Chapter 4 Manipulator Dynamics

Lecture Notes for

A Geometrical Introduction to Robotics and Manipulation

Richard Murray and Zexiang Li and Shankar S. Sastry
CRC Press

Zexiang Li[1] and Yuanqing Wu[1]

[1]ECE, Hong Kong University of Science & Technology

June 27, 2012

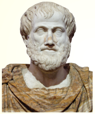
Table of Contents

Chapter 4 Manipulator Dynamics

Introduction

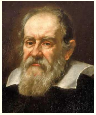
Definition: Dynamics
Physical laws governing the motions of bodies and aggregates of bodies.

☐ A short history:



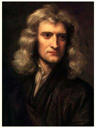
"Everything happens for a reason."

Aristotle (384 BC - 322 BC)



Experiments with cannon balls from the tower of Pisa.

G. Galilei (1564 - 1642)



Laws of motion.

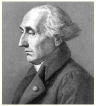
I. Newton (1642 - 1726)

(Continues next slide)

Introduction

Laws of motion from particles to rigid bodies.

L. Euler (1707 - 1783)



Calculus of Variation and the Principles of least action.

J. Lagrange (1736 - 1813)

Quaternions and Hamilton's Principle.

W. Hamilton (1805 - 1865)

† End of Section †

A simple example

Newton's Equation:

$$m\ddot{x} = F_x$$
$$m\ddot{y} = F_y - mg$$

Momentum: $P_x = m\dot{x}$

$$P_y = m\dot{y}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}P_x = F_x, \frac{\mathrm{d}}{\mathrm{d}t}P_y = F_y - mg$$

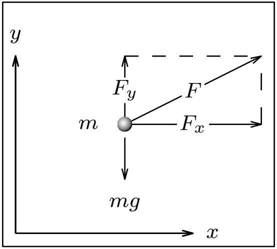


Figure 4.1

$\Leftrightarrow$

Lagrangian Equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = F_x$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{y}} - \frac{\partial L}{\partial y} = F_y$$

Lagrangian function:

$$L = T - V, P_x = \frac{\partial L}{\partial \dot{x}}, P_y = \frac{\partial L}{\partial \dot{y}}$$

Kinetic energy:

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$$

Potential energy:

$$V = mgy$$

Generalization to multibody systems

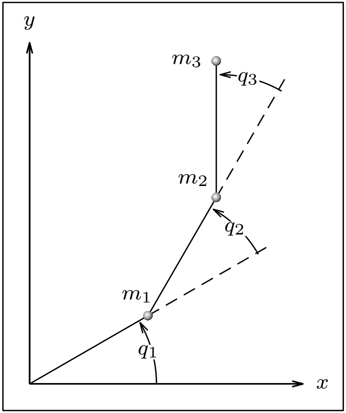


Figure 4.2

$q_i, i = 1, \dots, n$: generalized coordinates

Kinetic energy:

$$T = T(q, \dot{q})$$

Potential energy:

$$V = V(q)$$

Lagrangian:

$$L(q, \dot{q}) = T(q, \dot{q}) - V(q)$$

$\tau_i, i = 1, \dots, n$: external force on q_i

Lagrangian Equation:

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = \tau_i, i = 1, \dots, n}$$

Example: Pendulum equation

Generalized coordinate:

$$\theta \in S^1$$

Kinematics:

$$x = l\sin\theta, y = -l\cos\theta$$

$$\dot{x} = l\cos\theta\cdot\dot{\theta}, \dot{y} = l\sin\theta\cdot\dot{\theta}$$

Kinetic energy:

$$T(\theta,\dot{\theta}) = \frac{1}{2}m(\dot{x}^2+\dot{y}^2) = \frac{1}{2}ml^2\dot{\theta}^2$$

Potential energy:

$$V = mgl(1-\cos\theta)$$

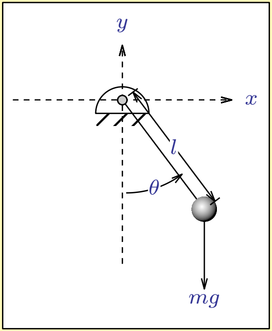


Figure 4.3

Lagrangian function:

$$L = T - V = \frac{1}{2}ml^2\dot{\theta}^2 - mgl(1-\cos\theta), \Rightarrow \frac{\partial L}{\partial\dot{\theta}} = ml^2\dot{\theta}, \frac{\partial L}{\partial\theta} = -mgl\sin\theta$$

Equation of motion:

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial\dot{\theta}} - \frac{\partial L}{\partial\theta} = \tau \Rightarrow ml^2\ddot{\theta} + mgl\sin\theta = \tau$$

◇

Example: a spherical pendulum

Generalized coordinate:

$$r(\theta, \phi) = \begin{bmatrix} l \sin\theta \cos\phi \\ l \sin\theta \sin\phi \\ -l \cos\theta \end{bmatrix}$$

Kinetic energy:

$$T = \frac{1}{2} m \|\dot{r}\|^2 = \frac{1}{2} m l^2 (\dot{\theta}^2 + (1 - \cos^2\theta)\dot{\phi}^2)$$

Potential energy:

$$V = -mgl\cos\theta$$

Lagrangian function:

$$L(q, \dot{q}) = \frac{1}{2} m l^2 (\dot{\theta} + (1 - \cos^2\theta)\dot{\phi}^2) + mgl\cos\theta$$

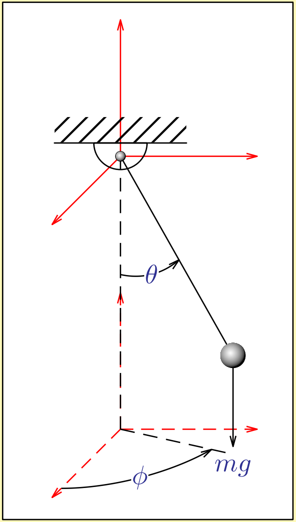


Figure 4.4
(continues next slide)

Example: a spherical pendulum

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\dfrac{\partial L}{\partial \dot{\theta}} = \dfrac{\mathrm{d}}{\mathrm{d}t}(ml^2\dot{\theta}) = ml^2\ddot{\theta}, \\ \dfrac{\partial L}{\partial \theta} = ml^2 \sin\theta\cos\theta\dot{\phi}^2 - mgl\sin\theta \end{cases}$$

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\dfrac{\partial L}{\partial \dot{\phi}} = \dfrac{\mathrm{d}}{\mathrm{d}t}(ml^2\sin^2\theta\dot{\phi}) = ml^2\sin^2\theta\ddot{\phi} + 2ml^2\sin\theta\cos\theta\dot{\theta}\dot{\phi}, \\ \dfrac{\partial L}{\partial \phi} = 0 \end{cases}$$

$$\begin{bmatrix} ml^2 & 0 \\ 0 & ml^2 s_\theta^2 \end{bmatrix}\begin{bmatrix} \ddot{\theta} \\ \ddot{\phi} \end{bmatrix} + \begin{bmatrix} -ml^2 s_\theta c_\theta \dot{\phi}^2 \\ 2ml^2 s_\theta c_\theta \dot{\theta}\dot{\phi} \end{bmatrix} + \begin{bmatrix} mgls_\theta \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

◇

Kinetic energy of a rigid body

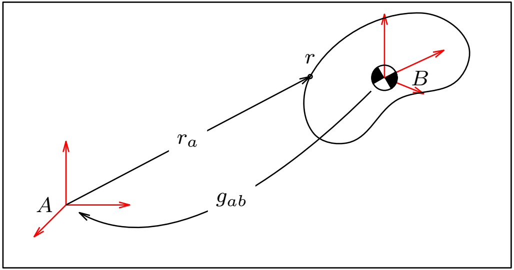


Figure 4.5

Volume occupied by the body:	V
Mass density:	$\rho(r)$
Mass:	$m = \int_V \rho(r) \mathrm{d}V$
Mass center:	$\overline{r} \triangleq \frac{1}{m} \int_V \rho(r) r \mathrm{d}V$
Relative to frame at the mass center:	$\overline{r} = 0$

(Continues next slide)

Kinetic energy of a rigid body

Kinetic energy (In A-frame):

$$T = \frac{1}{2}\int_V \rho(r)\|\dot{p} + \dot{R}r\|^2 \mathrm{d}V = \frac{1}{2}\int_V \rho(r)(\|\dot{p}\|^2 + 2\dot{p}^T\dot{R}r + \|\dot{R}r\|^2)\mathrm{d}V$$

$$= \frac{1}{2}m\|\dot{p}\|^2 + \underbrace{\dot{p}^T\dot{R}\int_V \rho(r)r\mathrm{d}V}_{=0} + \boxed{\frac{1}{2}\int_V \rho(r)\|\dot{R}r\|^2\mathrm{d}V}$$

$$\boxed{\begin{aligned}&\frac{1}{2}\int_V \rho(r)\|\dot{R}r\|^2\mathrm{d}V\\ &= \frac{1}{2}\int \rho(r)\|R^T\dot{R}r\|^2\mathrm{d}V = \frac{1}{2}\int \rho(r)\|\hat{\omega}r\|^2\mathrm{d}V = \frac{1}{2}\int \rho(r)\|\hat{r}\omega\|^2\mathrm{d}V\\ &= \frac{1}{2}\int \rho(r)(-\omega^T\hat{r}^2\omega)\mathrm{d}V = \frac{1}{2}\omega^T\left(-\int \rho(r)\hat{r}^2\mathrm{d}V\right)\omega \triangleq \frac{1}{2}\omega^T\mathcal{I}\omega\end{aligned}}$$

(Continues next slide)

Kinetic energy of a rigid body

where

$$\mathcal{I} = -\int \rho(r)\hat{r}^2 \mathrm{d}V \triangleq \begin{bmatrix} \mathcal{I}_{xx} & \mathcal{I}_{xy} & \mathcal{I}_{xz} \\ \mathcal{I}_{xy} & \mathcal{I}_{yy} & \mathcal{I}_{yz} \\ \mathcal{I}_{xz} & \mathcal{I}_{yz} & \mathcal{I}_{zz} \end{bmatrix}$$

is the *Inertia tensor* with

$$\mathcal{I}_{xx} = \int \rho(r)(y^2 + z^2)\mathrm{d}x\mathrm{d}y\mathrm{d}z, \quad \mathcal{I}_{xy} = -\int \rho(r)xy\mathrm{d}x\mathrm{d}y\mathrm{d}z$$

$$T = \frac{1}{2}m\|\dot{p}\|^2 + \frac{1}{2}(\omega^b)^T \mathcal{I}\omega^b = \frac{1}{2}m\|R^T\dot{p}\|^2 + \frac{1}{2}(\omega^b)^T\mathcal{I}\omega^b$$

$$= \frac{1}{2}(V^b)^T \underbrace{\begin{bmatrix} mI & 0 \\ 0 & \mathcal{I} \end{bmatrix}}_{M^b} V^b$$

M^b: Generalized inertia matrix in B-frame. ♢

Example: M^b for a rectangular object

$$
\begin{aligned}
\rho &= \frac{m}{l\omega h}\\
\mathcal{I}_{xx} &= \int_V \rho(y^2+z^2)\mathrm{d}x\mathrm{d}y\mathrm{d}z\\
&= \rho \int_{-\frac{h}{2}}^{\frac{h}{2}} \int_{-\frac{\omega}{2}}^{\frac{\omega}{2}} \int_{-\frac{l}{2}}^{\frac{l}{2}} (y^2+z^2)\mathrm{d}x\mathrm{d}y\mathrm{d}z\\
&= \rho\left(\frac{1}{12}(l\omega^3 h + l\omega h^3)\right) = \frac{m}{12}(\omega^2+h^2)\\
\mathcal{I}_{xy} &= -\int_V \rho xy\mathrm{d}V = -\rho \int_{-\frac{h}{2}}^{\frac{h}{2}} \int_{-\frac{\omega}{2}}^{\frac{\omega}{2}} \int_{-\frac{l}{2}}^{\frac{l}{2}} xy\mathrm{d}x\mathrm{d}y\mathrm{d}z\\
&= -\rho \int_{-\frac{h}{2}}^{\frac{h}{2}} \int_{-\frac{\omega}{2}}^{\frac{\omega}{2}} \frac{y}{2}x^2\Big|_{-\frac{l}{2}}^{\frac{l}{2}} \mathrm{d}y\mathrm{d}z = 0
\end{aligned}
$$

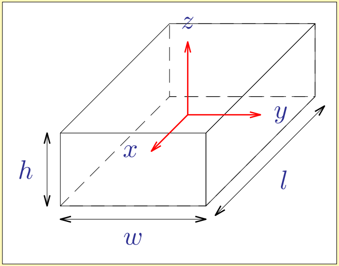


Figure 4.6

(Continues next slide)

Example: M^b for a rectangular object

$$\mathcal{I} = \begin{bmatrix} \frac{m}{12}(w^2+h^2) & 0 & 0 \\ 0 & \frac{m}{12}(l^2+h^2) & 0 \\ 0 & 0 & \frac{m}{12}(w^2+l^2) \end{bmatrix}, M^b = \begin{bmatrix} mI_{3\times 3} & 0 \\ 0 & \mathcal{I} \end{bmatrix}$$

