Chapter 3 Manipulator Kinematics

Lecture Notes for

A Geometrical Introduction to Robotics and Manipulation

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CRC Press

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June 29, 2012

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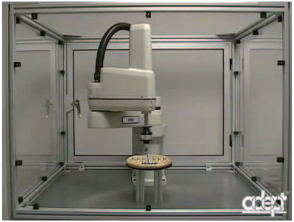
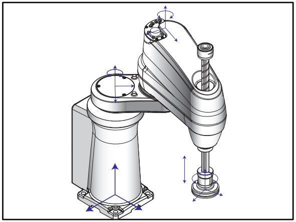
Chapter 3 Manipulator Kinematics

Forward kinematics



(a) Adept Cobra i600 (SCARA)



(b) Forward kinematics of SCARA

Figure 3.1

◊ Lower Pair Joints:

revolute joint $S^1 \mapsto SO(2)$

$$L_i \overset{\mathcal{R}}{\longrightarrow} L_{i+1}$$

prismatic joint $\mathbb{R} \mapsto T(1)$

$$L_i \overset{\mathcal{T}}{\longrightarrow} L_{i+1}$$

◊ Forward kinematics:

$$L_0 \xrightarrow{\text{joint } 1} L_1 \dashrightarrow^{\text{joint } n} L_n$$

Joint space

Revolute joint: $S^1, \theta_i \in S^1$ or $\theta_i \in [-\pi, \pi]$

Prismatic joint: $\mathbb{R}$

Joint space: $Q : \underbrace{S^1 \times \cdots \times S^1}_{\text{no. of } R \text{ joint}} \times \underbrace{\mathbb{R} \times \cdots \times \mathbb{R}}_{\text{no. of } P \text{ joint}}$

Adept $Q : S^1 \times S^1 \times S^1 \times \mathbb{R}$

Elbow $Q = \Gamma^6 : \underbrace{S^1 \times \cdots \times S^1}_{6}$

Reference (nominal) joint config: $\theta = (0, 0, \ldots, 0) \in Q$

Reference (nominal) end-effector config: $g_{st}(0) \in SE(3)$

Arbitrary configuration $g_{st}(\theta)$:

$$g_{st} : \theta \in Q \mapsto g_{st}(\theta) \in SE(3)$$

Two approaches of forward kinematics

□ Classical Approach:

$$g_{st}(\theta_1, \theta_2) = g_{st}(\theta_1) \cdot g_{l_1 l_2} \cdot g_{l_2 t}$$

Disadvantage: A coordinate frame needed for each link

□ The product of exponentials formula:

Consider Fig 3.2.

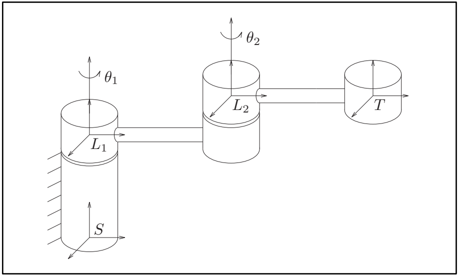


Figure 3.2: A two degree of freedom manipulator

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The product of exponentials formula

Step 1: Rotating about ω_2 by θ_2

$$\xi_2 = \begin{bmatrix} -\omega_2 \times q_2 \\ \omega_2 \end{bmatrix}$$

$$g_{st}(\theta_2) = e^{\hat{\xi}_2 \theta_2} \cdot g_{st}(0)$$

Step 2: Rotating about ω_1 by θ_1

$$\xi_1 = \begin{bmatrix} -\omega_1 \times q_1 \\ \omega_1 \end{bmatrix}$$

$$g_{st}(\theta_1, \theta_2) = e^{\hat{\xi}_1 \theta_1} \cdot \underbrace{e^{\hat{\xi}_2 \theta_2} \cdot g_{st}(0)}_{\text{offset}}$$

$$\theta : (0,0) \mapsto (0, \theta_2) \mapsto (\theta_1, \theta_2)$$

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The product of exponentials formula

What if another route is taken?

$$\theta : (0,0) \mapsto (\theta_1, 0) \mapsto (\theta_1, \theta_2)$$

Step 1: Rotating about ω_1 by θ_1

$$\xi_1 = \begin{bmatrix} -\omega_1 \times q_1 \\ \omega_1 \end{bmatrix}$$

$$g_{st}(\theta_1) = e^{\hat{\xi}_1 \theta_1} \cdot g_{st}(0)$$

Step 2: Rotating about ω_2' by θ_2

Let $e^{\hat{\xi}_1 \theta_1} = \begin{bmatrix} R_1 & p_1 \\ 0 & 1 \end{bmatrix}$

$$\omega_2' = R_1 \cdot \omega_2$$

$$q_2' = p_1 + R_1 \cdot q_2$$

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The product of exponentials formula

$$\xi_2' = \begin{bmatrix} -\omega_2' \times q_2' \\ \omega_2' \end{bmatrix} = \begin{bmatrix} -R_1\hat{\omega}_2 R_1^T(p_1 + R_1 q_2) \\ R_1\omega_2 \end{bmatrix}$$

$$= \begin{bmatrix} R_1 & \hat{p}_1 R_1 \\ 0 & R_1 \end{bmatrix} \begin{bmatrix} -\omega_2 \times q_2 \\ \omega_2 \end{bmatrix} = Ad_{e^{\hat{\xi}_1\theta_1}} \cdot \xi_2 \Rightarrow$$

$$\hat{\xi}_2' = e^{\hat{\xi}_1\theta_1} \cdot \hat{\xi}_2 \cdot e^{-\hat{\xi}_1\theta_1}$$

$$\begin{aligned} g_{st}(\theta_1, \theta_2) &= e^{\hat{\xi}_2'\theta_2} \cdot e^{\hat{\xi}_1\theta_1} \cdot g_{st}(0) \\ &= e^{e^{\hat{\xi}_1\theta_1} \cdot \hat{\xi}_2\theta_2 \cdot e^{-\hat{\xi}_1\theta_1}} \cdot e^{\hat{\xi}_1\theta_1} \cdot g_{st}(0) \\ &= e^{\hat{\xi}_1\theta_1} \cdot e^{\hat{\xi}_2\theta_2} \cdot e^{-\hat{\xi}_1\theta_1} \cdot e^{\hat{\xi}_1\theta_1} \cdot g_{st}(0) \\ &= \underbrace{e^{\hat{\xi}_1\theta_1} \cdot e^{\hat{\xi}_2\theta_2} \cdot g_{st}(0)}_{\text{Independent of the route taken}} \end{aligned}$$

Procedure for forward kinematic map

Identify a nominal configuration:

$$\Theta = (\theta_{10}, \dots, \theta_{n0}) = 0, g_{st}(0) \triangleq g_{st}(\theta_{10}, \dots, \theta_{n0})$$

Simplification of forward kinematics mapping:

Revolute joint: $\xi_i = \begin{bmatrix} -\omega_i \times q_i \\ \omega_i \end{bmatrix}$
Choose q_i s.t. ξ_i is simple.

Prismatic joint: $\xi_i = \begin{bmatrix} v_i \\ 0 \end{bmatrix}$

Write $g_{st}(\theta) = e^{\hat{\xi}_1 \theta_1} \dots e^{\hat{\xi}_n \theta_n} \cdot g_{st}(0)$ (product of exponential mapping)

Example: SCARA manipulator

$$g_{st}(0) = \left[\begin{array}{c|c} I & \begin{array}{c} 0 \\ l_1 + l_2 \\ l_0 \end{array} \\ \hline 0 & 1 \end{array}\right]$$

$$\omega_1 = \omega_2 = \omega_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

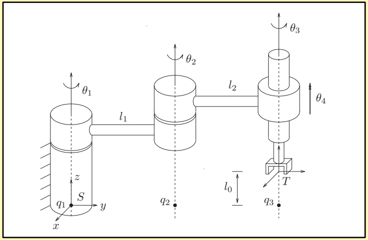


Figure 3.3

$$q_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, q_2 = \begin{bmatrix} 0 \\ l_1 \\ 0 \end{bmatrix}, q_3 = \begin{bmatrix} 0 \\ l_1 + l_2 \\ 0 \end{bmatrix}$$

$$\Rightarrow \xi_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \xi_2 = \begin{bmatrix} l_1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \xi_3 = \begin{bmatrix} l_1 + l_2 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \xi_4 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

(see next page)

