Chapter 5 Manipulator Control

Lecture Notes for

A Geometrical Introduction to Robotics and Manipulation

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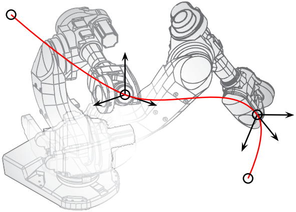
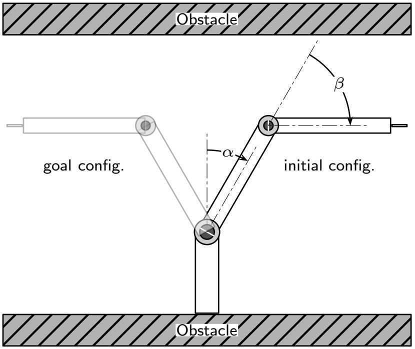
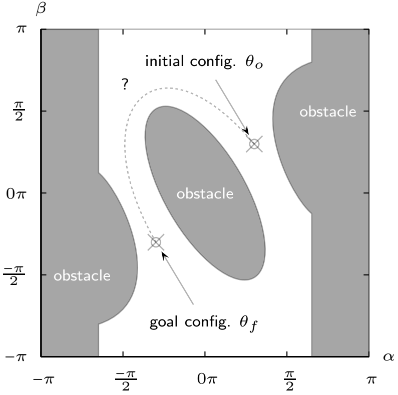


A Simple Robot Planning Example



A Two-DoF robot arm



Configuration space $\mathcal{C}$

Definition: Path planning

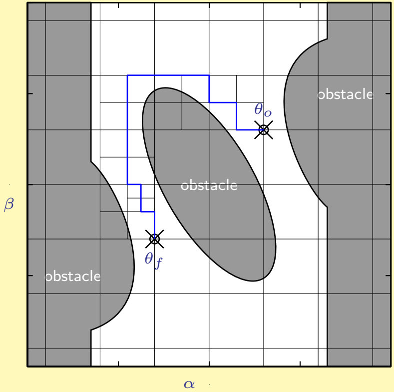
Given an initial and a final configuration θ_o and θ_f in the configuration space $\mathcal{C}$, find a collision-free path, $\theta : [0,1] \mapsto \mathcal{C}$ such that $\theta(0) = \theta_o$ and $\theta(1) = \theta_f$.

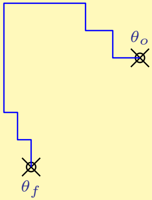
(see next page)

Path planning methods ([1, 2])

Probabilistic roadmap, Cell decomposition, numerical potential field,etc.

⋄ Example: Cell decomposition

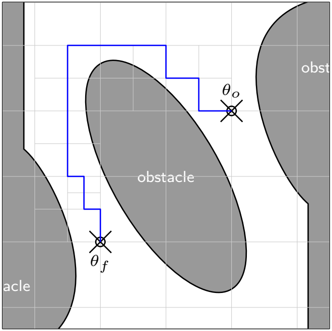




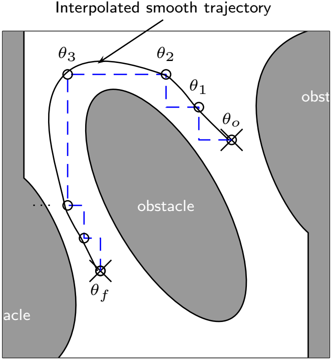
A collision free path

⋄ Note: The generated path may not be suitable for robot control. E.g. not smooth.

Trajectory generation



A collision free path



Generated via-points

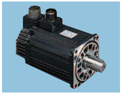
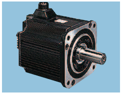
Definition: Trajectory generation

Given θ_o and θ_f, and a sequence of via points $\theta_k, k = 1, \ldots, n-1$, compute a trajectory $\theta : [t_0, t_n] \mapsto \mathcal{C}$ such that $\theta(t_0) = \theta_o$, $\theta(t_n) = \theta_f$, and $\theta(t_k) = \theta_k$, $k = 1, \ldots, n-1$.

$\diamond$ Note: the trajectory should be easy to specify, store and generate in real-time.