□ M^b under change of frames:

$$\hat{V}_1 = g_1^{-1}\cdot \dot{g}_1,\ T = \frac{1}{2}V_1^T M_1^b V_1$$

$$V_1 = \mathrm{Ad}_{g_0} V_2$$

$$T = \frac{1}{2}(\mathrm{Ad}_{g_0}V_2)^T M_1^b (\mathrm{Ad}_{g_0}V_2)$$

$$= \frac{1}{2}V_2^T \mathrm{Ad}_{g_0}^T M_1^b \mathrm{Ad}_{g_0} V_2 \triangleq \frac{1}{2}V_2^T M_2^b V_2$$

$$\boxed{M_2^b = \mathrm{Ad}_{g_0}^T M_1^b \mathrm{Ad}_{g_0}}$$

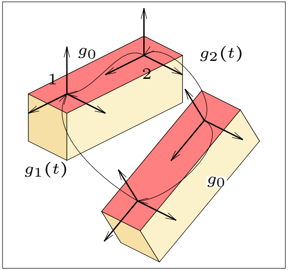


Figure 4.7

Example: Dynamics of a 2-dof planar robot

$$\mathcal{I}_i = \begin{bmatrix} \mathcal{I}_{xx_i} & 0 & 0 \\ 0 & \mathcal{I}_{yy_i} & 0 \\ 0 & 0 & \mathcal{I}_{zz_i} \end{bmatrix}, i = 1, 2$$

$$T(\theta, \dot{\theta}) = \frac{1}{2} m_1 \|v_1\|^2 + \frac{1}{2} \omega_1^T \mathcal{I}_1 \omega_1 + \frac{1}{2} m_2 \|v_2\|^2 + \frac{1}{2} \omega_2^T \mathcal{I}_2 \omega_2$$

$$\omega_1 = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix} \quad \omega_2 = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 + \dot{\theta}_2 \end{bmatrix}$$

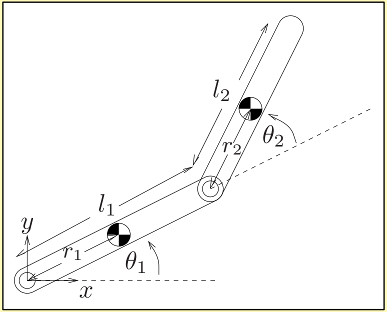


Figure 4.8

(Continues next slide)

Example: Dynamics of a 2-dof planar robot

$P_i = \begin{bmatrix} \overline{x}_i \\ \overline{y}_i \\ 0 \end{bmatrix}$: Mass center

γ_i : Distance from joint i to mass center

Change of Coordinates:

$$\begin{cases} \overline{x}_1 = r_1 c_1 \\ \overline{y}_1 = r_1 s_1 \end{cases} \Rightarrow \begin{cases} \dot{\overline{x}}_1 = -r_1 s_1 \dot{\theta}_1 \\ \dot{\overline{y}}_1 = r_1 c_1 \dot{\theta}_1 \end{cases}$$

$$\begin{cases} \overline{x}_2 = l_1 c_1 + r_2 c_{12} \\ \overline{y}_2 = l_1 s_1 + r_2 s_{12} \end{cases} \Rightarrow \begin{cases} \dot{\overline{x}}_2 = -(l_1 s_1 + r_2 s_{12})\dot{\theta}_1 - r_2 s_{12} \dot{\theta}_2 \\ \dot{\overline{y}}_2 = (l_1 c_1 + r_2 c_{12})\dot{\theta}_1 + r_2 c_{12} \dot{\theta}_2 \end{cases}$$

(Continues next slide)

Example: Dynamics of a 2-dof planar robot

Kinetic energy:

$$T(\theta,\dot{\theta}) = \frac{1}{2}m_1(\dot{\bar{x}}_1^2+\dot{\bar{y}}_1^2)+\frac{1}{2}\mathcal{I}_{zz_1}\dot{\theta}_1^2+\frac{1}{2}m_2(\dot{\bar{x}}_2^2+\dot{\bar{y}}_2^2)+\frac{1}{2}\mathcal{I}_{zz_2}(\dot{\theta}_1+\dot{\theta}_2)^2$$
$$= \frac{1}{2}[\dot{\theta}_1 \ \dot{\theta}_2]\underbrace{\begin{bmatrix}\alpha+2\beta c_2 & \delta+\beta c_2\\ \delta+\beta c_2 & \delta\end{bmatrix}}_{M(\theta)}\begin{bmatrix}\dot{\theta}_1\\ \dot{\theta}_2\end{bmatrix}$$

$$\alpha = \mathcal{I}_{zz_1}+\mathcal{I}_{zz_2}+m_1r_1^2+m_2(l_1^2+r_2^2),$$
$$\beta = m_2l_1r_2, \delta = \mathcal{I}_{zz_2}+m_2r_2^2, L = T$$

Equation of motion:

$$M(\theta)\begin{bmatrix}\ddot{\theta}_1\\ \ddot{\theta}_2\end{bmatrix}+\begin{bmatrix}-\beta s_2\dot{\theta}_2 & -\beta s_2(\dot{\theta}_1+\dot{\theta}_2)\\ \beta s_2\dot{\theta}_1 & 0\end{bmatrix}\begin{bmatrix}\dot{\theta}_1\\ \dot{\theta}_2\end{bmatrix}=\begin{bmatrix}\tau_1\\ \tau_2\end{bmatrix} \quad \diamond$$

† End of Section †

Dynamics of Open-chain Manipulators

Definition:

L_i: frame at mass center of link i, $g_{sl_i}(\theta) = e^{\hat{\xi}_1\theta_1}\cdots e^{\hat{\xi}_i\theta_i}g_{sl_i}(o)$

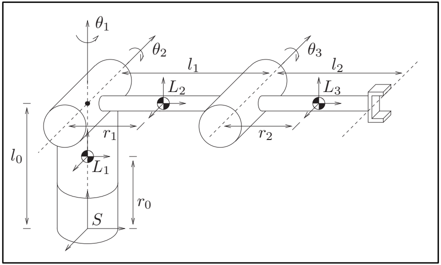


Figure 4.9

$$V^b_{sl_i} = J^b_{sl_i}(\theta)\dot{\theta} = [\xi_1^\dagger\ \xi_2^\dagger\ \cdots\ \xi_i^\dagger\ 0 \cdots 0]\begin{bmatrix}\dot{\theta}_1\\ \vdots\\ \dot{\theta}_i\\ \dot{\theta}_{i+1}\\ \vdots\\ \dot{\theta}_n\end{bmatrix} = J_i(\theta)\dot{\theta}$$

(Continues next slide)

Dynamics of Open-chain Manipulators

$$\xi_j^\dagger = \mathrm{Ad}^{-1}(e^{\hat{\xi}_{j+1}\theta_{j+1}}\cdots e^{\hat{\xi}_i\theta_i} g_{sl_i}(0))\xi_j, \ j \le i$$

$$T_i(\theta, \dot\theta) = \frac{1}{2}(V^b_{sl_i})^T M_i^b V^b_{sl_i} = \frac{1}{2}\dot\theta^T J_i^T(\theta) M_i^b J_i(\theta)\dot\theta$$

$$T(\theta) = \sum_{i=1}^n T_i(\theta, \dot\theta) = \frac{1}{2}\dot\theta^T M(\theta)\dot\theta,$$

$$M(\theta) = \sum_i J_i^T(\theta) M_i^b J_i(\theta) = \frac{1}{2}\sum_{i,j=1}^n M_{ij}(\theta)\dot\theta_i\dot\theta_j$$

$h_i(\theta)$: Height of L_i, $\quad V_i(\theta) = m_i g h_i(\theta), \quad V(\theta) = \sum_{i=1} m_i g h_i(\theta)$

Lagrange's Equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot\theta_i} - \frac{\partial L}{\partial \theta_i} = \tau_i, \ i = 1, \ldots, n,$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot\theta_i} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_{j=1}^n M_{ij}\dot\theta_j\right) = \sum_{j=1}^n M_{ij}\ddot\theta_j + \dot M_{ij}\dot\theta_j$$

(Continues next slide)

Dynamics of Open-chain Manipulators

$$\frac{\partial L}{\partial \theta_i} = \frac{1}{2}\sum_{j,k=1}^{n} \frac{\partial M_{kj}}{\partial \theta_i}\dot{\theta}_k\dot{\theta}_j - \frac{\partial V}{\partial \theta_i}, \qquad \dot{M}_{ij} = \sum_k \frac{\partial M_{ij}}{\partial \theta_k}\dot{\theta}_k$$

$$\Rightarrow \sum_{j=1}^{n} M_{ij}\ddot{\theta}_j + \sum_{j,k=1}^{n}\left(\frac{\partial M_{ij}}{\partial \theta_k}\dot{\theta}_j\dot{\theta}_k - \frac{1}{2}\frac{\partial M_{kj}}{\partial \theta_i}\dot{\theta}_k\dot{\theta}_j\right) + \frac{\partial V}{\partial \theta_i} = \tau_i$$

$$\Rightarrow \sum_{j=1}^{n} M_{ij}\ddot{\theta}_j + \sum_{j,k=1}^{n} \Gamma_{ijk}\dot{\theta}_k\dot{\theta}_j + \frac{\partial V}{\partial \theta_i} = \tau_i$$

$$\Gamma_{ijk} \triangleq \frac{1}{2}\left(\frac{\partial M_{ij}}{\partial \theta_k} + \frac{\partial M_{ik}}{\partial \theta_j} - \frac{\partial M_{kj}}{\partial \theta_i}\right)$$

$\dot{\theta}_i \cdot \dot{\theta}_j, i \neq j$: Coriolis term, $\dot{\theta}_i^2$: Centrifugal term

Define: $C_{ij}(\theta,\dot{\theta}) = \sum_{k=1}^{n} \Gamma_{ijk}\dot{\theta}_k = \frac{1}{2}\sum_{k=1}^{n}\left(\frac{\partial M_{ij}}{\partial \theta_k} + \frac{\partial M_{ik}}{\partial \theta_j} - \frac{\partial M_{kj}}{\partial \theta_i}\right)\dot{\theta}_k$

$$\Rightarrow \boxed{M(\theta)\ddot{\theta} + C(\theta,\dot{\theta})\dot{\theta} + N(\theta) = \tau}$$

Dynamics of Open-chain Manipulators

Property 1:

1. $M(\theta) = M^T(\theta), \dot{\theta}^T M(\theta)\dot{\theta} \geq 0, \dot{\theta}^T M(\theta)\dot{\theta} = 0 \Leftrightarrow \dot{\theta} = 0$
2. $\dot{M} - 2C \in \mathbb{R}^{n\times n}$ is skew symmetric

Proof :

$$
\begin{aligned}
(\dot{M} - 2C)_{ij} &= \dot{M}_{ij} - 2C_{ij}(\theta) \\
&= \sum_{k=1}^{n} \frac{\partial M_{ij}}{\partial \theta_k}\dot{\theta}_k - \frac{\partial M_{ij}}{\partial \theta_k}\dot{\theta}_k - \frac{\partial M_{ik}}{\partial \theta_j}\dot{\theta}_k + \frac{\partial M_{kj}}{\partial \theta_i}\dot{\theta}_k \\
&= \sum_{k=1}^{n} \frac{\partial M_{kj}}{\partial \theta_i}\dot{\theta}_k - \frac{\partial M_{ik}}{\partial \theta_j}\dot{\theta}_k
\end{aligned}
$$

Switching i and j shows $(\dot{M} - 2C)^T = -(\dot{M} - 2C)$ □

Example: Planar 2-DoF Robot (continued)

$$m_{11}(\theta) = \alpha + 2\beta\cos\theta_2, m_{22} = \delta$$

$$m_{12}(\theta) = m_{21}(\theta) = \delta + \beta\cos\theta_2$$

$$c_{11}(\theta,\dot\theta) = -\beta\sin\theta_2\cdot\dot\theta_2, c_{12}(\theta,\dot\theta) = -\beta\sin\theta_2(\dot\theta_1+\dot\theta_2)$$

$$c_{21}(\theta,\dot\theta) = \beta\sin\theta_2\cdot\dot\theta_1, c_{22}(\theta,\dot\theta) = 0$$

$$\Gamma_{111} = \frac{1}{2}(\frac{\partial M_{11}}{\partial\theta_1} + \frac{\partial M_{11}}{\partial\theta_1} - \frac{\partial M_{11}}{\partial\theta_1}) = \frac{1}{2}\frac{\partial M_{11}}{\partial\theta_1} = 0$$

$$\Gamma_{112} = \frac{1}{2}(\frac{\partial M_{11}}{\partial\theta_2} + \frac{\partial M_{12}}{\partial\theta_1} - \frac{\partial M_{21}}{\partial\theta_1}) = \frac{1}{2}\frac{\partial M_{11}}{\partial\theta_2} = -\beta\sin\theta_2$$

$$\Gamma_{121} = \frac{1}{2}(\frac{\partial M_{12}}{\partial\theta_1} + \frac{\partial M_{11}}{\partial\theta_2} - \frac{\partial M_{12}}{\partial\theta_1}) = \frac{1}{2}\frac{\partial M_{11}}{\partial\theta_2} = -\beta\sin\theta_2$$

$$\Gamma_{122} = \frac{1}{2}(\frac{\partial M_{12}}{\partial\theta_2} + \frac{\partial M_{12}}{\partial\theta_2} - \frac{\partial M_{22}}{\partial\theta_1}) = \frac{\partial M_{12}}{\partial\theta_2} - \frac{1}{2}\frac{\partial M_{22}}{\partial\theta_1} = -\beta\sin\theta_2$$

(Continues next slide)