Example: SCARA manipulator

$$g_{st}(\theta) = e^{\hat{\xi}_1\theta_1} \cdot e^{\hat{\xi}_2\theta_2} \cdot e^{\hat{\xi}_3\theta_3} \cdot e^{\hat{\xi}_4\theta_4} \cdot g_{st}(0) = \begin{bmatrix} R(\theta) & p(\theta) \\ 0 & 1 \end{bmatrix}$$

$$e^{\hat{\xi}_1\theta_1} = \begin{bmatrix} c_1 & -s_1 & 0 & 0 \\ s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \; e^{\hat{\xi}_2\theta_2} = \begin{bmatrix} c_2 & -s_2 & 0 & -l_1 s_1 \\ s_2 & c_2 & 0 & l_1 c_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$e^{\hat{\xi}_3\theta_3} = \begin{bmatrix} c_3 & -s_3 & 0 & -l_1 s_1 - l_2 c_{12} \\ s_3 & c_3 & 0 & l_1 c_1 + l_2 c_{12} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \; e^{\hat{\xi}_4\theta_4} = \begin{bmatrix} I & \begin{bmatrix} 0 \\ 0 \\ \theta_4 \end{bmatrix} \\ 0 & 1 \end{bmatrix}$$

$$g_{st}(\theta) = \begin{bmatrix} c_{123} & -s_{123} & 0 & -l_1 s_1 - l_2 s_{12} \\ s_{123} & c_{123} & 0 & l_1 c_1 + l_2 c_{12} \\ 0 & 0 & 1 & l_0 + \theta_4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

in which, $c_{123} = \cos(\theta_1 + \theta_2 + \theta_3)$ and $c_{12} = \cos(\theta_1 + \theta_2)$. ♢

Example: Elbow manipulator

$$g_{st}(0) = \begin{bmatrix} I & \begin{bmatrix} 0 \\ l_1 + l_2 \\ l_0 \end{bmatrix} \\ 0 & 1 \end{bmatrix}$$

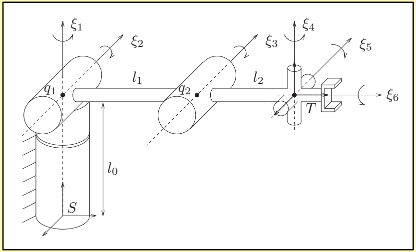


Figure 3.4

$$\xi_1 = \begin{bmatrix} -\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times \begin{bmatrix} 0 \\ 0 \\ l_0 \end{bmatrix} \\ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix},$$

(Continues next slide)

Example: Elbow manipulator

$$\xi_2 = \begin{bmatrix} -\begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} \times \begin{bmatrix} 0 \\ 0 \\ l_0 \end{bmatrix} \\ \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 0 \\ -l_0 \\ 0 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \xi_3 = \begin{bmatrix} 0 \\ -l_0 \\ l_1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \xi_4 = \begin{bmatrix} l_1 + l_2 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix},$$

$$\xi_5 = \begin{bmatrix} 0 \\ -l_0 \\ l_1 + l_2 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \xi_6 = \begin{bmatrix} -l_0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\rightarrow g_{st}(\theta_1, \ldots \theta_6) = e^{\hat{\xi}_1 \theta_1} \ldots e^{\hat{\xi}_6 \theta_6} \cdot g_{st}(0) = \begin{bmatrix} R(\theta) & p(\theta) \\ 0 & 1 \end{bmatrix}$$

$$p(\theta) = \begin{bmatrix} -s_1(l_2 c_2 + l_2 c_{23}) \\ c_1(l_1 c_2 + l_2 c_{23}) \\ l_0 - l_1 s_2 - l_2 s_{23} \end{bmatrix}, R(\theta) = [r_{ij}]$$

(see next page)

Example: Elbow manipulator

in which,

$$
\begin{aligned}
r_{11} &= c_6\big(c_1c_4 - s_1c_{23}s_4\big) + s_6\big(s_1s_{23}c_5 + s_1c_{23}c_4s_5 + c_1s_4s_5\big)\\
r_{12} &= -c_5\big(s_1c_{23}c_4 + c_1s_4\big) + s_1s_{23}s_5\\
r_{13} &= c_6\big(-c_5s_1s_{23} - (c_{23}c_4s_1 + c_1s_4)s_5\big) + \big(c_1c_4 - c_{23}s_1s_4\big)s_6\\
r_{21} &= c_6\big(c_4s_1 + c_1c_{23}s_4\big) - \big(c_1c_5s_{23} + (c_1c_{23}c_4 - s_1s_4)s_5\big)s_6\\
r_{22} &= c_5\big(c_1c_{23}c_4 - s_1s_4\big) - c_1s_{23}s_5\\
r_{23} &= c_6\big(c_1c_5s_{23} + (c_1c_{23}c_4 - s_1s_4)s_5\big) + \big(c_4s_1 + c_1c_{23}s_4\big)s_6\\
r_{31} &= -\big(c_6s_{23}s_4\big) - \big(c_{23}c_5 - c_4s_{23}s_5\big)s_6\\
r_{32} &= -\big(c_4c_5s_{23}\big) - c_{23}s_5\\
r_{33} &= c_6\big(c_{23}c_5 - c_4s_{23}s_5\big) - s_{23}s_4s_6
\end{aligned}
$$

Simplify forward Kinematics Map:

Choose base frame or ref. Config. s.t. $g_{st}(0) = I$ ♢

Manipulator Workspace

$W = \{g_{st}(\theta) | \forall \theta \in Q\} \subset SE(3)$

- Reachable Workspace:

$$W_R = \{p(\theta) | \forall \theta \in Q\} \subset \mathbb{R}^3$$

- Dextrous Workspace:

$$W_D = \{p \in \mathbb{R}^3 | \forall R \in SO(3), \exists \theta, g_{st}(\theta) = (p, R)\}$$

Example: A planar serial 3-bar linkage

(a) Workspace calculation:

$$g = (x, y, \phi)$$
$$x = l_1 c_1 + l_2 c_{12} + l_3 c_{123}$$
$$y = l_1 s_1 + l_2 s_{12} + l_3 s_{123}$$
$$\phi = \theta_1 + \theta_2 + \theta_3$$

(b) Construction of Workspace:

(c) Reachable Workspace:

(d) Dextrous Workspace:

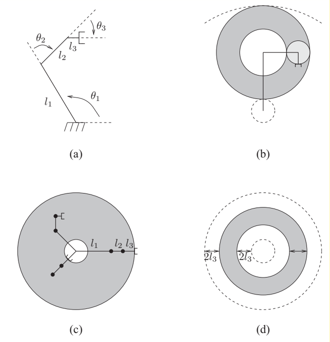


Figure 3.5

□ $6\mathcal{R}$ manipulator with max workspace (Paden):

Elbow manipulator and its kinematics inverse.

† End of Section †

Inverse kinematics

Definition: Inverse kinematics
Given $g \in SE(3)$, find $\theta \in Q$ s.t.

$$g_{st}(\theta) = g, \text{ where } g_{st} : Q \mapsto SE(3)$$

⋄ Example: A planar example

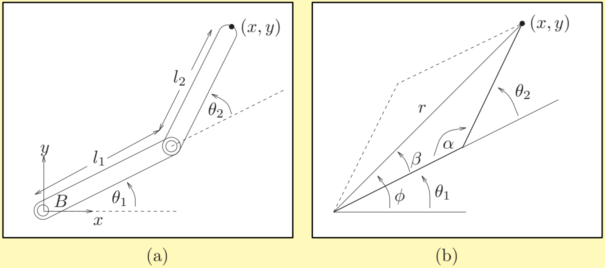


Figure 3.6

$$x = l_1 \cos\theta_1 + l_2 \cos(\theta_1 + \theta_2)$$
$$y = l_1 \sin\theta_1 + l_2 \sin(\theta_1 + \theta_2)$$

Given (x, y), solve for (θ_1, θ_2).

Inverse kinematics

♢ Review:

Polar Coordinates:

$$(r, \phi), r = \sqrt{x^2 + y^2}$$

Law of cosines:

$$\theta_2 = \pi \pm \alpha, \alpha = \cos^{-1} \frac{l_1^2 + l_2^2 - r^2}{2 l_1 l_2}$$

Flip solution: $\pi + \alpha$

$$\theta_1 = \operatorname{atan2}(y, x) \pm \beta, \beta = \cos^{-1} \frac{r^2 + l_1^2 - l_2^2}{2 l_1 r}$$

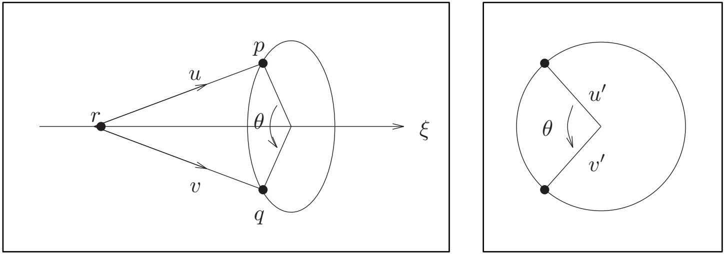
Hight Lights:

- Subproblems
- Each has zero, one or two solutions!