Main constraints on trajectory generation

1. Rated speed $|\dot{\theta}^i(t)| \leq \dot{\theta}^i_{\max}$
2. Rated Acceleration $|\ddot{\theta}^i(t)| \leq \ddot{\theta}^i_{\max}$
3. Bounded Jerk (avoiding excitation): $|\dddot{\theta}^i(t)| \leq \dddot{\theta}^i_{\max}$
4. Continuity in velocity, acceleration for bounded jerk

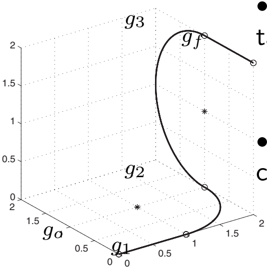
		Small Capacity		Medium Capacity		Large Capacity
		SGMAH	SGMPH	SGMSH	SGMGH	SGMBH
Yaskawa Σ series motor specification						
Rated Torque Range	[lb-in]	0.8-21	2.8-42	28.2-140	28-845	1239-3100
Peak Torque Range	[lb-in]	2.5-63	8.4-126	84.4-422	79-1988	2478-6120
Rated Speed	[rpm]	3000	3000	3000	1500	1500
Max. Speed	[rpm]	5000	5000	5000	3000	2000
Rated Acc.	[Rad/s^2]	57500	38500	12780	1575	1780
Power Range	[W]	30-750	100-1.5k	1k-5k	500-15k	22k-55k
Inertia		Low	Medium	Low	Medium	Medium

Generation of via-points

1. by a path planner (the previous example)
2. by a teach pendant:

via points directly recorded as joint angles, no inverse kinematics required.

3. by G-code (through CAM software):

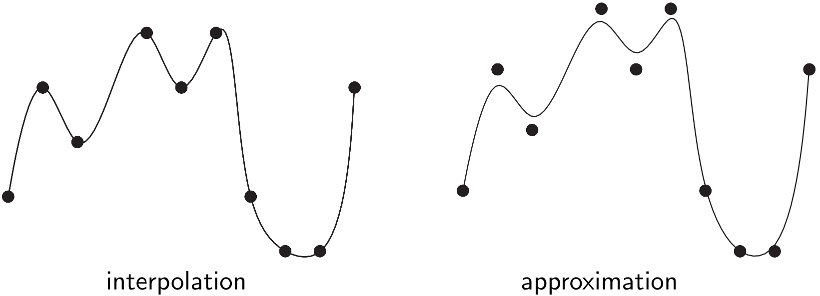


• use inverse kinematics and inverse Jacobian to obtain joint angles and velocity information:

$$g_i, V_i \xrightarrow{g^{-1}, J^{-1}} \theta_i, \dot{\theta}_i, i = 0, 1, 2, \dots$$

• constraints from both joint and workspace need be considered.