Example: Planar 2-DoF Robot (continued)

$$\Gamma_{211} = \frac{1}{2}\left(\frac{\partial M_{21}}{\partial \theta_1} + \frac{\partial M_{21}}{\partial \theta_1} - \frac{\partial M_{11}}{\partial \theta_2}\right) = \frac{\partial M_{21}}{\partial \theta_1} - \frac{1}{2}\frac{\partial M_{11}}{\partial \theta_2} = \beta \sin\theta_2$$

$$\Gamma_{212} = \frac{1}{2}\left(\frac{\partial M_{21}}{\partial \theta_2} + \frac{\partial M_{22}}{\partial \theta_1} - \frac{\partial M_{21}}{\partial \theta_2}\right) = \frac{1}{2}\frac{\partial M_{22}}{\partial \theta_1} = 0$$

$$\Gamma_{221} = \frac{1}{2}\left(\frac{\partial M_{22}}{\partial \theta_1} + \frac{\partial M_{21}}{\partial \theta_2} - \frac{\partial M_{12}}{\partial \theta_2}\right) = \frac{1}{2}\frac{\partial M_{22}}{\partial \theta_1} = 0$$

$$\Gamma_{222} = \frac{1}{2}\left(\frac{\partial M_{22}}{\partial \theta_2} + \frac{\partial M_{22}}{\partial \theta_2} - \frac{\partial M_{22}}{\partial \theta_2}\right) = \frac{1}{2}\frac{\partial M_{22}}{\partial \theta_2} = 0$$

$$\begin{aligned}\dot{M} - 2C &= \begin{bmatrix} -2\beta\sin\theta_2\cdot\dot\theta_2 & -\beta\sin\theta_2\cdot\dot\theta_2 \\ -\beta\sin\theta_2\cdot\dot\theta_2 & 0 \end{bmatrix} \\ &\quad - \begin{bmatrix} -2\beta\sin\theta_2\cdot\dot\theta_2 & -2\beta\sin\theta_2(\dot\theta_1+\dot\theta_2) \\ 2\beta\sin\theta_2\cdot\dot\theta_1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & \beta\sin\theta_2(2\dot\theta_1+\dot\theta_2) \\ -\beta\sin\theta_2(2\dot\theta_1+\dot\theta_2) & 0 \end{bmatrix} \Leftarrow \text{ skew-symmetric}\end{aligned}$$

♢

Dynamics of a 3-dof robot

$$\xi_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \xi_2 = \begin{bmatrix} 0 \\ -l_0 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}, \xi_3 = \begin{bmatrix} 0 \\ -l_0 \\ l_1 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

$$g_{sl_1}(0) = \begin{bmatrix} I & \begin{pmatrix} 0 \\ 0 \\ r_0 \end{pmatrix} \\ 0 & 1 \end{bmatrix}, g_{sl_2}(0) = \begin{bmatrix} I & \begin{pmatrix} 0 \\ r_1 \\ l_0 \end{pmatrix} \\ 0 & 1 \end{bmatrix},$$

$$g_{sl_3}(0) = \begin{bmatrix} I & \begin{pmatrix} 0 \\ l_1 + r_2 \\ l_0 \end{pmatrix} \\ 0 & 1 \end{bmatrix}$$

$$M_i = \left[\begin{array}{ccc|ccc} m_i & 0 & 0 & & & \\ 0 & m_i & 0 & & 0 & \\ 0 & 0 & m_i & & & \\ \hline & & & \mathcal{I}_{x_i} & 0 & 0 \\ & 0 & & 0 & \mathcal{I}_{y_i} & 0 \\ & & & 0 & 0 & \mathcal{I}_{z_i} \end{array}\right]$$

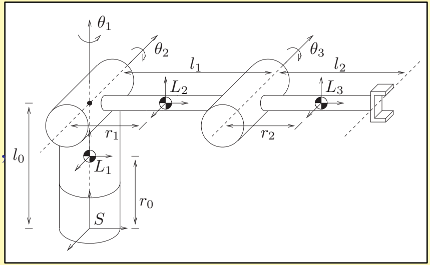


Figure 4.9

(Continues next slide)

Dynamics of a 3-dof robot

m_i: The mass of the object

$\mathcal{I}_{x_i}$: The moment of inertia about the x axis

$$\Gamma_{112} = (\mathcal{I}_{y_2} - \mathcal{I}_{z_2} - m_2 r_1^2) c_2 s_2 + (\mathcal{I}_{y_2} - \mathcal{I}_{z_3}) c_{23} s_{23} - t(l_1 s_2 + r_2 s_{23})$$
$$\Gamma_{113} = (\mathcal{I}_{y_3} - \mathcal{I}_{z_3}) c_{23} s_{23} - t r_2 s_{23}$$
$$\Gamma_{121} = (\mathcal{I}_{y_2} - \mathcal{I}_{z_2} - m_2 r_1^2) c_2 s_2 + (\mathcal{I}_{y_3} - \mathcal{I}_{z_3}) c_{23} s_{23} - t(l_1 s_2 + r_2 s_{23})$$
$$\Gamma_{131} = (\mathcal{I}_{y_3} - \mathcal{I}_{z_3}) c_{23} s_{23} - t r_2 s_{23}$$
$$\Gamma_{211} = (\mathcal{I}_{z_2} - \mathcal{I}_{y_2} + m_2 r_1^2) c_2 s_2 + (\mathcal{I}_{z_3} - \mathcal{I}_{y_3}) c_{23} s_{23} + t(l_1 s_2 + r_2 s_{23})$$
$$\Gamma_{223} = -l_1 m_3 r_2 s_3$$
$$\Gamma_{232} = -l_1 m_3 r_2 s_3$$
$$\Gamma_{233} = -l_1 m_3 r_2 s_3$$
$$\Gamma_{311} = (\mathcal{I}_{z_3} - \mathcal{I}_{y_3}) c_{23} s_{23} + t r_2 s_{23}$$
$$\Gamma_{322} = l_1 m_3 r_2 s_3$$

where $t = m_3(l_1 c_2 + r_2 c_{23})$.

(Continues next slide)

Dynamics of a 3-dof robot

$$N(\theta,\dot{\theta}) = \frac{\partial V}{\partial \theta}, V(\theta) = m_1 g h_1(\theta) + m_2 g h_2(\theta) + m_2 g h_3(\theta)$$

$$g_{sl_i}(\theta) = e^{\hat{\xi}_1\theta_1} \cdots e^{\hat{\xi}_i\theta_i} g_{sl_i}(0) \Rightarrow$$

$$h_1(\theta) = r_0,\ h_2(\theta) = l_0 - r_1 \sin\theta, h_3(\theta) = l_0 - l_1 \sin\theta_2 - r_2 \sin(\theta_2+\theta_3)$$

$$J_1 = J^b_{sl_1}(\theta) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, J_2 = J^b_{sl_2}(\theta) = \begin{bmatrix} -r_1 c_2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -r_1 & 0 \\ 0 & -1 & 0 \\ -s_2 & 0 & 0 \\ c_2 & 0 & 0 \end{bmatrix}$$

$$J_3 = J^b_{sl_3}(\theta) = \begin{bmatrix} -l_2 c_2 - r_2 c_{23} & 0 & 0 \\ 0 & l_1 s_3 & 0 \\ 0 & -r_2 - l_1 c_3 & -r_2 \\ 0 & -1 & -1 \\ -s_{23} & 0 & 0 \\ c_{23} & 0 & 0 \end{bmatrix}$$

(Continues next slide)

Dynamics of a 3-dof robot

$$M(\theta) = \begin{bmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{bmatrix} = J_1^T M_1 J_1 + J_2^T M_2 J_2 + J_3^T M_3 J_3$$

$$M_{11} = I_{y2} s_2^2 + I_{y3} s_{23}^2 + I_{z1} + I_{z2} c_2^2 + I_{z3} c_{23}^2 + m_2 r_1^2 c_2^2 + m_3 (l_1 c_2 + r_2 c_{23})^2$$

$$M_{12} = M_{13} = M_{21} = M_{31} = 0$$

$$M_{22} = I_{x2} + I_{x3} + m_2 l_1^2 + M_2 r_1^2 + m_3 r_2^2 + 2 m_3 l_1 r_2 c_3$$

$$M_{23} = I_{x3} + m_3 r_2^2 + m_3 l_1 r_2 c_3$$

$$M_{32} = I_{x3} + m_3 r_2^2 + m_3 l_1 r_2 c_3$$

$$M_{33} = I_{x_3} + m_3 r_2^2$$

$$C_{ij}(\theta, \dot{\theta}) = \sum_{k=1}^{n} \Gamma_{ijk} \dot{\theta}_k = \frac{1}{2} \sum_{k=1}^{n} \left(\frac{\partial M_{ij}}{\partial \theta_k} + \frac{\partial M_{ik}}{\partial \theta_j} - \frac{\partial M_{kj}}{\partial \theta_i} \right) \dot{\theta}_k \qquad \diamond$$

Additional Properties of the dynamics in terms of POE

Define:

$$A_{ij} = \begin{cases} \mathrm{Ad}^{-1}_{e^{\hat{\xi}_{j+1}\theta_{j+1}} \cdots e^{\hat{\xi}_i\theta_i}} & i > j \\ I & i = j \\ 0 & i < j \end{cases}$$

$$J_i(\theta) = \mathrm{Ad}_{g^{-1}_{sl_i}(0)}[A_{i1}\xi_1 \cdots A_{ii}\xi_i \; 0 \cdots 0]$$

$$M_i' = \mathrm{Ad}^T_{g^{-1}_{sl_i(0)}} M_i \mathrm{Ad}_{g^{-1}_{sl_i(0)}} \quad (\text{ intertia of } i^{th} \text{ link in } S)$$

Property 2:

$$M_{ij}(\theta) = \sum_{l=\max(i,j)}^{n} \xi_i^T A_{li}^T M_l' A_{lj} \xi_j, C_{ij}(\theta, \dot{\theta}) = \frac{1}{2} \sum_{k=1}^{n} \left(\frac{\partial M_{ij}}{\partial \theta_k} + \frac{\partial M_{ik}}{\partial \theta_j} - \frac{\partial M_{kj}}{\partial \theta_i} \right) \dot{\theta}_k$$

where

$$\frac{\partial M_{ij}}{\partial \theta_k} = \sum_{l=\max(i,j)}^{n} \left([A_{k-1,i}\xi_i, \xi_k]^T A_{lk}^T M_l' A_{lj} \xi_j + \xi_i^T A_{li}^T M_l' A_{lk} [A_{k-1,j}\xi_j, \xi_k] \right)$$

† End of Section †

Newton-Euler equations in spatial frame

Newton's Equation:

$$f^s = \frac{d}{dt}(m\dot{p}) = m\ddot{p}$$

Spatial angular momentum:

$$\mathcal{I}^s \cdot \omega^s = R(\mathcal{I} \cdot \omega^b) = \underbrace{R \cdot \mathcal{I} \cdot R^T}_{\mathcal{I}^s} \cdot \omega^s$$

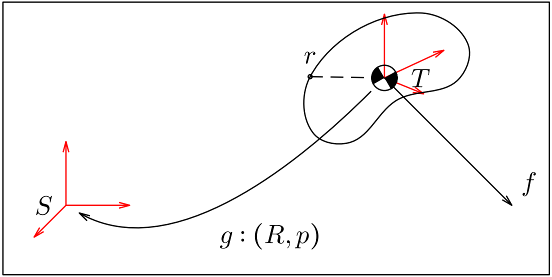


Figure 4.10

(Continues next slide)

Newton-Euler equations in spatial frame

$$\tau^s = \frac{\mathrm{d}}{\mathrm{d}t}(\mathcal{I}^s\omega^s) = \frac{\mathrm{d}}{\mathrm{d}t}(R\mathcal{I}R^T\omega^s) = \mathcal{I}^s\dot{\omega}^s + \dot{R}\mathcal{I}R^T\omega^s + R\mathcal{I}\dot{R}^T\omega^s$$
$$= \mathcal{I}^s\dot{\omega}^s + \underbrace{\dot{R}R^T}_{\hat{\omega}^s}\mathcal{I}^s\omega^s - R\mathcal{I}R^T\hat{\omega}^s\omega^s = \mathcal{I}^s\dot{\omega}^s + \omega^s \times (\mathcal{I}^s\omega^s)$$

- **Transform equations to body frame:**

$$\frac{\mathrm{d}}{\mathrm{d}t}(m\dot{p}) = \frac{\mathrm{d}}{\mathrm{d}t}\left(mRv^b\right) = m\dot{R}v^b + mR\dot{v}^b, R^T f^s = mR^T\dot{R}v^b + m\dot{v}^b$$
$$\Rightarrow f^b = m\omega^b \times v^b + m\dot{v}^b,$$
$$\tau^b = R^T\tau^s = R^T\frac{\mathrm{d}}{\mathrm{d}t}(R\mathcal{I}\omega^b) = \mathcal{I}\dot{\omega}^b + \omega^b \times \mathcal{I}\omega^b$$
$$\Rightarrow \underbrace{\begin{bmatrix} mI & 0 \\ 0 & \mathcal{I} \end{bmatrix}}_{M^b}\underbrace{\begin{bmatrix} \dot{v}^b \\ \dot{\omega}^b \end{bmatrix}}_{\dot{V}^b} + \begin{bmatrix} \omega^b \times mv^b \\ \omega^b \times \mathcal{I}\omega^b \end{bmatrix} = \begin{bmatrix} f^b \\ \tau^b \end{bmatrix} = F^b \qquad (*)$$

(Continues next slide)

Newton-Euler equations in spatial frame

Define:

$$[\ ,\ ] : se(3) \times se(3) \mapsto se(3), [\hat{\xi}_1, \hat{\xi}_2] \triangleq \hat{\xi}_1 \hat{\xi}_2 - \hat{\xi}_2 \hat{\xi}_1$$

if

$$\hat{\xi}_i = \begin{bmatrix} \hat{\omega}_i & v_i \\ 0 & 0 \end{bmatrix}, i = 1, 2$$

then

$$[\hat{\xi}_1, \hat{\xi}_2] = \begin{bmatrix} (\omega_1 \times \omega_2)^\wedge & \hat{\omega}_1 v_2 - \hat{\omega}_2 v_1 \\ 0 & 0 \end{bmatrix} = \mathrm{ad}_{\xi_1} \cdot \xi_2$$

where

$$\mathrm{ad}_{\xi_1} = \begin{bmatrix} \hat{\omega}_1 & \hat{v}_1 \\ 0 & \hat{\omega}_1 \end{bmatrix}$$