Paden-Kahan Subproblems

Subproblem 1: Rotation about a single axis

Let ξ be a zero-pitch twist, with unit magnitude and two points $p, q \in \mathbb{R}^3$.
Find θ s.t. $e^{\hat{\xi}\theta} p = q$



(a) (b)

Figure 3.6

Solution: Let $r \in l_\xi$, define $u = p - r, v = q - r, e^{\hat{\xi}\theta} r = r$

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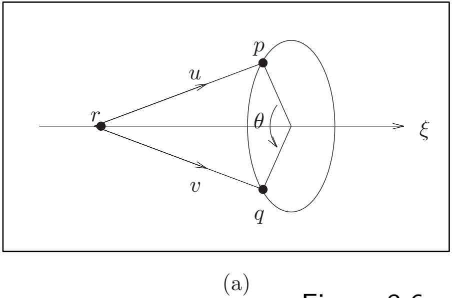
Paden-Kahan Subproblems

Moreover,

$$\Rightarrow e^{\hat{\xi}\theta} p = q \Rightarrow e^{\hat{\xi}\theta} \underbrace{(p-r)}_{u} = \underbrace{q-r}_{v} \Rightarrow \begin{bmatrix} e^{\hat{\omega}\theta} & * \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u \\ 0 \end{bmatrix} = \begin{bmatrix} v \\ 0 \end{bmatrix}$$

$$\Rightarrow e^{\hat{\omega}\theta} u = v \qquad \begin{cases} w^T u = w^T v \\ \|u\|^2 = \|v\|^2 \end{cases}$$



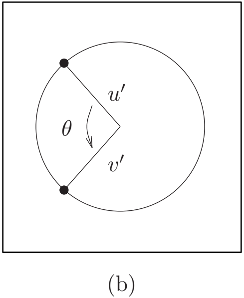


Figure 3.6

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Paden-Kahan Subproblems

$u' = (I - \omega\omega^T)u, v' = (I - \omega\omega^T)v$

The solution exists only if $\begin{cases} \|u'\|^2 = \|v'\|^2 \\ \omega^T u = \omega^T v \end{cases}$

- If $u' \neq 0$, then

$$u' \times v' = \omega \sin\theta \|u'\| \|v'\|$$
$$u' \cdot v' = \cos\theta \|u'\| \|v'\|$$

$$\Rightarrow \theta = \text{atan2}(\omega^T(u' \times v'), u'^T v')$$

- If $u' = 0$, $\Rightarrow$ Infinite number of solutions! $\diamondsuit$

Paden-Kahan Subproblems

Subproblem 2: Rotation about two subsequent axes

Let ξ_1 and ξ_2 be two zero-pitch, unit magnitude twists, with intersecting axes, and $p, q \in R^3$. find θ_1 and θ_2 s.t. $e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2} p = q$.

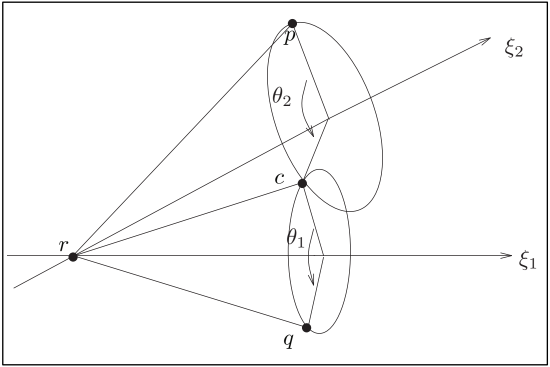

Figure 3.6

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Paden-Kahan Subproblems

Solution: If two axes of ξ_1 and ξ_2 coincide, then we get:

Subproblem 1: $\theta_1 + \theta_2 = \theta$

If the two axes are not parallel, $\omega_1 \times \omega_2 \neq 0$, then, let c satisfy:

$$e^{\hat{\xi}_2\theta_2} p = c = e^{-\hat{\xi}_1\theta_1} q$$

Set $r \in l_{\xi_1} \cap l_{\xi_2}$

$$e^{\hat{\xi}_2\theta_2} \underbrace{p - r}_{u} = \underbrace{c - r}_{z} = e^{-\hat{\xi}_1\theta_1} \underbrace{(q - r)}_{v}, \Rightarrow e^{\hat{\omega}_2\theta_2} u = z = e^{-\hat{\omega}_1\theta_1} v$$

$$\Rightarrow \begin{cases} \omega_2^T u = \omega_2^T z \\ \omega_1^T v = \omega_1^T z \end{cases}, \|u\|^2 = \|z\|^2 = \|v\|^2$$

As ω_1, ω_2 and $\omega_1 \times \omega_2$ are linearly independent,

$$z = \alpha\omega_1 + \beta\omega_2 + \gamma(\omega_1 \times \omega_2)$$
$$\Rightarrow \|z\|^2 = \alpha^2 + \beta^2 + 2\alpha\beta\omega_1^T\omega_2 + \gamma^2\|\omega_1 \times \omega_2\|^2$$

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Paden-Kahan Subproblems

$$\begin{matrix} \omega_2^T u = \alpha \omega_2^T \omega_1 + \beta \\ \omega_1^T v = \alpha + \beta \omega_1^T \omega_2 \end{matrix} \Rightarrow \begin{cases} \alpha & = \frac{(\omega_1^T \omega_2)\omega_2^T u - \omega_1^T v}{(\omega_1^T \omega_2)^2 - 1} \\ \beta & = \frac{(\omega_1^T \omega_2)\omega_1^T v - \omega_2^T u}{(\omega_1^T \omega_2)^2 - 1} \end{cases}$$

$$\|z\|^2 = \|u\|^2 \Rightarrow \gamma^2 = \frac{\|u\|^2 - \alpha^2 - \beta^2 - 2\alpha\beta\omega_1^T \omega_2}{\|\omega_1 \times \omega_2\|^2} \quad (*)$$

$(*)$ has zero, one or two solution(s):

$$\text{Given } z \Rightarrow c \Rightarrow \begin{cases} e^{\hat{\xi}_2 \theta_2} p = c \\ e^{\hat{\xi}_1 \theta_1} p = c \end{cases}$$

for θ_1 and θ_2

1. Two solutions when the two circles intersect.
2. One solution when they are tangent
3. Zero solution when they do not intersect

$\diamondsuit$

Paden-Kahan Subproblems

Subproblem 3: Rotation to a given point

Given a zero-pitch twist ξ, with unit magnitude and $p, q \in \mathbb{R}^3$, find θ s.t. $\|q - e^{\hat{\xi}\theta} p\| = \delta$

Define: $u = p - r, v = q - r, \|v - e^{\hat{\omega}\theta} u\|^2 = \delta^2$

$$u' = u - \omega\omega^T u$$
$$v' = v - \omega\omega^T v$$

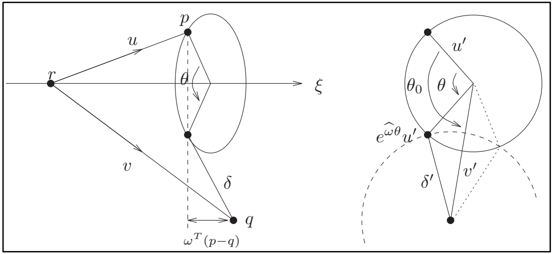

Figure 3.7

$$\Rightarrow u = u' + \omega\omega^T u, v = v' + \omega\omega^T v, \delta'^2 = \delta^2 - |\omega^T (p - q)|^2$$

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Paden-Kahan Subproblems

$$\|(v' + \omega\omega^T v) - e^{\hat{\omega}\theta}(u' + \omega\omega^T u)\|^2 = \delta^2 \Rightarrow$$
$$\|v' - e^{\hat{\omega}\theta}u' + \underbrace{\omega\omega^T(v-u)}_{\omega\omega^T(q-p)}\|^2 = \delta^2$$

$$\begin{aligned}
\|v' - e^{\hat{\omega}\theta}u'\|^2 &= \delta^2 - \|\omega^T(p-q)\|^2 = \delta'^2, \\
\theta_0 &= \operatorname{atan2}(\omega^T(u' \times v'), u'^T v'), \\
\phi = \theta_0 - \theta &\Rightarrow \|u'\|^2 + \|v'\|^2 - 2\|u'\| \cdot \|v'\| \cos\phi = \delta'^2, \\
\theta &= \theta_0 \pm \cos^{-1}\frac{\|u'\| + \|v'\| - \delta'^2}{2\|u'\| \cdot \|v'\|} \quad (*)
\end{aligned}$$

