为旋转轴任意一点

$$g_{st}(\theta_1,\theta_2) = e^{\hat{\xi}_2'\theta_2}\cdot e^{\hat{\xi}_1\theta_1}\cdot g_{st}(0)$$
$$= e^{e^{\hat{\xi}_1\theta_1}\hat{\xi}_2\theta_2 e^{-\hat{\xi}_1\theta_1}}\cdot e^{\hat{\xi}_1\theta_1}\cdot g_{st}(0)$$
$$= e^{\hat{\xi}_1\theta_1}\cdot e^{\hat{\xi}_2\theta_2}\cdot e^{-\hat{\xi}_1\theta_1}\cdot e^{\hat{\xi}_1\theta_1}\cdot g_{st}(0)$$
$$= e^{\hat{\xi}_1\theta_1}\cdot e^{\hat{\xi}_2\theta_2}\cdot g_{st}(0)$$

Independent of the route taken

Subproblem 1: Rotation about a single axis

Let ξ be a zero-pitch twist, with unit magnitude and two points $p, q\in\mathbb{R}^3$. Find θ s.t. $e^{\hat{\xi}\theta}p = q$

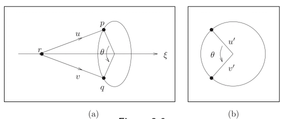


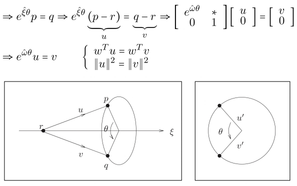
Figure 3.6

Solution: Let $r\in l_\xi$, define $u = p - r, v = q - r, e^{\hat{\xi}\theta}r = r$

Moreover,

$$\Rightarrow e^{\hat{\xi}\theta}p = q \Rightarrow e^{\hat{\xi}\theta}\underbrace{(p-r)}_{u} = \underbrace{q-r}_{v} \Rightarrow \begin{bmatrix} e^{\hat{\omega}\theta} & *\\ 0 & 1\end{bmatrix}\begin{bmatrix} u\\ 0\end{bmatrix} = \begin{bmatrix} v\\ 0\end{bmatrix}$$

$$\Rightarrow e^{\hat{\omega}\theta}u = v \qquad \begin{cases}\omega^Tu = \omega^Tv\\ \|u\|^2 = \|v\|^2\end{cases}$$



$u' = (I - \omega\omega^T)u, v' = (I - \omega\omega^T)v$

The solution exists only if $\begin{cases}\|u'\|^2 = \|v'\|^2\\ \omega^Tu = \omega^Tv\end{cases}$

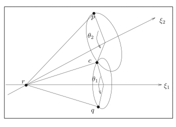
- If $u'\neq0$, then

$$u'\times v' = \omega\sin\theta\|u'\|\|v'\|$$
$$u'\cdot v' = \cos\theta\|u'\|\|v'\|$$
$$\Rightarrow \theta = \mathrm{atan2}(\omega^T(u'\times v'), u'^Tv')$$

- If $u' = 0$, $\Rightarrow$ Infinite number of solutions!

Subproblem 2: Rotation about two subsequent axes

Let ξ_1 and ξ_2 be two zero-pitch, unit magnitude twists, with intersecting axes, and $p, q\in\mathbb{R}^3$. find θ_1 and θ_2 s.t. $e^{\hat{\xi}_1\theta_1}e^{\hat{\xi}_2\theta_2}p = q$.



Solution: If two axes of ξ_1 and ξ_2 coincide, then we get:
Subproblem 1: $\theta_1 + \theta_2 = \theta$
If the two axes are not parallel, $\omega_1\times\omega_2\neq0$, then, let c satisfy:

$$e^{\hat{\xi}_2\theta_2}p = c = e^{-\hat{\xi}_1\theta_1}q$$

Set $r\in l_{\xi_1}\cap l_{\xi_2}$

$$e^{\hat{\xi}_2\theta_2}\underbrace{p-r}_{u} = \underbrace{c-r}_{z} = e^{-\hat{\xi}_1\theta_1}\underbrace{(q-r)}_{v}, \Rightarrow e^{\hat{\omega}_2\theta_2}u = z = e^{-\hat{\omega}_1\theta_1}v$$

$$\Rightarrow \begin{cases}\omega_2^Tu = \omega_2^Tz\\ \omega_1^Tv = \omega_1^Tz\end{cases}, \|u\|^2 = \|z\|^2 = \|v\|^2$$