It is straightforward computation to see that:

$$(*) \Leftrightarrow M^b \dot{V}^b - \mathrm{ad}^T_{V^b} M^b V^b = F^b$$

Property 3:

$$\mathrm{Ad}_g [\hat{\xi}_1, \hat{\xi}_2] = [\mathrm{Ad}_g \hat{\xi}_1, \mathrm{Ad}_g \hat{\xi}_2] \Rightarrow \mathrm{Ad}_g \mathrm{ad}_{\hat{\xi}_1} = \mathrm{ad}_{\mathrm{Ad}_g \hat{\xi}_1} \mathrm{Ad}_g$$

Coordinate invariance of Newton-Euler equations

$$M_1\dot{V}_1 - \mathrm{ad}^T_{V_1} M_1 V_1 = F_1$$

$$\begin{cases} V_1 = \mathrm{Ad}_{g_0} V_2, \\ F_1 = \mathrm{Ad}^T_{g_0^{-1}} F_2 \end{cases} \Rightarrow$$

$$F_1 = (\mathrm{Ad}^T_{g_0})^{-1} F_2 = (\mathrm{Ad}^{-1}_{g_0})^T F_2$$

$$M_1 = \mathrm{Ad}^{-T}_{g_0} M_2 \mathrm{Ad}^{-1}_{g_0}$$

$$\Rightarrow \mathrm{Ad}^{-T}_{g_0} M_2 \mathrm{Ad}^{-1}_{g_0} \mathrm{Ad}_{g_0} \dot{V}_2 - \mathrm{ad}^T_{(\mathrm{Ad}_{g_0} \dot{V}_2)} \mathrm{Ad}^{-T}_{g_0} M_2 \mathrm{Ad}^{-1}_{g_0} \mathrm{Ad}_{g_0} V_1 = \mathrm{Ad}^{-T}_{g_0} F_2$$

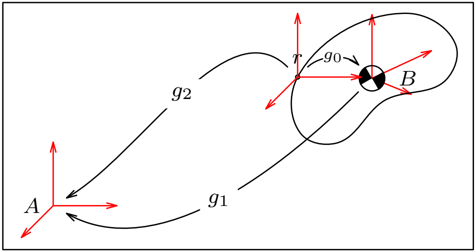


Figure 4.11

Since $\mathrm{ad}_{\mathrm{Ad}_{g_0} V} = \mathrm{Ad}_{g_0} \mathrm{ad}_V \mathrm{Ad}^{-1}_{g_0}$ by Property 3, we have

$$\boxed{M_2\dot{V}_2 - \mathrm{ad}^T_{V_2} M_2 V_2 = F_2}$$

Coordinate invariance of Newton-Euler equations

C_i: Frame fixed to link i, located along the i^{th} axis
F_i: Generalized force link $i-1$ exerting on link i, expressed in C_i
τ_i: Joint torque of link i
$g_{i-1,i}$: Transformation of C_i relative to C_{i-1}

$$g_{i-1,i}(\theta_i) = e^{\hat{\xi}'_i\theta_i} \cdot g_{i-1,i}(0) = g_{i-1,i}(0)e^{\hat{\xi}_i\theta_i}$$

$\xi_i = \mathrm{Ad}_{g^{-1}_{i-1,i}(0)} \cdot \xi'_i$: i^{th} axis in C_i frame.

$$\xi_i = \begin{cases} \begin{bmatrix} 0 \\ z_i \end{bmatrix} : & \text{Revolute joint.} \\ \begin{bmatrix} z_i \\ 0 \end{bmatrix} : & \text{Prismatic joint.} \end{cases}$$

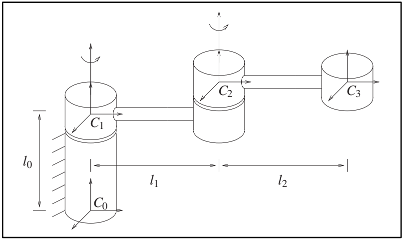


Figure 4.12

Coordinate invariance of Newton-Euler equations

$$\Rightarrow g_{i-1,i}^{-1} \cdot \dot{g}_{i-1,i} = \hat{\xi}_i \cdot \dot{\theta}_i$$

M_i: Moment of inertia in C_i

$$M_i = \begin{bmatrix} m_i I & -m_i \hat{r}_i \\ m_i \hat{r}_i & \mathcal{I}_i - m_i \hat{r}_i^2 \end{bmatrix}$$

m_i: Mass of link i

$\mathcal{I}_i$: inertia tensor

$$g_i = g_{i-1} g_{i-1,i}$$

$$\hat{V}_i = g_i^{-1} \cdot \dot{g}_i = g_{i-1,i}^{-1} \hat{V}_{i-1} g_{i-1,i} + \hat{\xi}_i \dot{\theta}_i$$

$$V_i = \mathrm{Ad}_{g_{i-1,i}^{-1}} V_{i-1} + \xi_i \dot{\theta}_i$$

$$\begin{aligned}
\dot{\hat{V}}_i &= \dot{g}_{i-1,i}^{-1} \hat{V}_{i-1} g_{i-1,i} + g_{i-1,i}^{-1} \hat{V}_{i-1} \dot{g}_{i-1,i} + g_{i-1,i}^{-1} \dot{\hat{V}}_{i-1} g_{i-1,i} + \hat{\xi}_i \ddot{\theta}_i \\
&= -g_{i-1,i}^{-1} \dot{g}_{i-1,i} g_{i-1,i}^{-1} \hat{V}_{i-1} g_{i-1,i} + g_{i-1,i}^{-1} \hat{V}_{i-1} g_{i-1,i} g_{i-1,i}^{-1} \dot{g}_{i-1,i} \\
&\quad + g_{i-1,i}^{-1} \dot{\hat{V}}_{i-1} g_{i-1,i} + \hat{\xi}_i \ddot{\theta}_i \\
&= -\hat{\xi}_i \dot{\theta}_i (\mathrm{Ad}_{g_{i-1,i}^{-1}} V_{i-1})^\wedge + (\mathrm{Ad}_{g_{i-1,i}^{-1}} V_{i-1})^\wedge \hat{\xi}_i \dot{\theta}_i + (\mathrm{Ad}_{g_{i-1,i}^{-1}} \dot{V}_{i-1})^\wedge + \hat{\xi}_i \ddot{\theta}_i \\
&\Rightarrow \dot{V}_i = \xi_i \ddot{\theta}_i + \mathrm{Ad}_{g_{i-1,i}^{-1}} \dot{V}_{i-1} - \mathrm{ad}_{\xi_i \dot{\theta}_i} (\mathrm{Ad}_{g_{i-1,i}^{-1}} V_{i-1})
\end{aligned}$$

(Continues next slide)

Coordinate invariance of Newton-Euler equations

□ Forward Recursion (kinematics):

$$
\begin{cases}
\text{init.}: V_0 = 0, \dot{V}_0 = \begin{bmatrix} g \\ 0 \end{bmatrix} \text{(gravity vector)} \\
g_{i-1,i} = g_{i-1,i}(0) e^{\hat{\xi}_i \theta_i} \\
V_i = \mathrm{Ad}_{g^{-1}_{i-1,i}} V_{i-1} + \xi_i \dot{\theta}_i \\
\dot{V}_i = \xi_i \ddot{\theta}_i + \mathrm{Ad}_{g^{-1}_{i-1,i}} \dot{V}_{i-1} - \mathrm{ad}_{\xi_i \dot{\theta}_i} (\mathrm{Ad}_{g^{-1}_{i-1,i}} V_{i-1})
\end{cases}
$$

□ Backward Recursion (inverse dynamics):

F_{n+1}: End-effector wrench, $g_{n,n+1}$: transform from tool frame to C_n

$$
\begin{aligned}
F_i &= \mathrm{Ad}^T_{g^{-1}_{i,i+1}} \cdot F_{i+1} + M_i \dot{V}_i - \mathrm{ad}^T_{V_i} \cdot M_i V_i \\
\tau_i &= \xi_i^T \cdot F_i
\end{aligned}
$$

Coordinate invariance of Newton-Euler equations

$$V_1 = \mathrm{Ad}_{g_{0,1}^{-1}} \cdot V_0 + \xi_1 \dot{\theta}_1$$

$$F_n = \mathrm{Ad}_{g_{n,n+1}^{-1}}^T \cdot F_{n+1} + M_n \dot{V}_n - \mathrm{ad}_{V_n}^T \cdot (M_n V_n)$$

Define:

$$V = \begin{bmatrix} V_1 \\ \vdots \\ V_n \end{bmatrix} \in \mathbb{R}^{6n \times 1}, \dot{\theta} = \begin{bmatrix} \dot{\theta}_1 \\ \vdots \\ \dot{\theta}_n \end{bmatrix} \in \mathbb{R}^n, \xi = \begin{bmatrix} \xi_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \xi_n \end{bmatrix} \in \mathbb{R}^{6n \times n}$$

$$F = \begin{bmatrix} F_1 \\ \vdots \\ F_n \end{bmatrix} \in \mathbb{R}^{6n \times 1}, \tau = \begin{bmatrix} \tau_1 \\ \vdots \\ \tau_n \end{bmatrix} \in \mathbb{R}^n, P_0 = \begin{bmatrix} \mathrm{Ad}_{g_{0,1}^{-1}} \\ 0 \\ \vdots \\ 0 \end{bmatrix} \in \mathbb{R}^{6n \times 6}$$

$$P_t = \begin{bmatrix} 0 \cdots 0 & \mathrm{Ad}_{g_{n,n+1}^{-1}} \end{bmatrix} \in \mathbb{R}^{6 \times 6n}$$

Coordinate invariance of Newton-Euler equations

$$V_1 = \mathrm{Ad}_{g_{0,1}^{-1}} \cdot V_0 + \xi_1 \dot{\theta}_1$$

$$V_2 - \mathrm{Ad}_{g_{1,2}^{-1}} V_1 = \xi_2 \dot{\theta}_2$$

$$\vdots$$

$$V_n - \mathrm{Ad}_{g_{n-1,n}^{-1}} V_{n-1} = \xi_n \dot{\theta}_n$$

$$\Rightarrow \underbrace{\begin{bmatrix} I & 0 & \cdots & 0 \\ -\mathrm{Ad}_{g_{1,2}^{-1}} & I & \ddots & \vdots \\ 0 & \ddots & \ddots & 0 \\ 0 & 0 & -\mathrm{Ad}_{g_{n-1,n}^{-1}} & I \end{bmatrix}}_{G^{-1}} \underbrace{\begin{bmatrix} V_1 \\ V_2 \\ \vdots \\ V_n \end{bmatrix}}_{V} = \underbrace{\begin{bmatrix} \mathrm{Ad}_{g_{0,1}^{-1}} \\ 0 \\ \vdots \\ 0 \end{bmatrix}}_{P_0} V_0 + \underbrace{\begin{bmatrix} \xi_1 & & & \\ & \xi_2 & & \\ & & \ddots & \\ & & & \xi_n \end{bmatrix}}_{\xi} \underbrace{\begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \vdots \\ \dot{\theta}_n \end{bmatrix}}_{\dot{\theta}}$$

Thus $V = GP_0V_0 + G\xi\dot{\theta}$

Coordinate invariance of Newton-Euler equations

where

$$G = \begin{bmatrix} I & 0 & 0 & \cdots & 0 \\ \mathrm{Ad}_{g_{1,2}^{-1}} & I & 0 & \cdots & 0 \\ \mathrm{Ad}_{g_{1,3}^{-1}} & \mathrm{Ad}_{g_{2,3}^{-1}} & I & \ddots & \vdots \\ \vdots & \vdots & \ddots & I & 0 \\ \mathrm{Ad}_{g_{1,n}^{-1}} & \mathrm{Ad}_{g_{2,n}^{-1}} & \cdots & \mathrm{Ad}_{g_{n-1,n}^{-1}} & I \end{bmatrix} \in \mathbb{R}^{6n\times 6n}$$

$$\begin{aligned} \dot{V}_1 &= \xi_1\ddot{\theta}_1 + \mathrm{Ad}_{g_{0,1}^{-1}}\dot{V}_0 - \mathrm{ad}_{\xi_1\dot{\theta}_1}(\mathrm{Ad}_{g_{0,1}^{-1}}V_0) \\ \dot{V}_2 - \mathrm{Ad}_{g_{1,2}^{-1}}\dot{V}_1 &= \xi_2\ddot{\theta}_2 - \mathrm{ad}_{\xi_2\dot{\theta}_2}(\mathrm{Ad}_{g_{1,2}^{-1}}V_1) \\ \dot{V}_n - \mathrm{Ad}_{g_{n-1,n}^{-1}}\dot{V}_{n-1} &= \xi_n\ddot{\theta}_n - \mathrm{ad}_{\xi_n\dot{\theta}_n}(\mathrm{Ad}_{g_{n-1,n}^{-1}}V_{n-1}) \end{aligned}$$

(Continues next slide)