Zero, one or two solutions! ♢

Solving inverse kinematics using subproblems

Technique 1: Eliminate the dependence on a joint

$e^{\hat{\xi}\theta}p = p$, if $p \in l_{\xi}$. Given $e^{\hat{\xi}_1\theta_1}e^{\hat{\xi}_2\theta_2}e^{\hat{\xi}_3\theta_3} = g$, select $p \in l_{\xi_3}$, $p \notin l_{\xi_1}$ or l_{ξ_2}, then:

$$gp = e^{\hat{\xi}_1\theta_1}e^{\hat{\xi}_2\theta_2}p$$

Technique 2: subtract a common point

$$e^{\hat{\xi}_1\theta_1}e^{\hat{\xi}_2\theta_2}e^{\hat{\xi}_3\theta_3} = g, q \in l_{\hat{\xi}_1} \cap l_{\hat{\xi}_2} \Rightarrow e^{\hat{\xi}_1\theta_1}e^{\hat{\xi}_2\theta_2}e^{\hat{\xi}_3\theta_3}p - q = gp - q \Rightarrow$$

$$\|e^{\hat{\xi}_3\theta_3}p - q\| = \|gp - q\|$$

Example: Elbow manipulator

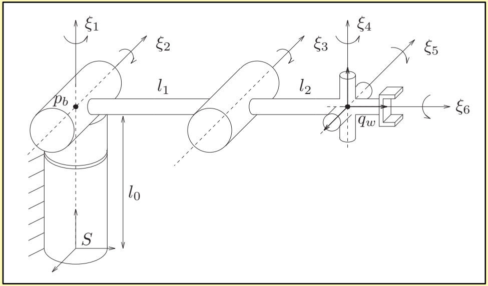

Figure 3.7

$$g_{st}(\theta) = e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2} e^{\hat{\xi}_3\theta_3} e^{\hat{\xi}_4\theta_4} e^{\hat{\xi}_5\theta_5} e^{\hat{\xi}_6\theta_6} g_{st}(0) = g_d$$

$$\Rightarrow e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2} e^{\hat{\xi}_3\theta_3} e^{\hat{\xi}_4\theta_4} e^{\hat{\xi}_5\theta_5} e^{\hat{\xi}_6\theta_6} = g_d \cdot g_{st}^{-1}(0) = g_1$$

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Example: Elbow manipulator

Step 1: Solve for θ_3

Let $e^{\hat{\xi}_1\theta_1}\dots e^{\hat{\xi}_6\theta_6}q_\omega = g_1 \cdot q_\omega$

$$\Rightarrow e^{\hat{\xi}_1\theta_1}e^{\hat{\xi}_2\theta_2}e^{\hat{\xi}_3\theta_3}q_\omega = g_1 \cdot q_\omega$$

Subtract p_b from $g_1 q_\omega$:

$$\left\| e^{\hat{\xi}_1\theta_1}e^{\hat{\xi}_2\theta_2}(e^{\hat{\xi}_3\theta_3}q_\omega - p_b) \right\| = \left\| g_1 q_\omega - p_b \right\|$$

$$\Rightarrow \left\| e^{\hat{\xi}_3\theta_3}q_\omega - p_b \right\| \triangleq \delta \leftarrow \text{Subproblem 3}$$

Step 2: Given θ_3, solve for θ_1, θ_2

$e^{\hat{\xi}_1\theta_1}e^{\hat{\xi}_2\theta_2}(e^{\hat{\xi}_3\theta_3}q_\omega) = g_1 q_\omega$, Subproblem 2 $\Rightarrow \theta_1, \theta_2$

(Continues next slide)

Elbow manipulator

Step 3: Given $\theta_1, \theta_2, \theta_3$, solve θ_4, θ_5

$$e^{\hat{\xi}_4\theta_4} e^{\hat{\xi}_5\theta_5} e^{\hat{\xi}_6\theta_6} = \underbrace{e^{-\hat{\xi}_3\theta_3} e^{-\hat{\xi}_2\theta_2} e^{-\hat{\xi}_1\theta_1} g_1}_{g_2}$$

let $p \in l_{\xi_6}, p \notin l_{\xi_4} \text{ or } l_{\xi_5}, e^{\hat{\xi}_4\theta_4} e^{\hat{\xi}_5\theta_5} p = g_2 p,$
Subproblem 2 $\Rightarrow \theta_4 \text{ and } \theta_5.$

Step 4: Given $(\theta_1, \dots, \theta_5)$, solve for θ_6

$$e^{\hat{\xi}_6\theta_6} = (e^{\hat{\xi}_1\theta_1} \dots e^{\hat{\xi}_5\theta_5})^{-1} \cdot g_1 \triangleq g_3$$

Let $p \notin l_{\xi_6} \Rightarrow e^{\hat{\xi}_6\theta_6} p = g_3 \cdot p = q \Leftarrow$ Subproblem 1
Maximum of solutions: 8

$\diamondsuit$

Example: Inverse Kinematics of SCARA

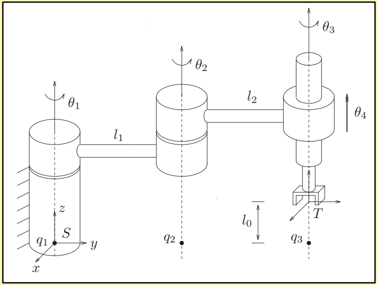


Figure 3.8

$$g_{st}(0) = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & l_1 + l_2 \\ 0 & 0 & 1 & l_0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right]$$

$$g_{st}(\theta) = e^{\hat{\xi}_1\theta_1} \ldots e^{\hat{\xi}_4\theta_4} g_{st}(0)$$

$$= \left[\begin{array}{ccc|c} c_\phi & -s_\phi & 0 & x \\ s_\phi & c_\phi & 0 & y \\ 0 & 0 & 1 & z \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \triangleq g_d$$

$$p = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \Rightarrow p(\theta) = \begin{bmatrix} -l_1 s_1 - l_2 s_{12} \\ l_1 c_1 + l_2 c_{12} \\ l_0 + \theta_4 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \Rightarrow \theta_4 = z - l_0$$

$$e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2} e^{\hat{\xi}_3\theta_3} = g_d g_{st}^{-1}(0) e^{-\hat{\xi}_4\theta_4} \triangleq g_1$$

(Continues next slide)

Example: Inverse Kinematics of SCARA

Let $p \in l_{\xi_3}, q \in l_{\xi_1} \Rightarrow e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2} p = g_1 p,$

$$\|e^{\hat{\xi}_1\theta_1}(e^{\hat{\xi}_2\theta_2}p - q)\| = \|g_1 p - q\|,$$

$$\|e^{\hat{\xi}_2\theta_2}p - q\| = \delta \leftarrow \text{ Subproblem 3 to get } \theta_2$$

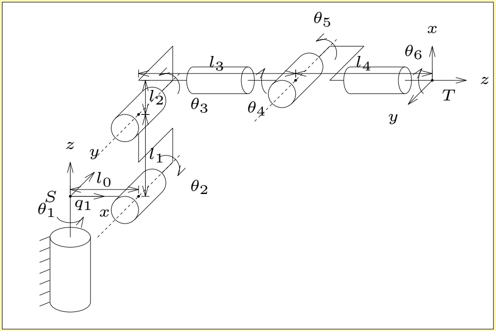
$$\Rightarrow e^{\hat{\xi}_1\theta_1}(e^{\hat{\xi}_2\theta_2}p) = g_1 p \Rightarrow \theta_1 \leftarrow \text{Subproblem 1 to get } \theta_1$$

$$\Rightarrow e^{\hat{\xi}_3\theta_3} = e^{-\hat{\xi}_2\theta_2} e^{-\hat{\xi}_1\theta_1} g_d g_{st}^{-1}(0) e^{-\hat{\xi}_4\theta_4} \triangleq g_2$$

$$e^{\hat{\xi}_3\theta_3}p = g_2 p, p \notin l_{\xi_3} \leftarrow \text{Subproblem 1 to get } \theta_3$$

There are a maximum of two solutions! ♢

Example: ABB IRB4400



$$\omega_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \omega_2 = -\omega_3 = -\omega_5 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \omega_4 = \omega_6 = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$$

$$q_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, q_2 = \begin{bmatrix} l_0 \\ 0 \\ 0 \end{bmatrix}, q_3 = \begin{bmatrix} l_0 \\ 0 \\ l_1 \end{bmatrix}, p_w := q_4 = q_5 = q_6 = \begin{bmatrix} l_0 + l_3 \\ 0 \\ l_1 + l_2 \end{bmatrix}$$

(Continues next slide)

Example: ABB IRB4400

$$g_{st}(0) = \left[\begin{array}{ccc|c} 0 & 0 & 1 & l_0 + l_3 + l_4 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & l_1 + l_2 \\ \hline 0 & 0 & 0 & 1 \end{array}\right], \xi_i = \begin{bmatrix} q_i \times \omega_i \\ \omega_i \end{bmatrix}$$

$$g_{st}(\theta) = e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2} e^{\hat{\xi}_3\theta_3} e^{\hat{\xi}_4\theta_4} e^{\hat{\xi}_5\theta_5} e^{\hat{\xi}_6\theta_6} g_{st}(0) := g_d$$

$$e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2} e^{\hat{\xi}_3\theta_3} p_w = g_d p_w =: q \Rightarrow e^{\hat{\xi}_2\theta_2} e^{\hat{\xi}_3\theta_3} p_w = e^{-\hat{\xi}_1\theta_1} q$$

$$\Rightarrow 0 = [0\ 1\ 0] \cdot e^{-\hat{\xi}_1\theta_1} q = \cos\theta_1 q_y - \sin\theta_1 q_x, q = \begin{bmatrix} q_x \\ q_y \\ q_z \end{bmatrix}$$

$$\Rightarrow \theta_1 = \tan^{-1}(q_y/q_x)$$

$$\|e^{\hat{\xi}_3\theta_3} p_w - q_2\| = \|e^{-\hat{\xi}_1\theta_1} q - q_2\| =: \delta \leftarrow \text{Subproblem 3 to get } \theta_3$$

$$e^{\hat{\xi}_2\theta_2}(e^{\hat{\xi}_3\theta_3} p_w) = e^{-\hat{\xi}_1\theta_1} q \leftarrow \text{Subproblem 1 to get } \theta_2$$

$$e^{\hat{\xi}_4\theta_4} e^{\hat{\xi}_5\theta_5} e^{\hat{\xi}_6\theta_6} = e^{-\hat{\xi}_3\theta_3} e^{-\hat{\xi}_2\theta_2} e^{-\hat{\xi}_1\theta_1} g_d g_{st}^{-1}(0) =: g_2$$