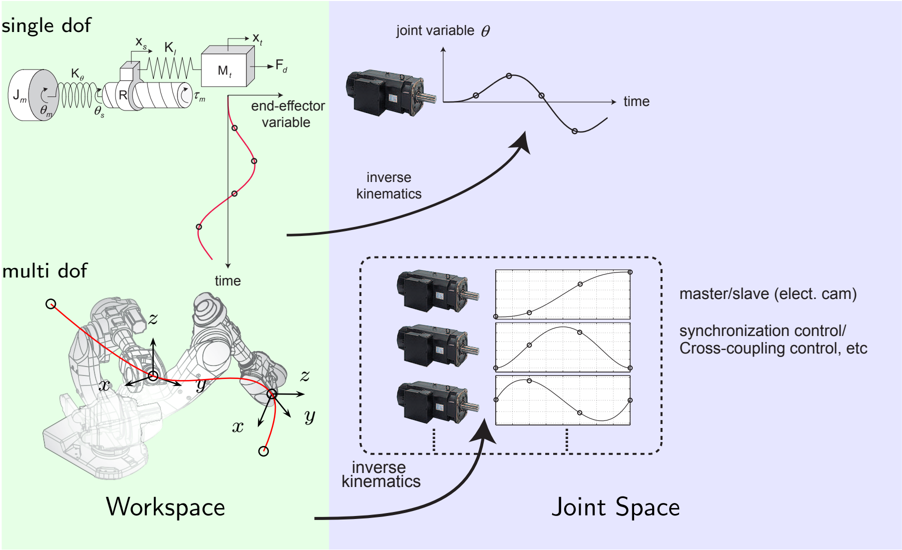
From via-points to trajectory



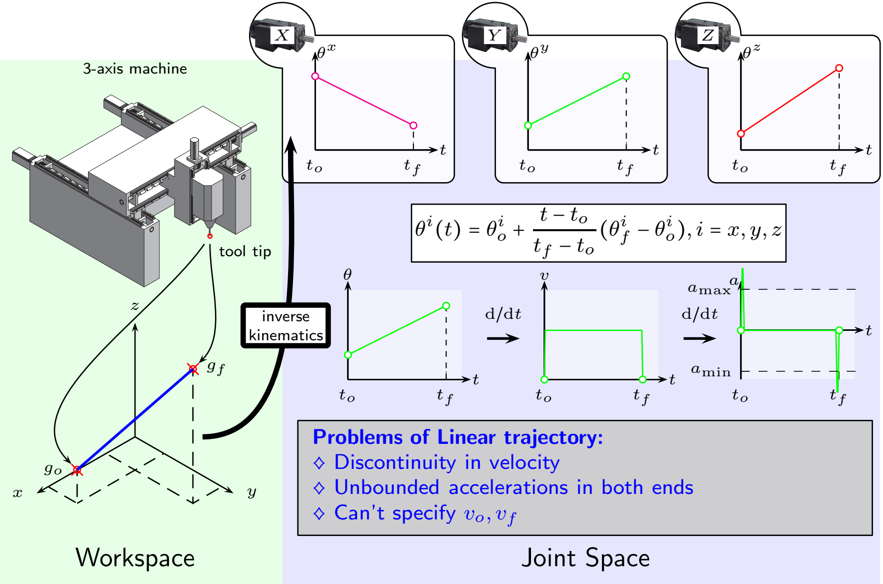
Definition: Interpolation
Constructing new data points within the range of a discrete set of known data points (exact fitting).

Definition: Approximation
Inexact fitting of a discrete set of known data points.

Types of trajectory



A simple example-Linear interpolation with no via-points



$$\theta^i(t) = \theta^i_o + \frac{t - t_o}{t_f - t_o}(\theta^i_f - \theta^i_o), i = x, y, z$$

Problems of Linear trajectory:

- ◇ Discontinuity in velocity
- ◇ Unbounded accelerations in both ends
- ◇ Can't specify v_o, v_f

Better approaches

1. Increase the order of the trajectory:

Linear trajectory:

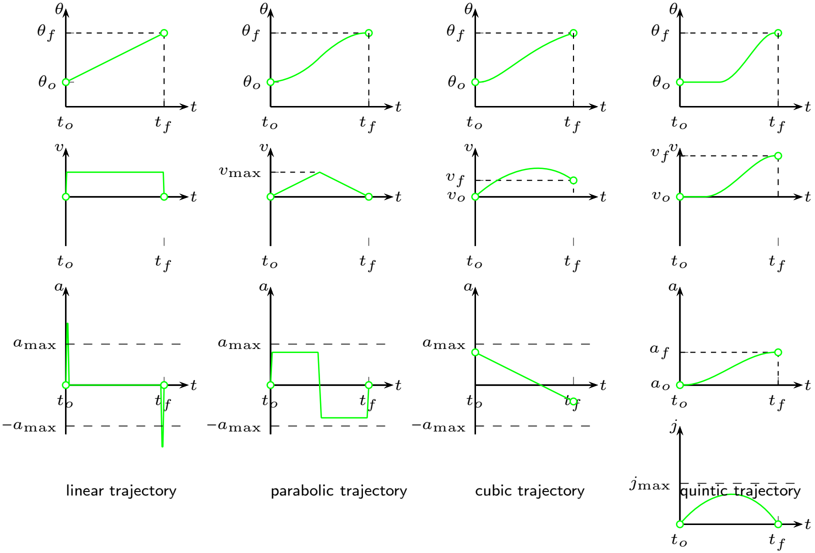
$$\theta(t) = \theta_o + \frac{t - t_o}{t_f - t_o}(\theta_f - \theta_o) = a_0 + a_1 t, a_0 = \frac{t_f\theta_o - t_o\theta_f}{t_f - t_o}, a_1 = \frac{\theta_f - \theta_o}{t_f - t_o}$$

$\Rightarrow 1^{st}$ order polynomial in t, $v(t)$ const., $a(t)$ impulse at t_o, t_f.