Coordinate invariance of Newton-Euler equations

$$\underbrace{\begin{bmatrix} I & 0 & \cdots & 0 \\ -\mathrm{Ad}_{g_{1,2}^{-1}} & I & \ddots & \vdots \\ 0 & \ddots & \ddots & 0 \\ 0 & 0 & -\mathrm{Ad}_{g_{n-1,n}^{-1}} & I \end{bmatrix}}_{G^{-1}} \begin{bmatrix} \dot{V}_1 \\ \dot{V}_2 \\ \vdots \\ \dot{V}_n \end{bmatrix} = \underbrace{\begin{bmatrix} \mathrm{Ad}_{g_{0,1}^{-1}} \\ 0 \\ \vdots \\ 0 \end{bmatrix}}_{P_0} \dot{V}_0 + \underbrace{\begin{bmatrix} \xi_1 & & & \\ & \xi_2 & & \\ & & \ddots & \\ & & & \xi_n \end{bmatrix}}_{\xi} \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \\ \vdots \\ \ddot{\theta}_n \end{bmatrix} +$$

$$\underbrace{\begin{bmatrix} -\mathrm{ad}_{\xi_1\dot{\theta}_1} & 0 & \cdots & 0 \\ 0 & -\mathrm{ad}_{\xi_2\dot{\theta}_2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & -\mathrm{ad}_{\xi_n\dot{\theta}_n} \end{bmatrix}}_{\mathrm{ad}_{\xi\dot{\theta}}} \begin{bmatrix} Ad_{g_{0,1}^{-1}} \\ 0 \\ \vdots \\ 0 \end{bmatrix} V_0 +$$

$$\begin{bmatrix} -\mathrm{ad}_{\xi_1\dot{\theta}_1} & 0 & \cdots & 0 \\ 0 & -\mathrm{ad}_{\xi_2\dot{\theta}_2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & -\mathrm{ad}_{\xi_n\dot{\theta}_n} \end{bmatrix} \underbrace{\begin{bmatrix} 0 & 0 & \cdots & 0 \\ \mathrm{Ad}_{g_{1,2}^{-1}} & 0 & \ddots & \vdots \\ 0 & \ddots & \ddots & 0 \\ 0 & 0 & \mathrm{Ad}_{g_{n-1,n}^{-1}} & 0 \end{bmatrix}}_{\Gamma} \begin{bmatrix} V_1 \\ V_2 \\ \vdots \\ V_n \end{bmatrix}$$

(Continues next slide)

Coordinate invariance of Newton-Euler equations

Thus

$$\dot{V} = G \cdot \xi \ddot{\theta} + G \cdot P_0 \dot{V}_0 + G \cdot \mathrm{ad}_{\xi \dot{\theta}} P_0 V_0 + G \cdot \mathrm{ad}_{\xi \dot{\theta}} \Gamma V$$

Finally the backward recursion:

$$\begin{aligned} F_n &= \mathrm{Ad}^T_{g^{-1}_{n,n+1}} F_{n+1} + M_n \dot{V}_n - \mathrm{ad}^T_{V_n} \cdot (M_n V_n) \\ F_{n-1} &= \mathrm{Ad}^T_{g^{-1}_{n-1,n}} F_n + M_{n-1} \dot{V}_{n-1} - \mathrm{ad}^T_{V_{n-1}} \cdot (M_{n-1} V_{n-1}) \\ &\vdots \\ F_1 &= \mathrm{Ad}^T_{g^{-1}_{1,2}} F_2 + M_1 \dot{V}_1 - \mathrm{ad}^T_{V_1} (M_1 V_1) \end{aligned}$$

(Continues next slide)

Coordinate invariance of Newton-Euler equations

$$\Rightarrow \begin{bmatrix} I & -\mathrm{Ad}^T_{g^{-1}_{1,2}} & 0 & 0 \\ 0 & I & \ddots & 0 \\ \vdots & \ddots & \ddots & -\mathrm{Ad}^T_{g^{-1}_{n-1,n}} \\ 0 & \cdots & 0 & I \end{bmatrix} \begin{bmatrix} F_1 \\ F_2 \\ \vdots \\ F_n \end{bmatrix} = \underbrace{\begin{bmatrix} M_1 & & & \\ & M_2 & & \\ & & \ddots & \\ & & & M_n \end{bmatrix}}_{M} \begin{bmatrix} \dot{V}_1 \\ \dot{V}_2 \\ \vdots \\ \dot{V}_n \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ \vdots \\ 0 \\ \mathrm{Ad}^T_{g^{-1}_{n,n+1}} \end{bmatrix}}_{P_t^T} F_{n+1} +$$

$$\underbrace{\begin{bmatrix} -\mathrm{ad}^T_{V_1} & & \\ & \ddots & \\ & & -\mathrm{ad}^T_{V_n} \end{bmatrix}}_{\mathrm{ad}^T_V} \begin{bmatrix} M_1 & & \\ & \ddots & \\ & & M_n \end{bmatrix} \begin{bmatrix} V_1 \\ \vdots \\ V_n \end{bmatrix}$$

$$\boxed{F = G^T M \dot{V} + G^T P_t^T \underbrace{F_{n+1}}_{F_t} + G^T \cdot \mathrm{ad}^T_V M V}$$

(Continues next slide)

Coordinate invariance of Newton-Euler equations

$$\tau = \xi^T \cdot F$$

$$\Rightarrow \begin{aligned} \tau &= \xi^T G^T M\dot{V} + \xi^T G^T P_t^T F_t + \xi^T G^T \cdot \mathrm{ad}_V^T MV \\ &= \xi^T G^T M(G\xi\ddot{\theta} + GP_0\dot{V}_0 + G\cdot \mathrm{ad}_{\xi\dot{\theta}} P_0 V_0 + G \cdot \mathrm{ad}_{\xi\dot{\theta}}\Gamma V) + \\ &\xi^T G^T P_t^T F_t + \xi^T G^T \cdot \mathrm{ad}_V^T MV \\ &= \xi^T G^T MG\xi\ddot{\theta} + \xi^T G^T MGP_0\dot{V}_0 + \xi^T G^T MG\cdot \mathrm{ad}_{\xi\dot{\theta}}\Gamma V + \\ &\xi^T G^T P_t^T F_t + \xi^T G^T \cdot \mathrm{ad}_V^T MV \end{aligned}$$

Finally we get (recall that $V_0 = 0$, we have $V = G\xi\dot{\theta}$):

$$\overbrace{\xi^T G^T MG\xi}^{M(\theta)}\ddot{\theta} + \overbrace{\xi^T G^T(MG\cdot \mathrm{ad}_{\xi\dot{\theta}}\Gamma + \mathrm{ad}_V^T M)G\xi}^{C(\theta,\dot{\theta})}\dot{\theta} + \underbrace{\xi^T G^T MGP_0\dot{V}_0}_{\phi(\theta)} + \underbrace{\xi^T G^T P_t^T}_{J_t^T(\theta)} F_t = \tau$$

Coordinate invariance of Newton-Euler equations

$$\boxed{M(\theta)\ddot{\theta} + C(\theta,\dot{\theta}) + \phi(\theta) + J_t^T(\theta)F_t = \tau}$$

$$\begin{aligned} M(\theta) &= \xi^T G^T \mathcal{M} G \xi \\ C(\theta,\dot{\theta}) &= \xi^T G^T (\mathcal{M} G \mathrm{ad}_{\xi\dot{\theta}} \Gamma + \mathrm{ad}_V^T \mathcal{M}) G \xi \dot{\theta} \\ \phi(\theta) &= \xi^T G^T \mathcal{M} G P_0 \dot{V}_0 \\ J_t &= P_t G \xi \end{aligned}$$

$$\mathcal{M} = \begin{bmatrix} M_1 & & 0 \\ & \ddots & \\ 0 & & M_n \end{bmatrix}$$

Property 4:

$$\Gamma^n = 0,$$

$$G = (I - \Gamma)^{-1} = I + \Gamma + \Gamma^2 + \cdots + \Gamma^{n-1}$$

$$I + \Gamma G = G$$

† End of Section †

Square root factorization of M

Definition: Articulated body inertia algorithm (Featherstone)

$$\hat{M}_n = M_n, b_n = -\mathrm{ad}^T_{V_n}(M_n V_n)$$

$$F_i = \hat{M}_i \dot{V}_i + b_i, i = n, \ldots, 1$$

$$b_i = b_i(V_i, V_{i+1}, \xi_{i+1}, \hat{M}_{i+1}, \tau_{i+1}) \text{ (bias force)}$$

$$\begin{cases} F_i = \mathrm{Ad}^T_{g^{-1}_{i,i+1}} F_{i+1} + M_i \dot{V}_i - \mathrm{ad}^T_{V_i} M_i V_i \\ F_{i+1} = \hat{M}_{i+1} \dot{V}_{i+1} + b_{i+1} \\ \dot{V}_{i+1} = \mathrm{Ad}_{g^{-1}_{i,i+1}} \dot{V}_i + \xi_{i+1} \ddot{\theta}_{i+1} - \mathrm{ad}_{\xi_{i+1}\dot{\theta}_{i+1}} \mathrm{Ad}_{g^{-1}_{i,i+1}} V_i \end{cases} \Rightarrow$$

(Continues next slide)

Square root factorization of M

$$F_{i+1} = \hat{M}_{i+1}(\mathrm{Ad}_{g^{-1}_{i,i+1}} \dot{V}_i + \xi_{i+1}\ddot{\theta}_{i+1} - \mathrm{ad}_{\xi_{i+1}\dot{\theta}_{i+1}} \mathrm{Ad}_{g^{-1}_{i,i+1}} V_i) + b_{i+1} \Rightarrow$$

$$\tau_{i+1} = \xi^T_{i+1} F_{i+1} = \xi^T_{i+1}(\hat{M}_{i+1}(\mathrm{Ad}_{g^{-1}_{i,i+1}} \dot{V}_i + \xi_{i+1}\ddot{\theta}_{i+1} - \mathrm{ad}_{\xi_{i+1}\dot{\theta}_{i+1}} \mathrm{Ad}_{g^{-1}_{i,i+1}} V) + b_{i+1}) \Rightarrow$$

$$\ddot{\theta}_{i+1} = \frac{\tau_{i+1} + \xi^T_{i+1}(\hat{M}_{i+1}(-\mathrm{Ad}_{g^{-1}_{i,i+1}} \dot{V}_i + \mathrm{ad}_{\xi_{i+1}\dot{\theta}_{i+1}} \mathrm{Ad}_{g^{-1}_{i,i+1}} V_i) - b_{i+1})}{\xi^T_{i+1}\hat{M}_{i+1}\xi_{i+1}}$$

$$\Rightarrow$$

$$\begin{aligned} F_i = {} & \mathrm{Ad}^T_{g^{-1}_{i,i+1}}(\hat{M}_{i+1}(\mathrm{Ad}_{g^{-1}_{i,i+1}} \dot{V}_i + \\ & \xi_{i+1}\left(\frac{\tau_{i+1} + \xi^T_{i+1}(\hat{M}_{i+1}(-\mathrm{Ad}_{g^{-1}_{i,i+1}} \dot{V}_i + \mathrm{ad}_{\xi_{i+1}\dot{\theta}_{i+1}} \mathrm{Ad}_{g^{-1}_{i,i+1}} V_i) - b_{i+1})}{\xi^T_{i+1}\hat{M}_{i+1}\xi_{i+1}}\right) \\ & - \mathrm{ad}_{\xi_{i+1}\dot{\theta}_{i+1}} \mathrm{Ad}_{g^{-1}_{i,i+1}} V_i) + b_{i+1}) + M_i\dot{V}_i - \mathrm{ad}^T_{V_i} M_i V_i \end{aligned}$$

$$\Rightarrow$$

(Continues next slide)

Square root factorization of M

$$\hat{M}_i = M_i + \mathrm{Ad}^T_{g^{-1}_{i,i+1}} \hat{M}_{i+1} \mathrm{Ad}_{g^{-1}_{i,i+1}} - \frac{\mathrm{Ad}^T_{g^{-1}_{i,i+1}} \hat{M}_{i+1} \xi_{i+1} \xi^T_{i+1} \hat{M}_{i+1} \mathrm{Ad}_{g^{-1}_{i,i+1}}}{\xi^T_{i+1} \hat{M}_{i+1} \xi_{i+1}}$$

$$b_i = \mathrm{Ad}^T_{g^{-1}_{i,i+1}} b_{i+1} - \mathrm{ad}^T_{V_i} M_i V_i - \mathrm{Ad}^T_{g^{-1}_{i,i+1}} \hat{M}_{i+1} \mathrm{ad}_{\xi_{i+1}\dot{\theta}_{i+1}} \mathrm{Ad}_{g^{-1}_{i,i+1}} V_i +$$
$$\frac{\mathrm{Ad}^T_{g^{-1}_{i,i+1}} \hat{M}_{i+1} \xi_{i+1} (\tau_{i+1} - \xi^T_{i+1} b_{i+1} + \xi^T_{i+1} \hat{M}_{i+1} \mathrm{ad}_{\xi_{i+1}\dot{\theta}_{i+1}} \mathrm{Ad}_{g^{-1}_{i,i+1}} V_i)}{\xi^T_{i+1} \hat{M}_{i+1} \xi_{i+1}}$$

$$\Rightarrow \hat{M} = M + \Gamma^T \hat{M} \Gamma - \Gamma^T \hat{M} \xi (\xi^T \hat{M} \xi)^{-1} \xi^T \hat{M} \Gamma$$

$$\Rightarrow M = \hat{M} - \Gamma^T \hat{M} \Gamma + \Gamma^T \hat{M} \xi (\xi^T \hat{M} \xi)^{-1} \xi^T \hat{M} \Gamma$$

$$\Rightarrow M(\theta) = \xi^T G^T (\hat{M} - \Gamma^T \hat{M} \Gamma + \Gamma^T \hat{M} \xi (\xi^T \hat{M} \xi)^{-1} \xi^T \hat{M} \Gamma) G \xi$$

$$\Rightarrow M(\theta) = \xi^T G^T \hat{M}^T \xi (\xi^T \hat{M} \xi)^{-1} \xi^T \hat{M} G \xi \triangleq W W^T$$

$$\boxed{W = \xi^T G^T \hat{M}^T \xi (\xi^T \hat{M} \xi)^{-\frac{1}{2}}}$$

† End of Section †

General formulation

Definition: Holonomic constraints

Let $q = (q_1, \ldots, q_n) \in E$, Ambient Space. A *holonomic constraint* is a set of constraint equations:

$$h_i(q) = 0, i = 1, \ldots, k$$

If $\{\mathrm{d}h_i(q)\}_{i=1}^{k}$ is linearly independent, then $Q = h^{-1}(0)$ is a manifold of $\dim\, m \triangleq n - k$.

$$T_qQ : \{V \in T_qE | \mathrm{d}h_i \cdot V = 0, \forall i = 1, \ldots, k\} \subset T_qE :$$

subspace of permissible velocities.