Use subproblem 1,2 to solve for $\theta_4, \theta_5, \theta_6$ $\diamondsuit$

† End of Section †

Manipulator Jacobian

Given $g_{st} : Q \to SE(3)$,

$$\theta(t) = (\theta_1(t) \dots \theta_n(t))^T \to e^{\hat{\xi}_1 \theta_1} \dots e^{\hat{\xi}_n \theta_n} g_{st}(0)$$

and $\dot{\theta}(t) = (\dot{\theta}_1(t) \dots \dot{\theta}_n(t))^T$,

What is the velocity of the tool frame?

$$\begin{aligned}
\hat{V}_{st}^s &= \dot{g}_{st}(\theta) g_{st}^{-1}(\theta) = \sum_{i=1}^{n} \left(\frac{\partial g_{st}}{\partial \theta_i} \dot{\theta}_i\right) g_{st}^{-1}(\theta) \\
&= \sum_{i=1}^{n} \left(\frac{\partial g_{st}}{\partial \theta_i} g_{st}^{-1}(\theta)\right) \dot{\theta}_i \Rightarrow V_{st}^s = \sum_{i=1}^{n} \left(\frac{\partial g_{st}}{\partial \theta_i} g_{st}^{-1}(\theta)\right)^\vee \dot{\theta}_i \\
&= \underbrace{\left[\left(\frac{\partial g_{st}}{\partial \theta_1} g_{st}^{-1}(\theta)\right)^\vee, \dots, \left(\frac{\partial g_{st}}{\partial \theta_n} g_{st}^{-1}(\theta)\right)^\vee\right]}_{J_{st}^s(\theta) \in \mathbb{R}^{6 \times n}} \begin{bmatrix} \dot{\theta}_1 \\ \vdots \\ \dot{\theta}_n \end{bmatrix}
\end{aligned}$$

(Continues next slide)

Manipulator Jacobian

$$g_{st}(\theta) = e^{\hat{\xi}_1\theta_1}\dots e^{\hat{\xi}_n\theta_n} g_{st}(0)$$

$$\frac{\partial g_{st}}{\partial \theta_1} g_{st}^{-1}(\theta) = (\hat{\xi}_1 e^{\hat{\xi}_1\theta_1}\dots e^{\hat{\xi}_n\theta_n} g_{st}(0))(g_{st}(\theta))^{-1} = \hat{\xi}_1 \Rightarrow$$

$$(\frac{\partial g_{st}}{\partial \theta_1} g_{st}^{-1}(\theta))^{\vee} = \xi_1$$

$$\frac{\partial g_{st}}{\partial \theta_2} g_{st}^{-1}(\theta) = (e^{\hat{\xi}_1\theta_1}\hat{\xi}_2 e^{\hat{\xi}_2\theta_2}\dots e^{\hat{\xi}_n\theta_n} g_{st}(0))(g_{st}(\theta))^{-1}$$

$$= e^{\hat{\xi}_1\theta_1}\hat{\xi}_2 e^{\hat{\xi}_2\theta_2}\dots e^{\hat{\xi}_n\theta_n} g_{st}(0) g_{st}^{-1}(\theta) = e^{\hat{\xi}_1\theta_1}\hat{\xi}_2 e^{-\hat{\xi}_1\theta_1} \triangleq \hat{\xi}_2'$$

$$(\frac{\partial g_{st}}{\partial \theta_2} g_{st}^{-1}(\theta))^{\vee} = \mathrm{Ad}_{e^{\hat{\xi}_1\theta_1}}\xi_2 = \xi_2' \dots\dots$$

$$(\frac{\partial g_{st}}{\partial \theta_i} g_{st}^{-1}(\theta))^{\vee} = \mathrm{Ad}_{e^{\hat{\xi}_1\theta_1}\dots e^{\hat{\xi}_{i-1}\theta_{i-1}}}\xi_i \triangleq \xi_i'$$

$$\Rightarrow J_{st}^s(\theta) = [\xi_1, \xi_2', \dots, \xi_n']$$

(Continues next slide)

Manipulator Jacobian

Interpretation of ξ_i':

- ξ_i' is only affected by $\theta_1 \ldots \theta_{i-1}$
- The twist associated with joint i, at the present configuration.

Body jacobian:

$$V_{st}^b = J_{st}^b(\theta) \cdot \dot{\theta}$$
$$J_{st}^b(\theta) = [\xi_1^\dagger \ldots \xi_{n-1}^\dagger, \xi_n^\dagger]$$
$$\xi_i^\dagger = \mathrm{Ad}^{-1}_{e^{\hat{\xi}_{i+1}\theta_{i+1}} \ldots e^{\hat{\xi}_n \theta_n} g_{st}(0)} \xi_i$$

Joint twist written with respect to the body frame at the current configuration!

$$J_{st}^s(\theta) = \mathrm{Ad}_{g_{st}(\theta)} \cdot J_{st}^b(\theta)$$

If J_{st}^s is invertible, $\dot{\theta}(t) = (J_{st}^s(\theta))^{-1} \cdot V_{st}^s(t)$

(Continues next slide)

Manipulator Jacobian

Given $g(t)$, how to find $\theta(t)$?

$$\left.\begin{array}{ll} 1) & \hat{V}^s_{st} = \dot{g}(t)g^{-1}(t) \\ 2) & \left\{\begin{array}{l} \dot{\theta}(t) = (J^s_{st}(\theta))^{-1}V^s_{st}(t) \\ \theta(0) = \theta_0 \end{array}\right. \end{array}\right\} \Rightarrow \theta(t)$$

⋄ Example: Jacobian for a SCARA manipulator

$$q_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, q_2' = \begin{bmatrix} -l_1 s_1 \\ l_1 c_1 \\ 0 \end{bmatrix},$$

$$q_3' = \begin{bmatrix} -l_1 s_1 - l_2 s_{12} \\ l_1 c_1 + l_2 c_{12} \\ 0 \end{bmatrix},$$

$$\omega_1 = \omega_2' = \omega_3' = [0\ 0\ 1]^T$$

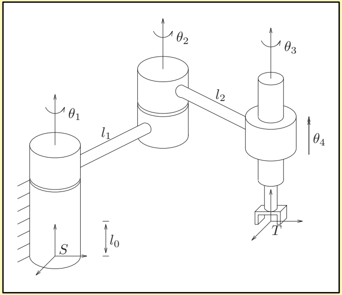


Figure 3.9

(Continues next slide)

Example: Jacobian for a SCARA manipulator

$$\xi_1 = \begin{bmatrix} -\omega_1 \times q_1 \\ \omega_1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T$$

$$\xi_2' = \begin{bmatrix} -\omega_2' \times q_2' \\ \omega_2' \end{bmatrix} = \begin{bmatrix} l_1 c_1 & l_1 s_1 & 0 & 0 & 0 & 0 \end{bmatrix}^T$$

$$\xi_3' = \begin{bmatrix} -\omega_3' \times q_3' \\ \omega_3' \end{bmatrix} = \begin{bmatrix} l_1 c_1 + l_2 c_{12} & l_1 s_1 + l_2 s_{12} & 0 & 0 & 0 & 1 \end{bmatrix}^T$$

$$\xi_4' = \begin{bmatrix} v_4' \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}^T$$

$$J_{st}^s(\theta) = \begin{bmatrix} 0 & l_1 c_1 & l_1 c_1 + l_1 c_{12} & 0 \\ 0 & l_1 s_1 & l_1 s_1 + l_1 s_{12} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