As ω_1, ω_2 and $\omega_1\times\omega_2$ are linearly independent,

$$z = \alpha\omega_1 + \beta\omega_2 + \gamma(\omega_1\times\omega_2)$$
$$\Rightarrow \|z\|^2 = \alpha^2 + \beta^2 + 2\alpha\beta\omega_1^T\omega_2 + \gamma^2\|\omega_1\times\omega_2\|^2$$

$$\begin{matrix}\omega_2^Tu = \alpha\omega_2^T\omega_1 + \beta\\ \omega_1^Tv = \alpha + \beta\omega_1^T\omega_2\end{matrix} \Rightarrow \begin{cases}\alpha = \frac{(\omega_1^T\omega_2)\omega_2^Tu - \omega_1^Tv}{(\omega_1^T\omega_2)^2 - 1}\\ \beta = \frac{(\omega_1^T\omega_2)\omega_1^Tv - \omega_2^Tu}{(\omega_1^T\omega_2)^2 - 1}\end{cases}$$

$$\|z\|^2 = \|u\|^2 \Rightarrow \gamma^2 = \frac{\|u\|^2 - \alpha^2 - \beta^2 - 2\alpha\beta\omega_1^T\omega_2}{\|\omega_1\times\omega_2\|^2} \quad (*)$$

$(*)$ has zero, one or two solution(s):

$$\text{Given } z \Rightarrow c \Rightarrow \begin{cases}e^{\hat{\xi}_2\theta_2}p = c\\ e^{\hat{\xi}_1\theta_1}p = c\end{cases}$$

for θ_1 and θ_2

1. Two solutions when the two circles intersect.
2. One solution when they are tangent
3. Zero solution when they do not intersect

Subproblem 3: Rotation to a given point

Given a zero-pitch twist ξ, with unit magnitude and $p, q\in\mathbb{R}^3$, find θ s.t. $\|q - e^{\hat{\xi}\theta}p\| = \delta$

Define: $u = p - r, v = q - r, \|v - e^{\hat{\omega}\theta}u\|^2 = \delta^2$

$u' = u - \omega\omega^Tu$

$v' = v - \omega\omega^Tv$

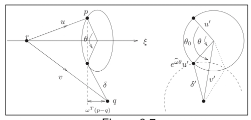


Figure 3.7

$$\Rightarrow u = u' + \omega\omega^Tu, v = v' + \omega\omega^Tv, \delta'^2 = \delta^2 - |\omega^T(p-q)|^2$$

$$\|(v' + \omega\omega^Tv) - e^{\hat{\omega}\theta}(u' + \omega\omega^Tu)\|^2 = \delta^2 \Rightarrow$$
$$\|v' - e^{\hat{\omega}\theta}u' + \underbrace{\omega\omega^T(v-u)}_{\omega\omega^T(q-p)}\|^2 = \delta^2$$

$$\|v' - e^{\hat{\omega}\theta}u'\|^2 = \delta^2 - \|\omega^T(p-q)\|^2 = \delta'^2,$$
$$\theta_0 = \mathrm{atan2}(\omega^T(u'\times v'), u'^Tv'),$$
$$\phi = \theta_0 - \theta \Rightarrow \|u'\|^2 + \|v'\|^2 - 2\|u'\|\cdot\|v'\|\cos\phi = \delta'^2,$$
$$\theta = \theta_0 \pm \cos^{-1}\frac{\|u'\| + \|v'\|^2 - \delta'^2}{2\|u'\|\cdot\|v'\|} \quad (*)$$

Technique 1: Eliminate the dependence on a joint

$e^{\hat{\xi}\theta}p = p$, if $p\in l_\xi$. Given $e^{\hat{\xi}_1\theta_1}e^{\hat{\xi}_2\theta_2}e^{\hat{\xi}_3\theta_3} = g$, select $p\in l_{\xi_3}$, $p\notin l_{\xi_1}$ or l_{ξ_2}, then:

$$gp = e^{\hat{\xi}_1\theta_1}e^{\hat{\xi}_2\theta_2}p$$

Technique 2: subtract a common point

$$e^{\hat{\xi}_1\theta_1}e^{\hat{\xi}_2\theta_2}e^{\hat{\xi}_3\theta_3} = g, q\in l_{\xi_1}\cap l_{\xi_2} \Rightarrow e^{\hat{\xi}_1\theta_1}e^{\hat{\xi}_2\theta_2}e^{\hat{\xi}_3\theta_3}p - q = gp - q \Rightarrow$$
$$\|e^{\hat{\xi}_3\theta_3}p - q\| = \|gp - q\|$$