(Continues next slide)

$$T_q^*Q^\perp : \{f \in T_q^*E | \langle f, v\rangle = 0, \forall V \in T_qQ\}$$
$$= \mathrm{span}\{\mathrm{d}h_1, \dots, \mathrm{d}h_k\} \subset T_q^*E :$$

the subspace of constraint forces (Annihilator of T_qQ).

$$\dim(T_qQ) = m, \dim(T_q^*Q^\perp) = n - m = k$$

Definition: Constraint forces

$$\Gamma = \frac{\partial h}{\partial q}^T \cdot \lambda$$

$\lambda \in \mathbb{R}^k$: the vector of relative magnitudes of constraint forces.

Definition: Pfaffian Constraints

A Pfaffian constraint has the form:

$$A(q)\dot{q} = 0, A(q) \in \mathbb{R}^{k\times n}$$

Given a Pfaffian constraint,

$$\Delta_q = \{v_q \in T_qE | A(q) \cdot v_q = 0\} \subset T_qE :$$

distribution of permissible velocities.

Definition:

$A(q)\dot{q} = 0$ is holonomic (or integrable) iff Δ_q is an involutive distribution, iff

$$\exists h_i : E \mapsto \mathbb{R}, i = 1, \ldots, k \text{ s.t. } \Delta_q = T_qQ, Q = h^{-1}(0)$$

§ Constraint forces:

$$\Gamma = A^T(q) \cdot \lambda, \lambda \in \mathbb{R}^k$$

§ Kinetic energy:

$$T(q, \dot{q}) = \frac{1}{2}\dot{q}^T \cdot M(q) \cdot \dot{q}$$

§ Potential energy:

$$V(q)$$

§ Lagrangian:

$$L(q, \dot{q}) = T(q, \dot{q}) - V(q)$$

Lagrange's equations with constraints:

$$M(q)\ddot{q} + C(q,\dot{q}) + N(q) + A^T(q)\lambda = F$$

Explicit solution for constraint forces:

$$A(q)\ddot{q} + \dot{A}(q)\dot{q} = 0$$

$$(AM^{-1}A^T)\lambda = AM^{-1}(F - C - N) + \dot{A}\dot{q}$$

$$\lambda = (AM^{-1}A^T)^{-1}(AM^{-1}(F - C - N) + \dot{A}\dot{q})$$

Example:

$$x^2 + y^2 = l^2$$

$$\underbrace{\begin{bmatrix} x & y \end{bmatrix}}_{A(q)} \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = 0$$

$$L(q, \dot{q}) = \frac{1}{2} m(\dot{x}^2 + \dot{y}^2) - mgy$$

$$\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \begin{bmatrix} \ddot{x} \\ \ddot{y} \end{bmatrix} + \begin{bmatrix} 0 \\ mg \end{bmatrix} + \begin{bmatrix} x \\ y \end{bmatrix} \lambda = 0$$

$$\lambda = (AM^{-1}A^T)^{-1}(AM^{-1}(F - C - N) + \dot{A}\dot{q})$$

$$= -\frac{m}{l^2}(gy + \dot{x}^2 + \dot{y}^2)$$

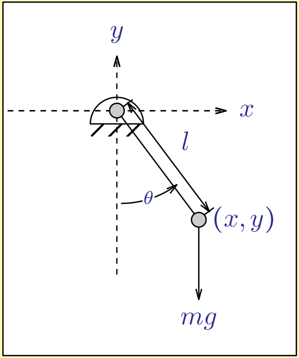


Figure 4.13

$$\text{Tension} = \left\| \begin{bmatrix} x \\ y \end{bmatrix} \lambda \right\| = \frac{mg}{l} y + \frac{m}{l}(\dot{x}^2 + \dot{y}^2)$$

Lagrange-d'Alembert formulation

Given the Pfaffian constraint $A(q)\dot{q} = 0$ and virtual displacement $\delta q \in \mathbb{R}^k$, we have:

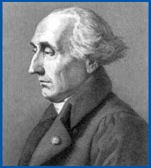
Theorem 1 (D'alembert Principle):
Forces of constraints do no virtual work!

$$(A^T(q)\lambda) \cdot \delta q = 0 \text{ for } A(q)\delta q = 0$$

1717–1783

Theorem 2 (Lagrange-d'Alembert Equation):

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} - \tau\right) \cdot \delta q = 0, A(q)\delta q = 0$$



1736-1813

Let $A(q) = [A_1(q) \;\; A_2(q)]$, and $A_2(q) \in \mathbb{R}^{k\times k}$ is invertible, then $\delta q_1 \in \mathbb{R}^{n-k}$ are free variables:

$$\begin{aligned}
&\delta q_2 = -A_2^{-1}(q)A_1(q)\delta q_1 \\
&\Rightarrow \left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot q} - \frac{\partial L}{\partial q} - \tau\right)\cdot \delta q \\
&= \left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot q_1} - \frac{\partial L}{\partial q_1} - \tau_1\right)\cdot \delta q_1 + \left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot q_2} - \frac{\partial L}{\partial q_2} - \tau_2\right)\cdot \delta q_2 \\
&= \left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot q_1} - \frac{\partial L}{\partial q_1} - \tau_1\right)\cdot \delta q_1 + \left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot q_2} - \frac{\partial L}{\partial q_2} - \tau_2\right)\cdot (-A_2^{-1}A_1)\delta q_1
\end{aligned}$$

As $\delta q_1 \in \mathbb{R}^{n-k}$ is free,

$$\boxed{\left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot q_1} - \frac{\partial L}{\partial q_1} - \tau_1\right) - A_1^T A_2^{-T}\left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot q_2} - \frac{\partial L}{\partial q_2} - \tau_2\right) = 0}$$

Lagrange-d'Alembert equation

Example: Dynamics of a rolling disk

Pfaffian constraint:

$$\begin{cases} \dot{x} - \rho\cos\theta\dot{\phi} = 0 \\ \dot{y} - \rho\sin\theta\dot{\phi} = 0 \end{cases}$$

$$\Rightarrow A(q)\dot{q} = \begin{bmatrix} 1 & 0 & 0 & -\rho\cos\theta \\ 0 & 1 & 0 & -\rho\sin\theta \end{bmatrix}\dot{q} = 0$$

$$L(q,\dot{q}) = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}\mathcal{I}_1\ddot{\theta}^2 + \frac{1}{2}\mathcal{I}_2\dot{\phi}^2$$

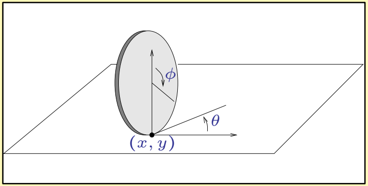


Figure 4.14

Lagrange-d'Alembert equation:

$$\left(\begin{bmatrix} m & & & \\ & m & & \\ & & \mathcal{I}_1 & \\ & & & \mathcal{I}_2 \end{bmatrix}\ddot{q} - \begin{bmatrix} 0 \\ 0 \\ \tau_\theta \\ \tau_\phi \end{bmatrix}\right) \cdot \delta q = 0$$

where $A(q) \cdot \delta q = 0$.

(Continues next slide)

Example: Dynamics of a rolling disk

As

$$\begin{cases} \delta x = \rho\cos\theta\cdot\delta\phi \\ \delta y = \rho\sin\theta\cdot\delta\phi \end{cases}$$

the equation of motion is:

$$\left(\begin{bmatrix} 0 & 0 \\ m\rho c_\theta & m\rho s_\theta \end{bmatrix}\begin{bmatrix} \ddot{x} \\ \ddot{y} \end{bmatrix} + \begin{bmatrix} \mathcal{I}_1 & 0 \\ 0 & \mathcal{I}_2 \end{bmatrix}\begin{bmatrix} \ddot{\theta} \\ \ddot{\phi} \end{bmatrix} - \begin{bmatrix} \tau_\theta \\ \tau_\phi \end{bmatrix}\right)\cdot\begin{bmatrix} \delta\theta \\ \delta\phi \end{bmatrix} = 0$$

$$\Rightarrow \begin{bmatrix} 0 & 0 \\ m\rho c_\theta & m\rho s_\theta \end{bmatrix}\begin{bmatrix} \ddot{x} \\ \ddot{y} \end{bmatrix} + \begin{bmatrix} \mathcal{I}_1 & 0 \\ 0 & \mathcal{I}_2 \end{bmatrix}\begin{bmatrix} \ddot{\theta} \\ \ddot{\phi} \end{bmatrix} = \begin{bmatrix} \tau_\theta \\ \tau_\phi \end{bmatrix}$$

As

$$\begin{cases} \ddot{x} = \rho\cos\theta\cdot\ddot{\phi} - \rho\sin\theta\cdot\dot{\theta}\dot{\phi} \\ \ddot{y} = \rho\sin\theta\cdot\ddot{\phi} + \rho\cos\theta\cdot\dot{\theta}\dot{\phi} \end{cases}$$

$$\Rightarrow \begin{bmatrix} \mathcal{I}_1 & 0 \\ 0 & \mathcal{I}_2 + m\rho^2 \end{bmatrix}\begin{bmatrix} \ddot{\theta} \\ \ddot{\phi} \end{bmatrix} = \begin{bmatrix} \tau_\theta \\ \tau_\phi \end{bmatrix}$$

Solve for $(\theta(t), \phi(t))$, and then solve for $(x(t), y(t))$ from:

$$\begin{cases} \dot{x} = \rho\cos\theta\cdot\dot{\phi} \\ \dot{y} = \rho\sin\theta\cdot\dot{\phi} \end{cases} \Leftarrow \text{ 1st order differential equation}$$

Nature of nonholonomic constraints

Consider $q = (r, s) \in \mathbb{R}^2 \times \mathbb{R}$ with Pfaffian constraint,

$$\dot{s} + a^T(r)\dot{r} = 0, a(r) \in \mathbb{R}^2$$

Lagrangian:

$$L = L(r, \dot{r}, \dot{s})$$

and **constrained Lagrangian:**

$$L_c(r, \dot{r}) = L(r, \dot{r}, -a^T(r)\dot{r})$$

⇒ **Lagrange's equation:**

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L_c}{\partial \dot{r}_i} - \frac{\partial L_c}{\partial r_i} = 0, i = 1, 2$$

$$\Rightarrow \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{r}_i} - a_i(r)\frac{\partial L}{\partial \dot{s}}\right) - \left(\frac{\partial L}{\partial r_i} - \frac{\partial L}{\partial \dot{s}}\sum_j \frac{\partial a_j}{\partial r_i}\dot{r}_j\right) = 0$$

(Continues next slide)

Nature of nonholonomic constraints

$$\Rightarrow \left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{r}_i} - \frac{\partial L}{\partial r_i}\right) - a_i(r)\left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{s}} - \frac{\partial L}{\partial s}\right) = \underbrace{\frac{\partial L}{\partial \dot{s}}\left(\dot{a}_i(r) - \sum_j \frac{\partial a_j}{\partial r_i}\dot{r}_j\right)}_{\neq 0}(*)$$

If the constraint is holonomic, i.e.

$$a_i(r) = \frac{\partial h}{\partial r_i} \text{ for some } h : E \mapsto \mathbb{R}$$

then RHS (right hand side) of $(*)$ equals

$$\frac{\partial L}{\partial \dot{s}}\left(\sum_j \frac{\partial^2 h}{\partial r_i \partial r_j}\dot{r}_j - \sum_j \frac{\partial^2 h}{\partial r_j \partial r_i}\dot{r}_j\right) = 0$$

Metric,duality and orthogonality on T_qE

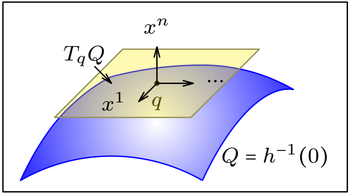


Figure 4.15

$$\mathcal{K} \triangleq \frac{1}{2} \ll \dot{q}, \dot{q} \gg_M = \frac{1}{2}\dot{q}^T M(q)\dot{q}$$

$$T_qQ^\perp = \{V_1 \in T_qE | \ll V_1, V_2 \gg_M \triangleq V_1^T M V_2 = 0, \forall V_2 \in T_qQ\}$$

$$T_q^*Q^\perp = \{f \in T_q^*E | \langle f, V \rangle = 0, \forall V \in T_qQ\} : \quad \text{constraint forces}$$

$$T_qE = T_qQ \oplus T_qQ^\perp, T_q^*E = T_q^*Q \oplus T_q^*Q^\perp$$

Definition:

$$M^\flat : T_qE \mapsto T_q^*E, \langle M^\flat V_1, V_2 \rangle = V_1^T M V_2 = \ll V_1, V_2 \gg_M$$

$$M^\sharp : T_q^*E \mapsto T_qE, M^\sharp = {M^\flat}^{-1}$$

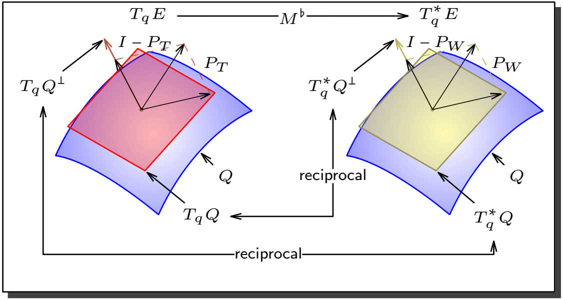


Figure 4.16

Property 5:
Under the basis $\frac{\partial}{\partial q_i}$ and $\mathrm{d}q_i, i = 1, \ldots, n$ of T_qE and T_q^*E respectively, the matrix representation of $M^\flat$ and $M^\sharp$ is M and M^{-1} respectively.