◊

Example: Jacobian of Stanford Arm

$$q_1 = q_2 = \begin{bmatrix} 0 \\ 0 \\ l_0 \end{bmatrix},$$

$$\omega_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

$$\omega_2' = \begin{bmatrix} -c_1 \\ -s_1 \\ 0 \end{bmatrix}$$

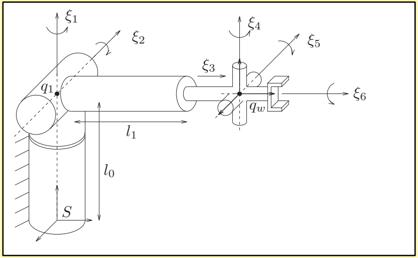


Figure 3.10

$$\xi_1 = \begin{bmatrix} -\omega_1 \times q_1 \\ \omega_1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\xi_2' = \begin{bmatrix} -\omega_2' \times q_2 \\ \omega_2' \end{bmatrix} = \begin{bmatrix} l_0 s_1 & l_0 c_1 & 0 & -c_1 & -s_1 & 0 \end{bmatrix}$$

$$\xi_3' = \begin{bmatrix} e^{\hat{z}\theta_1} \cdot e^{-\hat{x}\theta_2} \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \\ 0 \end{bmatrix} = [-s_1 c_2 \ c_1 c_2 \ -s_2 \ 0 \ 0 \ 0]^T = \begin{bmatrix} v_3 \\ 0 \end{bmatrix}$$

(Continues next slide)

Example: Jacobian of Stanford Arm

$$q'_\omega = \begin{bmatrix} 0 \\ 0 \\ l_0 \end{bmatrix} + e^{\hat{z}\theta_1} \cdot e^{-\hat{x}\theta_2} \cdot \begin{bmatrix} 0 \\ l_1 + \theta_3 \\ 0 \end{bmatrix} = \begin{bmatrix} -(l_1 + \theta_3)s_1 c_2 \\ (l_1 + \theta_3)c_1 c_2 \\ l_0 - (l_1 + \theta_3)s_2 \end{bmatrix}$$

$$\omega'_4 = e^{\hat{z}\theta_1} \cdot e^{-\hat{x}\theta_2} \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -s_{1 s_2} \\ c_1 s_2 \\ c_2 \end{bmatrix}$$

$$\omega'_5 = e^{\hat{z}\theta_1} \cdot e^{-\hat{x}\theta_2} \cdot e^{\hat{z}\theta_4} \cdot \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -c_1 c_4 + s_1 c_2 s_4 \\ -s_1 c_4 - c_1 c_2 s_4 \\ s_2 s_4 \end{bmatrix}$$

$$\omega'_6 = e^{\hat{z}\theta_1} \cdot e^{-\hat{x}\theta_2} \cdot e^{\hat{z}\theta_4} \cdot e^{-\hat{x}\theta_5} \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -c_5(s_1 c_2 c_4) + s_1 s_2 s_5 \\ c_5(c_1 c_2 c_4 - s_1 s_4) - c_1 s_2 s_5 \\ -s_2 c_4 c_5 - c_2 s_5 \end{bmatrix}$$

$$J^s_{st} = \begin{bmatrix} 0 & -\omega'_2 \times q_1 & v'_3 & -\omega'_5 \times q'_\omega & -\omega'_5 \times q'_\omega & -\omega'_6 \times q'_\omega \\ \omega_1 & \omega'_2 & 0 & \omega'_4 & \omega'_5 & \omega'_6 \end{bmatrix}$$

$\diamondsuit$

End-effector force

$$F_t = \begin{bmatrix} \text{force} \\ \text{torque} \end{bmatrix}$$

$$W = \int_{t_1}^{t_2} V_{st}^b \cdot F_t \mathrm{d}t = \int_{t_1}^{t_2} \dot{\theta} \cdot \tau \mathrm{d}t = \int_{t_1}^{t_2} \dot{\theta}^T (J_{st}^b(\theta))^T \cdot F_t \mathrm{d}t$$

$$\Rightarrow \tau = (J_{st}^b)^T F_t = (J_{st}^s)^T F_s$$

- Given F_t, what τ is required to balance that force?
- If we apply a set of joint torques, what is the resulting end-effector wrench?

Structural force

Structural force: that produces no work on admissible velocity space V^b

$$F^b \cdot V^b = 0, \forall V^b \in \mathrm{Im} J^b_{st}(\theta) \Rightarrow F^b \in (\mathrm{Im} J^b_{st})^\perp$$

◇ Review:

$$\forall A \in \mathbb{R}^{m\times n}, \begin{cases} (\mathrm{Im} A)^\perp = \ker A^T \\ (\ker A)^\perp = \mathrm{Im} A^T \end{cases}$$

$$(\mathrm{Im} J^b_{st})^\perp = \ker (J^b_{st})^T, \tau = (J^b_{st})^T F^b \equiv 0, \forall F^b \in \ker (J^b_{st})^T$$

Example: SCARA manipulator

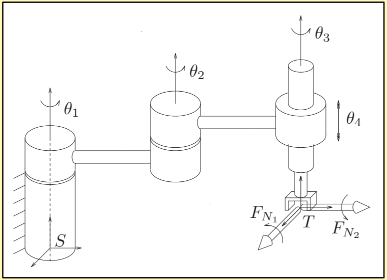


Figure 3.11

$$(J_{st}^s(\theta))^T = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ l_1c_1 & l_1s_1 & 0 & 0 & 0 & 1 \\ l_1c_1 + l_1c_{12} & l_1s_1 + l_1s_{12} & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

$\ker((J_{st}^s(\theta))^T)$: spanned by

$$F_{N_1} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$
$$F_{N_2} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$\diamondsuit$

Singularities

Definition:
θ is called a singular configuration if there $\exists\dot{\theta} \neq 0$ s.t.

$$V_{st}^s = J_{st}^s(\theta)\dot{\theta} = 0$$

or, a singularity config. is a point θ at which J_{st}^s drops rank.

Consequence: $(n = 6)$

1. Can't move in certain directions.
2. Large joint motion is required.
3. Large structural force.
4. Can't apply end-effector force in certain direction force!

Singularities for $6R$-manipulators

Case 1: Two collinear revolute joints

$J(\theta)$ is singular if there exists two joints

$$\xi_1 = \begin{bmatrix} -\omega_1 \times q_1 \\ \omega_1 \end{bmatrix}, \xi_2 = \begin{bmatrix} -\omega_2 \times q_2 \\ \omega_2 \end{bmatrix}$$

s.t.

1. The axes are parallel, $\omega_1 = \pm\omega_2$
2. The axes are collinear, $\omega_i \times (q_1 - q_2) = 0, i = 1, 2$

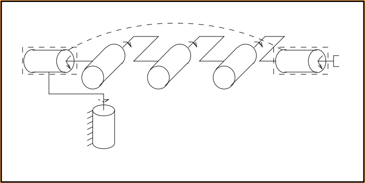


Figure 3.12

Proof :
Elementary row or column operation do not change rank of $J(\theta)$:

$$J(\theta) = \begin{bmatrix} -\omega_1 \times q_1 & -\omega_2 \times q_2 & \cdots \\ \omega_1 & \omega_2 & \cdots \end{bmatrix} \in \mathbb{R}^{6\times n} \xrightarrow{\omega_1=\omega_2}$$

$$J(\theta) \sim \begin{bmatrix} -\omega_1 \times q_1 & -\omega_2 \times (q_2 - q_1) & \cdots \\ \omega_1 & 0 & \cdots \end{bmatrix}$$

$$= \begin{bmatrix} -\omega_1 \times q_1 & 0 & \cdots \\ \omega_1 & 0 & \cdots \end{bmatrix}$$

□

(Continues next slide)

Singularities for 6R-manipulators

Case 2: Three parallel coplanar revolute joint axes

$J(\theta)$ is singular if there exists three joints s.t.

1. The axes are parallel, $\omega_i = \pm\omega_j, i, j = 1, 2, 3$
2. The axes are coplanar, i.e. there exists a plane with normal n s.t.

$$n^T \omega_i = 0, n^T (q_i - q_j) = 0, i, j = 1, 2, 3$$

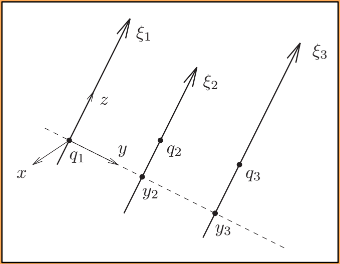


Figure 3.13

(Continues next slide)

Singularities for $6R$-manipulators

Proof :

Using change of frame $J \sim \mathrm{Ad}_g J$ and assume

$$J(\theta) = \begin{bmatrix} -\omega_1 \times q_1 & -\omega_2 \times (q_2 - q_1) & \cdots \\ \omega_1 & \omega_2 & \cdots \end{bmatrix},$$

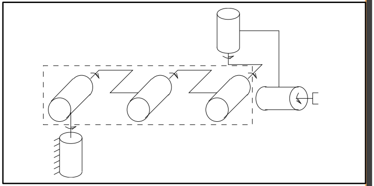


Figure 3.14

$$\mathrm{Ad}_g J(\theta) = \underbrace{\begin{bmatrix} 0 & \pm y_2 & \pm y_3 & \cdots \\ 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ 1 & \pm 1 & \pm 1 & \cdots \end{bmatrix}}_{\text{Linearly dependent}}$$