Example: Elbow manipulator

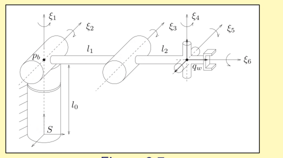


Figure 3.7

$$g_{st}(\theta) = e^{\hat{\xi}_1\theta_1}e^{\hat{\xi}_2\theta_2}e^{\hat{\xi}_3\theta_3}e^{\hat{\xi}_4\theta_4}e^{\hat{\xi}_5\theta_5}e^{\hat{\xi}_6\theta_6}g_{st}(0) = g_d$$
$$\Rightarrow e^{\hat{\xi}_1\theta_1}e^{\hat{\xi}_2\theta_2}e^{\hat{\xi}_3\theta_3}e^{\hat{\xi}_4\theta_4}e^{\hat{\xi}_5\theta_5}e^{\hat{\xi}_6\theta_6} = g_d\cdot g_{st}^{-1}(0) = g_1$$

Step 1: Solve for θ_3

Let $e^{\hat{\xi}_1\theta_1}\ldots e^{\hat{\xi}_6\theta_6}q_\omega = g_1\cdot q_\omega$

$$\Rightarrow e^{\hat{\xi}_1\theta_1}e^{\hat{\xi}_2\theta_2}e^{\hat{\xi}_3\theta_3}q_\omega = g_1\cdot q_\omega$$

Subtract p_b from g_1q_ω:

$$\|e^{\hat{\xi}_1\theta_1}e^{\hat{\xi}_2\theta_2}(e^{\hat{\xi}_3\theta_3}q_\omega - p_b)\| = \|g_1q_\omega - p_b\|$$
$$\Rightarrow \|e^{\hat{\xi}_3\theta_3}q_\omega - p_b\| \triangleq \delta \leftarrow \text{Subproblem 3}$$

Step 2: Given θ_3, solve for θ_1, θ_2

$e^{\hat{\xi}_1\theta_1}e^{\hat{\xi}_2\theta_2}(e^{\hat{\xi}_3\theta_3}q_\omega) = g_1q_\omega$, Subproblem 2 $\Rightarrow \theta_1, \theta_2$

Step 3: Given $\theta_1, \theta_2, \theta_3$, solve θ_4, θ_5

$$e^{\hat{\xi}_4\theta_4}e^{\hat{\xi}_5\theta_5}e^{\hat{\xi}_6\theta_6} = \underbrace{e^{-\hat{\xi}_3\theta_3}e^{-\hat{\xi}_2\theta_2}e^{-\hat{\xi}_1\theta_1}g_1}_{g_2}$$

let $p\in l_{\xi_6}, p\notin l_{\xi_4}$ or $l_{\xi_5}, e^{\hat{\xi}_4\theta_4}e^{\hat{\xi}_5\theta_5}p = g_2p$,
Subproblem 2 $\Rightarrow \theta_4$ and θ_5.

Step 4: Given $(\theta_1,\ldots,\theta_5)$, solve for θ_6

$e^{\hat{\xi}_6\theta_6} = (e^{\hat{\xi}_1\theta_1}\ldots e^{\hat{\xi}_5\theta_5})^{-1}\cdot g_1 \triangleq g_3$
Let $p\notin l_{\xi_6} \Rightarrow e^{\hat{\xi}_6\theta_6}p = g_3\cdot p = q \Leftarrow$ Subproblem 1
Maximum of solutions: 8

Given $g_{st} : Q \to SE(3)$,

$$\theta(t) = (\theta_1(t)\ldots\theta_n(t))^T \to e^{\hat{\xi}_1\theta_1}\ldots e^{\hat{\xi}_n\theta_n}g_{st}(0)$$

and $\dot{\theta}(t) = (\dot{\theta}_1(t)\ldots\dot{\theta}_n(t))^T$,

What is the velocity of the tool frame?

$$\hat{V}_{st}^s = \dot{g}_{st}(\theta)g_{st}^{-1}(\theta) = \sum_{i=1}^n(\frac{\partial g_{st}}{\partial\theta_i}\dot{\theta}_i)g_{st}^{-1}(\theta)$$
$$= \sum_{i=1}^n(\frac{\partial g_{st}}{\partial\theta_i}g_{st}^{-1}(\theta))\dot{\theta}_i \Rightarrow V_{st}^s = \sum_{i=1}^n(\frac{\partial g_{st}}{\partial\theta_i}g_{st}^{-1}(\theta))^\vee\dot{\theta}_i$$
$$= \underbrace{[(\frac{\partial g_{st}}{\partial\theta_1}g_{st}^{-1}(\theta))^\vee, \ldots, (\frac{\partial g_{st}}{\partial\theta_n}g_{st}^{-1}(\theta))^\vee]}_{J_{st}^s(\theta)\in\mathbb{R}^{6\times n}}\begin{bmatrix}\dot{\theta}_1\\ \vdots\\ \dot{\theta}_n\end{bmatrix}$$