Property 6:

$$M^\sharp(T_q^*Q) = T_qQ$$
$$M^\sharp(T_q^*Q^\perp) = T_qQ^\perp$$

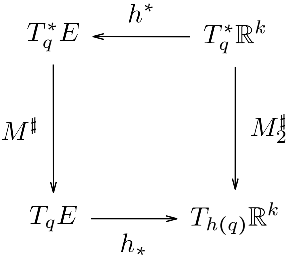
Given

$$h: E \mapsto \mathbb{R}^k, m = n - k$$
$$h_* \triangleq T_q h : T_q E \mapsto T_{h(q)} \mathbb{R}^k$$
$$h^* \triangleq T_q^* h : T_{h(q)}^* \mathbb{R}^k \mapsto T_q^* E$$

we have:

Property 7:

$$\ker h_* = T_q Q, h_*(T_q Q^\perp) = T_{h(q)} \mathbb{R}^k, h^*(T_{h(q)}^* \mathbb{R}^k) = T_q^* Q^\perp$$



$$M_2^\sharp = h_* \circ M^\sharp \circ h^*$$

Lemma 1:
The map $(I - P_\omega) : T_q^* E \mapsto T_q^* Q^\perp$ given by

$$(I - P_\omega) = h^* \circ M_2^\flat \circ h_* \circ M^\sharp$$

is a well-defined projection map, with the property:

$$(I - P_\omega) f_1 = 0, \forall f_1 \in T_q^* Q$$

$$(I - P_\omega) f_2 = f_2, \forall f_2 \in T_q^* Q^\perp$$

Proof :
Given $f_1 \in T_q^* Q, M^\sharp(f_1) \in T_q Q = \ker h_*$, then $(I - P_\omega) f_1 = 0$. For $f_2 \in T_q^* Q^\perp, \exists \lambda \in \mathbb{R}^{n-m}$ s.t. $f_2 = h^* \lambda$, and

$$(I - P_\omega) f_2 = h^* M_2^\flat h_* M^\sharp h^* \lambda = h^* \lambda = f_2$$

thus $P_\omega : T_q^* E \mapsto T_q^* Q$ is a well-defined projection map. Similarly,

$$P_T : T_q E \mapsto T_q Q, P_T = I - M^\sharp h^* M_2^b h_*$$

and

$$(I - P_T) : T_q E \mapsto T_q Q^\perp$$

are projection maps. □

Lemma 2:

$$P_\omega M = M P_T$$
$$P_\omega h^* = h_* P_T = 0$$
$$P_T = P_\omega^T$$

For nonholonomic constraints:

$$h_* \leftarrow A(q)$$
$$h^* \leftarrow A^*(q)$$
$$T_q Q \leftarrow \Delta_q$$
$$T_q^* Q^\perp \leftarrow \mathrm{span}\{a_i(q), i = 1, \ldots, k\}$$

application in hybrid velocity/force control.

□ Lagrange's equations of motion:

$$M(q)\ddot{q} + C(q,\dot{q}) + N = \tau + A^T(q)\lambda \Rightarrow$$
$$\lambda = (AM^{-1}A^T)^{-1}AM^{-1}(M\ddot{q} + (C + N - \tau))$$

Define $P_\omega = I - A^T(AM^{-1}A^T)^{-1}AM^{-1}$, then:

$$P_\omega M\ddot{q} + P_\omega C + P_\omega N = P_\omega \tau$$

Denote $\tilde{C} = P_\omega C, \tilde{N} = P_\omega N, \tilde{\tau} = P_\omega \tau$, $P_\omega M\ddot{\theta} \triangleq \tilde{M}\ddot{\theta}$ is the intertia force in T_q^*Q.

Definition: Dynamics in T_q^*Q

$$\tilde{M}\ddot{\theta} + \tilde{C} + \tilde{N} = \tilde{\tau}$$

(Continues next slide)

Similarly

$$(I - P_\omega)(M\ddot{q} + C + N) = (I - P_\omega)\tau + A^T\tau$$

Let

$$P_T = I - M^{-1}A^T(AM^{-1}A^T)^{-1}A$$

then since $P_\omega M = MP_T$, we have:

Definition: Dynamics in $T_q^*Q^\perp$

$$M(I - P_T)(\ddot{q} + M^{-1}C) = (I - P_\omega)(\tau - N) + A^T\lambda$$

Geometric Interpretation:

$$\nabla \leftrightarrow M$$

$$M\ddot{q} + C + N = \tau + A^T\lambda \Leftrightarrow M\nabla_{\dot{q}}\dot{q} = \tau - N + A^T\lambda$$

$$\tilde{\nabla} \leftrightarrow \text{ induced metric on } T_qQ$$

$$S : TQ \otimes TQ \mapsto N(Q) : 2^{\text{nd}} \text{ fundamental form}$$

$$TQ : \text{ tangent vector field}$$

$$N(Q) : \text{ normal vector field}$$

$$\nabla_X Y = \tilde{\nabla}_X Y + S(X, Y)$$
$$M \underbrace{(I - P_T)(\ddot{q} + M^{-1}C)}_{S(\dot{q},\dot{q})} = (I - P_\omega)(\tau - N) + A^T\lambda$$

$MS(\dot{q}, \dot{q})$: centrifugal force due to curvature of Q in E

Definition: Hybrid position/force control

$$M\tilde{\nabla}_{\dot{q}}\dot{q} = \tilde{\tau} - \tilde{N}$$
$$MS(\dot{q}, \dot{q}) = (I - P_\omega)(\tau - N) + A^T\lambda$$

Dynamics of a Spherical Pendulum

$$K = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) = \frac{1}{2}\dot{q}^T M \dot{q}$$

$$q = (x, y, z)^T, M = mI$$

$$h(q) = q^T q - r^2 = 0$$

$$A = (x, y, z), M_2^\sharp = AM^\sharp A^T = r^2/m$$

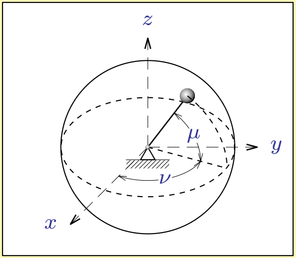


Figure 4.17

$$P_\omega = \frac{1}{r^2}\begin{bmatrix} y^2 + z^2 & -xy & -xz \\ -yz & x^2 + z^2 & -yz \\ -zx & -zy & x^2 + y^2 \end{bmatrix}$$

$$I - P_\omega = \frac{1}{r^2}\begin{bmatrix} x^2 & xy & xz \\ yx & y^2 & yz \\ zx & zy & z^2 \end{bmatrix}$$

(Continues next slide)

$$P_T = P_\omega^T = P_\omega$$

(μ, ν): Spherical coordinates

$$q = (r\cos\mu\cos\nu, r\cos\mu\sin\nu, r\sin\mu)^T$$

$$\begin{aligned}
\tilde{\nabla}_{\dot{q}}\dot{q} &= P_T(\nabla_{\dot{q}}\dot{q}) \\
&= \begin{bmatrix} -r\sin\mu\cos\nu & -r\sin\nu \\ -r\sin\mu\sin\nu & r\cos\nu \\ r\cos\mu & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \\
S(\dot{q},\dot{q}) &= (I - P_T)(\nabla_{\dot{q}}\dot{q}) \\
&= (-\dot{\mu}^2 - \cos^2\mu\dot{\nu}^2) \begin{bmatrix} r\cos\mu\cos\nu \\ r\cos\mu\sin\nu \\ r\sin\mu \end{bmatrix}
\end{aligned}$$

where

$$\begin{aligned}
v_1 &= \ddot{\mu} + \sin\mu\cos\mu\dot{\nu}^2 \\
v_2 &= \cos\mu\ddot{\nu} - 2\sin\mu\dot{\mu}\dot{\nu}
\end{aligned}$$

♢

Control Algorithm

1. holonomic constraints:

$$\tilde{q} : \text{coordinates of } Q$$

$$q = \psi(\tilde{q}) \Rightarrow \dot{q} = J \cdot \dot{\tilde{q}}$$

$$\tau = MJ(\ddot{\tilde{q}}_d - K_v \dot{\tilde{e}} - K_p \tilde{e}) + C_1 + N + A^T(-\lambda_d + K_I \int (\lambda - \lambda_d))$$

2. nonholonomic constraints:

Let $J(q) \in \mathbb{R}^{n \times m}$ be s.t. $AJ = 0$. Write $\dot{q} = J \cdot u$ for some u

$$\tau = MJ(\dot{u}_d - K_p(u - u_d)) + M\dot{J}u + C + N + A^T(-\lambda_d + K_I \int (\lambda - \lambda_d))$$

Example: 6-DoF manipulator on a sphere with frictionless point contact

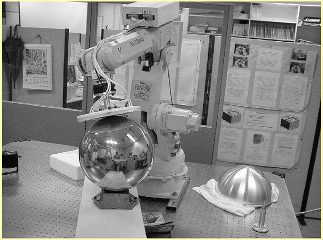
- Contact constraint:

$$v_z = 0 \Leftrightarrow [0\ 0\ 1\ 0\ 0\ 0]\mathrm{Ad}_{g^{-1}_{f l_f}} V_{of} = 0$$

$\Rightarrow$ Holonomic constraint:

$\eta = (\alpha_o^T, \alpha_f^T, \psi)$: Parametrization of Q

$P_\omega = \mathrm{diag}(1,1,0,1,1,1)$



- Newton-Euler Equations of motion:

$$M\dot{V}_{of} - \mathrm{ad}^T_{V_{of}} M V_{of} = F_m + G + A^T\lambda$$

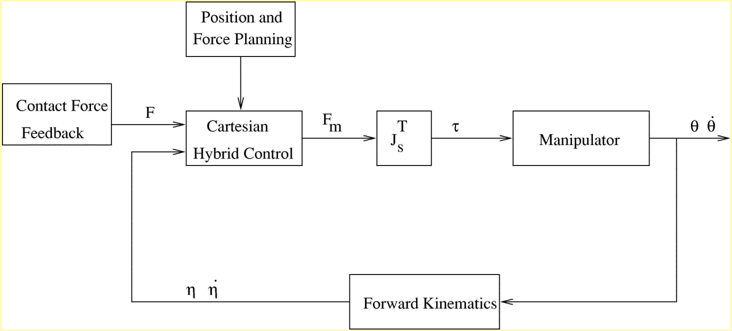
$$V_{of} = \begin{bmatrix} R_\psi M_o & -M_f & 0 \\ 0 & 0 & 0 \\ R_\psi R_o K_o M_o & -R_o K_f M_f & 0 \\ -T_o M_o & -T_f M_f & 1 \end{bmatrix} \begin{bmatrix} \dot{\alpha}_o \\ \dot{\alpha}_f \\ \dot{\psi} \end{bmatrix} \triangleq J\dot{\eta}$$

$$M J\ddot{\eta} + C_1 = F_m + G + A^T\lambda \qquad (*)$$

(Continues next slide)

$P_\omega(*)$:

$$\tilde{M}\ddot{\eta} + \tilde{C}_1 = B_1 F_m + B_1 G$$
$$-\phi_3 - \lambda = b_2 F_m + b_2 G$$
$$\hat{F} = [f_1 \quad f_2 \quad f_4 \quad f_5 \quad f_6]^T = \tilde{M}(\ddot{\eta}_d - K_v \dot{\tilde{e}} - K_p \tilde{e}) + \tilde{C}_1 - B_1 G$$
$$f_3 = -\phi_3 - \lambda_d + K_I \int (\lambda - \lambda_d) - b_2 G$$
$$\tau = J_s^T F_m$$



Example: 6-DoF manipulator rolling on a sphere

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}}_{\dot{A}_1(q)} V_{l_o l_f} = 0$$

$$\begin{aligned} f_c &= A_1^T \lambda, \lambda \in \mathbb{R}^4 \\ \begin{bmatrix} \omega_x \\ \omega_y \end{bmatrix} &= -R_0(K_f + R_\psi K_o R_\psi) M_f \dot{\alpha}_f \\ V_{of} &= \mathrm{Ad}_{g_{f l_f}} \cdot V_{l_o l_f} \\ &= \mathrm{Ad}_{g_{f l_f}} \begin{bmatrix} 0 \\ 0 \\ 0 \\ -R_o(K_f + R_\psi K_o R_\psi) M_f \\ o \end{bmatrix} \dot{\alpha}_f \\ &\triangleq J_f \dot{\alpha}_f \end{aligned}$$

$\mathrm{span}\{J_f\}$: Not involutive

(Continues next slide)

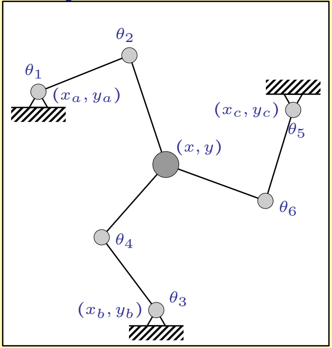
$$M J_f \ddot{\alpha}_f + (M \dot{J}_f \dot{\alpha}_f - \mathrm{ad}^T_{J_f \dot{\alpha}_f} M J_f \dot{\alpha}_f) = F_m + G + A^T \lambda$$

$$F_m = M J_f (\ddot{\alpha}_{fd} - K_p(\dot{\alpha}_f - \dot{\alpha}_{fd})) + (M \dot{J}_f \dot{\alpha}_f - \mathrm{ad}^T_{J_f \dot{\alpha}_f} M J_f \dot{\alpha}_f) + A^T(-\lambda_d + \int (\lambda - \lambda_d)) - G$$

◇ Example: Redundant parallel manipulator

$$\theta = (\theta_1, \ldots, \theta_6) \in E$$
$$\theta_a = (\theta_1, \theta_3, \theta_5)$$
$$\theta_p = (\theta_2, \theta_4, \theta_6)$$
$$H(\theta) = \begin{bmatrix} x_a + lc_1 + lc_{12} - x_b - lc_3 - lc_{34} \\ y_a + ls_1 + ls_{12} - y_b - ls_3 - ls_{34} \\ x_a + lc_1 + lc_{12} - x_c - lc_5 - lc_{56} \\ y_a + ls_1 + ls_{12} - y_c - ls_5 - ls_{56} \end{bmatrix} = 0$$

where $c_{ij} = \cos(\theta_i + \theta_j), s_{ij} = \sin(\theta_i + \theta_j)$.