Examples are such as the Elbow manipulator in its reference configuration. □

(Continues next slide)

Singularities for 6R-manipulators

Case 3: Four intersecting revolute joints axes

$J(\theta)$ is singular if there exists four concurrent revolute joints with intersection point q s.t.:

$$\omega_i \times (q_i - q) = 0, i = 1, \ldots, 4$$

Proof :

Choose the frame origin at q,

$$p = q_i, i = 1, \ldots, 4$$

$$J(\theta) = \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots \\ \omega_1 & \omega_2 & \omega_3 & \omega_4 & \cdots \end{bmatrix}$$

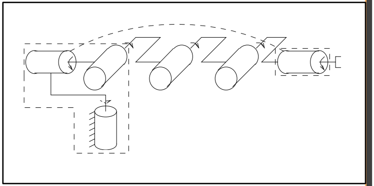


Figure 3.15

Manipulability

- **Jacobian relation of $g : \theta \in Q \mapsto g(\theta) \in SE(3)$**

$$V = J(\theta)\dot{\theta} \qquad (*)$$

- **Inverse Jacobian:**

  Given $v \in \mathbb{R}^n$, solve for $\dot{\theta} \in \mathbb{R}^n$ from $(*)$

- **Application: Kinematic control by Inverse Jacobian**
  - Input: A desired $g_d(t) \in SE(3), t \in [0, T]$
  - Output: $\theta(k) = \theta(k\Delta T), \Delta T:$ Sampling period, $k = 1, \dots, N = [T/\Delta T]$
  - Step 1: Let $g_d(k+1) = g(k)e^{\hat{V}\Delta T} = g(\theta(k))e^{\hat{V}\Delta T}$, solve for

$$\hat{V}\Delta T = \log(g^{-1}(k) \cdot g_d(k+1))$$

  - Step 2: Solve for $\dot{\theta}(k)$ from $V = J(\theta(k)) \cdot \dot{\theta}(k)$ and update

$$\theta(k+1) = \theta(k) + \dot{\theta}(k)\Delta T$$

Local manipulability measures $\Leftrightarrow$ Properties of J, or $(*)$

Singular Value Decomposition

Given $A : \mathbb{R}^n \mapsto \mathbb{R}^m$, let $r = \mathrm{rank}(A)$, then:

$$\dim(R(A)) = \dim(R(A^T)) = r$$

$$\dim(\eta(A)) = n - r, \dim(\eta(A^T)) = m - r$$

$$\mathbb{R}^n = R(A^T) \oplus \eta(A)$$

$$\mathbb{R}^m = R(A) \oplus \eta(A^T)$$

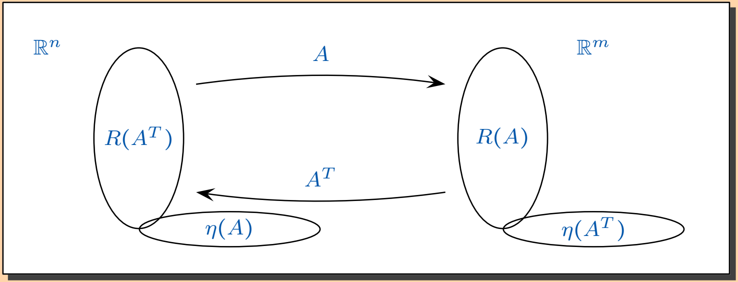


Figure 3.16

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Singular Value Decomposition

SVD of A: $A = U\Sigma V^T$

where: $U = [u_1 \cdots u_r, u_{r+1} \cdots u_m] \triangleq [U_1 | U_2] \in \mathbb{R}^{m\times m}$
$V = [v_1 \cdots v_r, v_{r+1} \cdots v_n] \triangleq [V_1 | V_2] \in \mathbb{R}^{n\times n}$

are orthogonal, i.e. $U \in O(m), V \in O(n)$, or $U^T U = I_m$, $V^T V = I_n$, and

$$\Sigma = \begin{bmatrix} \sigma_1 & & & & & \\ & \ddots & & & & \\ & & \sigma_p & & & \\ & & & 0 & \cdots & 0 \\ & & & \vdots & \ddots & \vdots \\ & & & 0 & \cdots & 0 \end{bmatrix},$$

$p = \min(m, n)$, where $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > \sigma_{r+1} = \cdots = \sigma_p = 0$

- σ_i: Singular value of A, $\sigma_{\max}(A) = \sigma_1$
- u_i, v_i: i^{th} left (right) singular vector of A: $\begin{aligned} Av_i &= \sigma_i u_i \\ A^T u_i &= \sigma_i v_i \end{aligned}$

Properties of SVD

- $A = \sum_{i=1}^{r} \sigma_i u_i v_i^T \Rightarrow (A^T A) v_k = \sigma_k^2 v_k$ or $\lambda(A^T A) = \{ \sigma_1^2, \sigma_2^2, \ldots, \sigma_r^2, 0, \ldots, 0\}$

- $\mathrm{span}(V_1) = R(A^T), \mathrm{span}(V_2) = \eta(A)$
  $\mathrm{span}(U_1) = R(A), \mathrm{span}(U_2) = \eta(A^T)$

- Let $n = m = r$, then A maps the unit sphere $S^{n-1} = \{x \in \mathbb{R}^n | \|x\|_2 = 1\}$ to an ellipsoid with semi-axes $\sigma_i u_i$.

- $\|A\|_F^2 = \sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}|^2 = \sum_{i=1}^{r} \sigma_i^2, \|A\|_2 = \sigma_1$

- Sensitivity Analysis for $Ax = b, m = n = r$

$$A(x + \delta x) = b + \delta b \Rightarrow A\delta x = \delta b$$

(Continues next slide)

Properties of SVD

$$\frac{\|\delta x\|}{\|x\|} \Big/ \frac{\|\delta b\|}{\|b\|} = \frac{\|b\|}{\|x\|}\frac{\|\delta x\|}{\|\delta b\|} = \frac{\|Ax\|}{\|x\|}\frac{\|A^{-1}\delta b\|}{\|\delta b\|}$$
$$\leq \|A\|\|A^{-1}\|$$
$$\triangleq k(A) := \frac{\sigma_1(A)}{\sigma_r(A)}, \text{ condition number}, k(A) \geq 1$$

- Frobenius condition number: $k_F(A) = \frac{1}{n}\sqrt{\mathrm{tr}(AA^T)\mathrm{tr}(AA^T)^{-1}}$
- Manipulability Measures:

$$\mu_1(\theta) = \sigma_{\min}(J(\theta))$$

$$\mu_2(\theta) = \frac{\sigma_{\min}(J(\theta))}{\sigma_{\max}(J(\theta))} \triangleq k^{-1}(J(\theta))$$

$$\mu_3(\theta) = \det(J(\theta)) = \prod_{i=1}^{n}\sigma_i(J(\theta))$$

$$\mu_4(\theta) = k_F^{-1}(J(\theta))$$

♢

† End of Section †

Redundant manipulator

Definition:
A manipulator is kinematically redundant if the number of independently controllable joints is greater than the dimension of the task space.

⋄ Example:

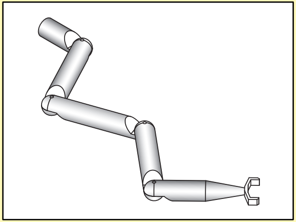


Figure 3.17

$$T = SE(2)$$
$$Q = S^1 \times S^1 \times S^1 \times S^1$$

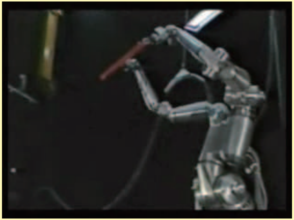


Figure 3.18

$$T = SE(3), Q = \Gamma^n$$

Redundant manipulator

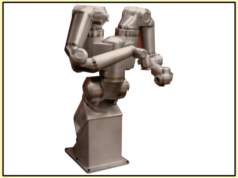


Figure 3.19: 17-DoF manipulator

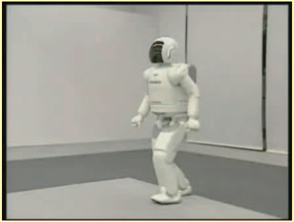


Figure 3.21: Honda's Asimo with 34 DoF (3 in the head, 7 in each arm, 2 in each hand, 1 in the torso, 6 in each leg.)

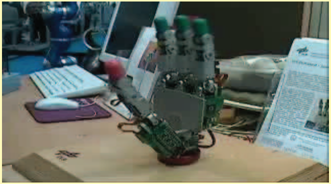


Figure 3.20: DLR hand of 4 identical fingers with 4 joints and 3 degrees of freedom each.