(Continues next slide)

$$M_i(\theta) \in \mathbb{R}^{2\times 2} : i^{th} \text{ chain}$$

$$M(\theta) = \mathrm{diag}(M_1(\theta), \ldots, M_3(\theta))$$

$$M(\theta)\ddot{\theta} + C + N = \tau + A^T\lambda$$

If all joints are actuated, we can achieve:

Position control of end-effector
+
internal grasping force

As $\tau_2, \tau_4, \tau_6 = 0$,

$$\tilde{\theta} \in \mathbb{R}^2 : \text{ local parametrization of } Q = H^{-1}(0)$$

$$\theta = \psi(\tilde{\theta}) : \text{ embedding of } Q \text{ in } E$$

$$\dot{\theta} = J\dot{\tilde{\theta}}$$

(Continues next slide)

Given $P_\omega : T^*_\theta E \mapsto T^*_\theta Q$, the dynamics in $T^*_\theta Q$ is given by:

$$
\begin{aligned}
P_\omega M J \ddot{\tilde{\theta}} + P_\omega (C_1 + N) &= P_\omega \tau \\
\tilde{\tau} &= (\tau_1, \tau_3, \tau_5) \\
\tilde{P}_\omega &= (P_1, P_3, P_5) \\
\hat{\tau} &= \hat{P}_\omega \tilde{\tau} = P_\omega \tau \in \mathbb{R}^6 \\
\hat{\tau} &= P_\omega M J (\ddot{\tilde{\theta}}_d - K_v \dot{\tilde{e}} - K_p \tilde{e}) + P_\omega (C_1 + N)
\end{aligned}
$$

♢

Gauge-invariant Formulation (Aghili)

◇ **Square root factorization of inertia matrix:**

$$M = WW^T \text{(square root factorization)}$$

$$\begin{cases} v \triangleq W^T \dot{q} \in \mathbb{R}^n \\ u \triangleq W^{-1} \tau \in \mathbb{R}^n \end{cases} \qquad T = \frac{1}{2}\dot{q}^T M \dot{q} = \frac{1}{2} v^T v$$

◇ **Lagrange's Equation:**

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T}{\partial \dot{q}} - \frac{\partial T}{\partial q} = \tau \Rightarrow \frac{\mathrm{d}}{\mathrm{d}t}(Wv) - \frac{\partial v^T}{\partial q} v = \tau \Rightarrow$$

$$W\dot{v} + \dot{W}v - \frac{\partial v^T}{\partial q} v = \tau \Rightarrow \dot{v} + W^{-1}(\dot{W} - \frac{\partial v^T}{\partial q})v = W^{-1}\tau = u$$

Define $\Gamma \triangleq W^{-1}(\dot{W} - \frac{\partial v^T}{\partial q})$, then:

$$\boxed{\dot{v} + \Gamma v = u}$$

(Continues next slide)

Gauge-invariant Formulation (Aghili)

◇ **Change of coordinates:**

$$\bar{v} = V^T v, \bar{u} = V^T u, V \in U(n) \Rightarrow \frac{\mathrm{d}}{\mathrm{d}t}(V\bar{v}) + \Gamma(V\bar{v}) = V\bar{u} \Rightarrow$$

$$V\dot{\bar{v}} + \dot{V}\bar{v} + \Gamma V\bar{v} = V\bar{u} \Rightarrow \dot{\bar{v}} + V^T(\Gamma + \dot{V}V^T)V\bar{v} = \bar{u}$$

$$VV^T = I \Rightarrow \dot{V}V^T + V\dot{V}^T = 0 \Rightarrow \dot{V}V^T = -(\dot{V}V^T)^T =: -\Omega \Rightarrow$$

$$\bar{\Gamma} \triangleq V^T(\Gamma - \Omega)V, \Rightarrow \boxed{\dot{\bar{v}} + \bar{\Gamma}\bar{v} = \bar{u}}$$

◇ **Pfaffian constraint:**

$$A(q)\dot{q} = 0, A(q) \in \mathbb{R}^{m\times n}, \Lambda \triangleq AW^{-T} \Rightarrow \Lambda v = 0,$$

◇ **Lagrange's equation with constraint:**

$$\boxed{\dot{v} + \Gamma v = u + \Lambda^T\lambda}$$

(Continues next slide)

Gauge-invariant Formulation (Aghili)

◇ **SVD of Λ:**

$$\boxed{\Lambda = U\Sigma V^T, \bar{v} = V^T v, \bar{u} = V^T u, \bar{\Lambda} \triangleq \Lambda V}$$

where $\bar{\Lambda}\bar{v} = 0$ and:

$$\sigma = \begin{bmatrix} S & 0 \\ 0 & 0 \end{bmatrix}, S = \mathrm{diag}(\sigma_1, \ldots, \sigma_r), \sigma_1 \geq \cdots \geq \sigma_r, r \leq m$$

$$\begin{cases} U = [U_1 \ U_2], U_1 \in \mathbb{R}^{m\times r}, U_2 \in \mathbb{R}^{m\times(m-r)} \\ V = [V_1 \ V_2], V_1 \in \mathbb{R}^{n\times r}, V_2 \in \mathbb{R}^{n\times(n-r)} \end{cases} \Rightarrow$$

$$\bar{\Lambda} = [\Lambda_r \ 0_{m\times(n-r)}], \Lambda_r \triangleq U_1 S, \bar{\Lambda}\bar{v} = 0 \Rightarrow$$

$$\bar{v} = \begin{bmatrix} 0_{r\times 1} \\ v_r \end{bmatrix}, v_r \triangleq V_2^T v = V_2^T W^T \dot{q}$$

(Continues next slide)

Gauge-invariant Formulation (Aghili)

$$\bar{u} = \begin{bmatrix} u_o \\ u_r \end{bmatrix}, \begin{cases} u_o = V_1^T W^{-1} \tau \\ u_r = V_2^T W^{-1} \tau \end{cases}$$

$$\bar{\Gamma}_{ij} \triangleq V_i^T (\Gamma - \Omega) V_j, i, j = 1, 2, \Gamma_r \triangleq \bar{\Gamma}_{22}, \Gamma_o \triangleq \bar{\Gamma}_{12}$$

◇ **Decoupled equation of motion/constrained force:**

$$\boxed{\begin{aligned} \dot{v}_r + \Gamma_r v_r &= u_r \\ \Gamma_o v_r &= u_o + \Gamma_r^T \lambda \end{aligned}}$$

◇ **Combined equation of motion (Kane's equation):**

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t} \begin{bmatrix} q \\ v_r \end{bmatrix} = \begin{bmatrix} W^{-T} V_2 \\ -\Gamma_r \end{bmatrix} v_r + \begin{bmatrix} 0 \\ I \end{bmatrix} u_r}$$

(Continues next slide)

Gauge-invariant Formulation (Aghili)

◇ **Composite error vector** ε

$$q = q(\theta), \theta \in \mathbb{R}^{n-r} : \text{generalized coordinate (of } Q \subset \mathbb{R}^n\text{)}$$

$$v_r = B(\theta)\dot{\theta}, B(\theta) \triangleq V_2^T W^T J, J \triangleq \frac{\partial q}{\partial \theta}, \tilde{\theta} \triangleq \theta - \theta_d, \tilde{v}_r \triangleq v_r - v_{r_d} \Rightarrow$$

$$\varepsilon \triangleq \tilde{v}_r + BK_p\tilde{\theta} = B(\dot{\tilde{\theta}} + K_p\tilde{\theta}) : \text{ composite error}$$

$$s \triangleq v_{r_d} - BK_p\tilde{\theta} = v_r - \varepsilon = B(\dot{\theta}_d - K_p\tilde{\theta}), \tilde{\lambda} \triangleq \lambda - \lambda_d$$

◇ **Hybrid position/force control**

$$\boxed{\begin{aligned} u_r &= \dot{s} + \Gamma_r s - K_d\varepsilon \\ u_o &= -\Lambda_r^T \lambda_d + \Gamma_o v_r \end{aligned}}$$

Note: integration term $K_I \int (\lambda - \lambda_d)$ **is missing from** u_o**.**

(Continues next slide)

Gauge-invariant Formulation (Aghili)

$$\dot{v}_r + \Gamma_r v_r = u_r = \frac{\mathrm{d}}{\mathrm{d}t}(v_r - \varepsilon) + \Gamma_r(v_r - \varepsilon) - K_d\varepsilon \Rightarrow \dot{\varepsilon} = -(\Gamma_r + K_d)\varepsilon$$

$$\Lambda_r^T \lambda + \Gamma_o v_r = u_o = -\Lambda_r^T \lambda_d + \Gamma_o v_r \Rightarrow \Lambda_r^T \tilde{\lambda} + K_I \int \tilde{\lambda} = 0$$

◊ **Equivalence to the Geometric approach**

$$\boxed{\begin{aligned} P_\omega &= W V_2 V_2^T W^{-1} \\ I - P_\omega &= W V_1 V_1^T W^{-1} \end{aligned}} \Rightarrow \begin{aligned} V^T W^{-1} P_\omega \tau &= \begin{bmatrix} 0 \\ V_2^T W^{-1} \tau \end{bmatrix} = \begin{bmatrix} 0 \\ u_r \end{bmatrix} \\ V^T W^{-1}(I - P_\omega)\tau &= \begin{bmatrix} V_1^T W^{-1} \tau \\ 0 \end{bmatrix} = \begin{bmatrix} u_o \\ 0 \end{bmatrix} \end{aligned}$$

(Continues next slide)

Gauge-invariant Formulation (Aghili)

1. Geometric approach (K_p, K_I, K_d)**:**

$$u = \underbrace{\begin{bmatrix} V_1^T W^{-1} A^T(-\lambda_d + K_I \int(\lambda - \lambda_d)) \\ V_2^T W^T J(\ddot{\theta}_d - K_d \dot{\tilde{\theta}} - K_p \tilde{\theta}) \end{bmatrix}}_{fb} + \underbrace{V^T W^{-1} C_1}_{ff}$$

2. Gauge-invariant formulation $(\tilde{K}_p, \tilde{K}_d)$**:**

$$u = \begin{bmatrix} u_o \\ u_r \end{bmatrix} = \underbrace{\begin{bmatrix} V_1^T W^{-1} A^T(-\lambda_d) \\ V_2^T W^T J(\ddot{\theta}_d - K_d' \dot{\tilde{\theta}} - K_p' \tilde{\theta}) \end{bmatrix}}_{fb} + \underbrace{C_1'}_{ff} \Rightarrow$$

$$K_d' \triangleq \tilde{K}_p + (V_2^T W^T J)^{-1} \tilde{K}_d (V_2^T W^T J)$$

$$K_p' \triangleq (V_2^T W^T J)^{-1} \tilde{K}_d (V_2^T W^T J) \tilde{K}_p$$

† End of Section †

4.5 References

□ Reference:

[1] D.T., Greenwood, **Classical Dynamics**. Dover Publications, 1997.
[2] V.I., Arnold, **Mathematical Methods of Classical Mechanics (Graduate Texts in Mathematics)**, 2nd ed. Springer, 1989.
[3] J.E., Marsden and T. Ratiu, **Introduction to Mechanics and Symmetry: A Basic Exposition of Classical Mechanical Systems (Texts in Applied Mathematics)**, 2nd ed. Springer, 1999.
[4] G., Rodriguez and K.K., Delgado, **Spatial operator factorization and inversion of the manipulator mass matrix**. Transactions on Robotics and Automation, 8(1), pp.65-76, 1992.
[5] F.C., Park and J.E., Bobrow and S.R. Ploen, **A Lie Group Formulation of Robot Dynamics.** International Journal of Robotics Research, Vol.14(6), pp.609-618, 1995.
[6] S.R., Ploen, **Geometric Algorithms for the Dynamics and Control of Multibody Systems**. Phd Thesis, University of California Irvine, 1997.
[7] S.R., Ploen and F.C., Park, **Coordinate-Invariant Algorithms for Robot Dynamics.** IEEE Transactions on Robotics and Automation, Vol. 15(6), pp.1130-1135, 1999.
[8] G.F., Liu and Z.X., Li, **A Unified Approach to Modeling and Control of Constrained Mechanical Systems**. IEEE Transactions on Robotics and Automation, Vol.18(4), pp.574-587, 2002.
[9] F., Aghili, **A Gauge-Invariant Formulation for Constrained Mechanical Systems Using Square-Root Factorization and Unitary Transformation**. Journal of Computational and Nonlinear Dynamics, Vol.4, pp.031010-1~031010-10, 2009.