Figure 3.22: OCTARM, a hyperredundant (continuum) manipulator with 27 DoF

Redundant manipulator

- **Main use of Redundancy**
  - Avoid singularities, joint limits and workspace obstacles;
  - Optimize certain cost such as joint torque and energy

□ Self-Motion Manifold and Internal Motion:

- **Forward Kinematic Map**

$$g_{st}(\theta) = e^{\hat{\xi}_1\theta_1}\cdots e^{\hat{\xi}_n\theta_n} g_{st}(0), n > p, \text{ task space dimension.}$$

$$r = n - p(\geq 1): \text{ Degree of redundancy}$$

$$r \gg 1: \text{Hyperredundant}$$

- **Jacobian**

$$J(\theta)\dot{\theta} = V, V \in \mathbb{R}^p, \dot{\theta} \in \mathbb{R}^n$$

(Continues next slide)

Redundant manipulator

- **Self-motion manifold**

$$Q_s = \{\theta \in Q | g_{st}(\theta) = g_d\}$$

- **Internal motion space**

$$T_\theta Q_s = \{\dot\theta \in T_\theta Q | J(\theta)\dot\theta = 0\} \subset T_\theta Q$$

◇ Example:

$$\begin{cases} l_1 \cos\theta_1 + l_2\cos(\theta_1+\theta_2) \\ +l_3\cos(\theta_1+\theta_2+\theta_3) = x \\ l_1\sin\theta_1 + l_2\sin\theta_1 + \theta_2 \\ +l_3\sin(\theta_1+\theta_2+\theta_3) = y \end{cases}$$

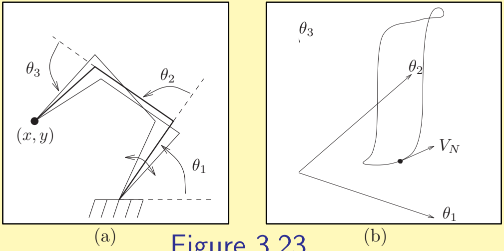


Figure 3.23

$$\frac{\partial p}{\partial\theta} = \left[\begin{array}{c|c|c} -l_1 s_1 - l_2 s_{12} - l_3 s_{123} & -l_2 s_{12} - l_3 s_{123} & -l_3 s_{123} \\ \hline l_1 c_1 + l_2 c_{12} + l_3 c_{123} & l_2 c_{12} + l_3 c_{123} & l_3 c_{123} \end{array}\right],$$

$$v_N = \begin{bmatrix} l_2 l_3 s_3 \\ -l_2 l_3 s_3 - l_1 l_3 s_{23} \\ l_1 l_2 s_3 + l_1 l_3 s_{23} \end{bmatrix}$$

◇

Redundant manipulator

◇ **Example:**

A representation Fix your palm on the table, and then move your shoulder and elbow joints. This gives the self-motion manifold and the internal motion of the 7-DoF redundant robot shown in Fig. 1.

□ Redundancy Resolution:

$$V = J\dot{\theta}, J \in \mathbb{R}^{p\times n}, n > p, \mathrm{rank}(J) = k \leq p < n$$

Strategy for $\dot{\theta} \in \mathbb{R}^n$ **given** $V \in \mathbb{R}^p$**.**

Case 1: $k = p$

Case 2: $k < p$

Review: Least Square Problems

Consider

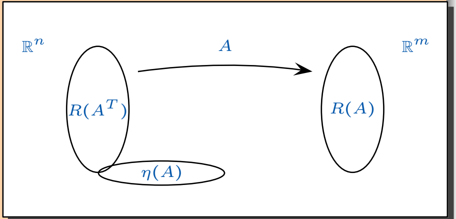
$$Ax = b, A \in \mathbb{R}^{m\times n}, n > m,$$
$$\mathrm{rank}(A) = k.$$

Case 1:

$$k = m \Rightarrow \dim(\eta(A)) = n - k > 0,$$
$$R(A) = \mathbb{R}^m$$

P1: $$\min_{x\in\mathbb{R}^n} \frac{1}{2}\|x\|^2$$

s.t. $$Ax - b = 0$$



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Review: Least Square Problems

Solution: $\varphi(x,\lambda) = \frac{1}{2}\|x\|^2 - \lambda(Ax-b), \lambda \in \mathbb{R}^m$

$$\left(\frac{\partial\varphi}{\partial x}\right)^T = x - A^T\lambda = 0 \Rightarrow x = A^T\lambda$$

$$\left(\frac{\partial\varphi}{\partial \lambda}\right)^T = Ax - b = 0 \Rightarrow AA^T\lambda = b$$

$$\Rightarrow x = A^T(AA^T)^{-1}b$$

$$\triangleq A^+b$$

$A^+ = A^T(AA^T)^{-1} \in \mathbb{R}^{m\times m}$: Moore-Penrose Inverse

In terms of SVD: $A = U\Sigma V^T = \sum_{i=1}^m \sigma_i u_i v_i^T \Rightarrow A^+ = \sum_{i=1}^m \frac{1}{\sigma_i} v_i u_i^T$

$$x = \left(\sum_{i=1}^m \frac{1}{\sigma_i} v_i u_i^T\right)b = \sum_{i=1}^m \frac{u_i^T b}{\sigma_i} v_i$$

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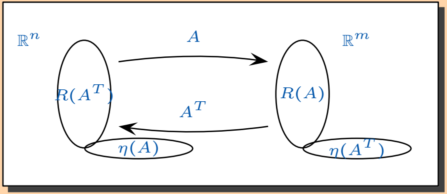
Review: Least Square Problems

Case 2:

$$k < m \Rightarrow \dim(\eta(A^T)) = m - k$$



P2: $$\min_{x\in\mathbb{R}^n} f(x)$$

$$f(x) = \|Ax - b\|^2 + \lambda^2\|x\|^2$$

Solution: $$\left(\frac{\partial f}{\partial x}\right)^T = (AA^T + \lambda^2 I)x - A^T b = 0$$

$$\Rightarrow x = (A^T A + \lambda^2 I)A^T b$$

$$= \sum_{i=1}^{k} \frac{\sigma_i}{\sigma_i^2 + \lambda^2} v_i u_i^T b$$

Redundancy Resolution

$$J\dot{\theta} = V, J \in \mathbb{R}^{p\times n}$$

Case 1: If $\mathrm{rank}(J) = p$, the minimum-norm solution is given by

$$\boxed{\dot{\theta} = J^{+}V = \sum_{i=1}^{p} \frac{u_i^T V}{\sigma_i} v_i} \qquad (*)$$

- If $\sigma_i \ll 1$, then for $V = u_i, \|V\| = 1$, $\dot{\theta} = \frac{1}{\sigma_i} v_i \Rightarrow \|\dot{\theta}\| \frac{1}{\sigma_i} \gg 1$, large joint rate needed.
- For cyclic trajectory in task space, $(*)$ does not give cyclic trajectory in joint space (see [13] Chapter 2)

Redundancy Resolution

General Solution:

$$\boxed{\dot{\theta} = \underbrace{J^{+}V}_{\in R(J^T)} + \underbrace{(I - J^{+}J)\dot{\theta}_0}_{\in \eta(J)}, \dot{\theta}_0 \in \mathbb{R}^n}$$

- **How to select $\dot{\theta}_0 \in \mathbb{R}^n$ so as to stay away from singularity, joint limits or workspace obstacles?**

Let

$$\varphi(\theta) = \begin{cases} \mu_1^{-1}(\theta) = \sigma_{\min}^{-1}(J) \\ \mu_2^{-1}(\theta) = \frac{\sigma_{\max}(J)}{\sigma_{\min}(J)} \\ \mu_3^{-1}(\theta) = \frac{1}{\det^{1/2}(JJ^{-1})} \end{cases}$$

for *singularity avoidance*.

(Continues next slide)

Redundancy Resolution

or

$$\varphi(\theta) = \frac{1}{2}\sum_{i=1}^{n}\left(\frac{\theta_i - \theta_{i,\mathrm{mid}}}{\theta_{i,\max} - \theta_{i,\min}}\right)^2$$

for *avoiding joint limits*. Then,

$$\dot{\theta} = J^{+}V - \lambda_{\varphi}\nabla\varphi(\theta) \qquad (\Delta)$$

where $\nabla\varphi(\theta) \in \mathbb{R}^n$: gradient of φ,
$\lambda_{\varphi} \in \mathbb{R}$: step size (see [13] on selection of λ_{φ})

Note (Δ) minimizes

$$L(\theta, \dot{\theta}) = \frac{1}{2}\dot{\theta}^T\dot{\theta} + K_{\varphi}\dot{\theta}^T\nabla\varphi(\theta)$$

Redundancy Resolution

- **Damped Least-Square:**

$$J\dot{\theta} = V, J \in \mathbb{R}^{p \times n}, p < n$$

$$\begin{aligned}\dot{\theta} &= (J^T J + \lambda^2 I)^{-1} J^T V \\ &= J^T (J J^T + \lambda^2 I)^{-1} V\end{aligned}$$

λ : Dampening coefficient. See [10] on selection of λ.
In terms of SVD:

$$\dot{\theta} = \sum_{i=1}^{k} \frac{\sigma_i}{\sigma_i^2 + \lambda^2} v_i u_i^T V$$

† End of Section †

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