Higher order trajectories:

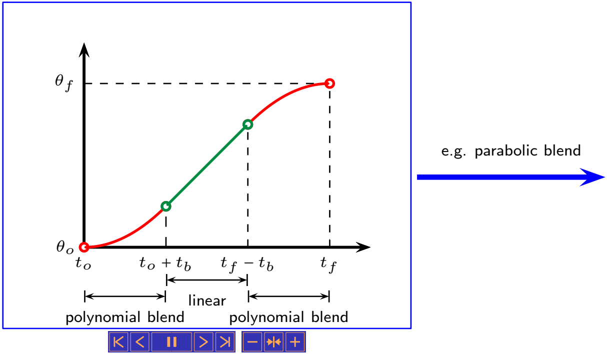
order of θ	order of v	order of a	allowable design variables
2 (parabolic)	1	0	$\theta_o, \theta_f, v_{\max}$
3 (cubic)	2	1	$\theta_o, \theta_f, v_o, v_f$
5 (quintic)	4	3	$\theta_o, \theta_f, v_o, v_f, a_o, a_f$

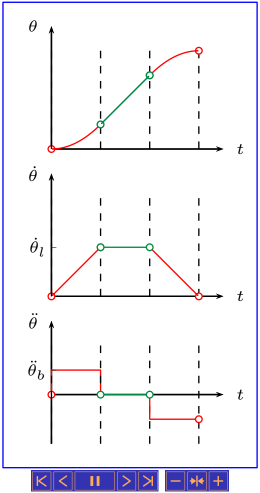
Higher order trajectories: time profile



Better approaches

2. Composition of elementary trajectories:
E.g., linear trajectory with polynomial blends:

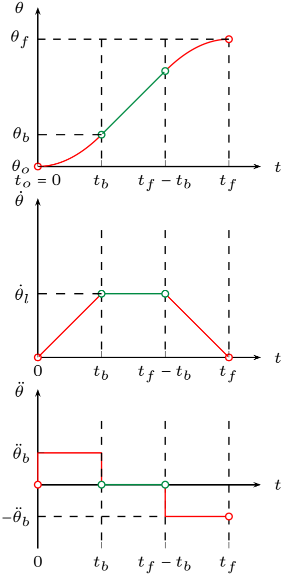




◇ A list of composite trajectories:

- ☐ Linear with parabolic (Trapezoidal): 2-1-2
- ☐ Linear with circular
- ☐ Linear with quintic: 5-1-5
- ☐ Linear with S (Double S): 3-2-3-1-3-2-3

Linear Function with Parabolic Blends (LFPB)



◇ Input: $\theta(t_o) = \theta(0) = \theta_o$
$\theta(t_f) = \theta_f$
$t_d = t_f - t_o$: Duration of travel

◇ Control Parameters:

$$\dot{\theta}_l(\leq \dot{\theta}_{\max}) : \text{ linear velocity}$$

$$\ddot{\theta}_b(\leq \ddot{\theta}_{\max}) : \text{ blend acceleration}$$

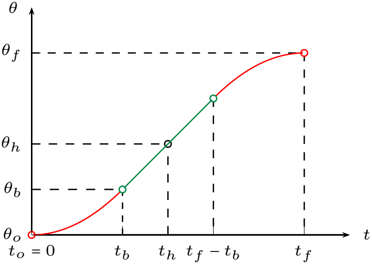
$$t_b(0 < t_b \leq \frac{t_d}{2}) : \text{ blend time}$$

$$t_d - 2t_b : \text{ linear time}$$

◇ LFPB Trajectory:

$$\theta(t) = \begin{cases} \theta_o + \frac{1}{2}\ddot{\theta}_b(t - t_o)^2 & t_o \leq t < t_o + t_b \\ \theta_o + \ddot{\theta}_b t_b \left(t - t_o - \frac{t_b}{2}\right) & t_o + t_b \leq t < t_f - t_b \\ \theta_f - \frac{1}{2}\ddot{\theta}_b(t_f - t)^2 & t_f - t_b \leq t \leq t_f \end{cases}$$

Derivation of the LFPB



◇ Velocity match condition:

$$\ddot{\theta}_b t_b = \frac{\theta_h - \theta_b}{t_h - t_b}, (t_h \triangleq \frac{t_d}{2}, \theta_h \triangleq \theta(t_h)) \quad (5.1.1)$$

◇ Blend region:

$$\theta_b \triangleq \theta(t_b) = \theta_o + \frac{1}{2}\ddot{\theta}_b t_b^2 \quad (5.1.2)$$

$$(5.1.1) + (5.1.2) : \Rightarrow \ddot{\theta}_b t_b^2 - \ddot{\theta}_b t_d t_b + (\theta_f - \theta_o) = 0$$

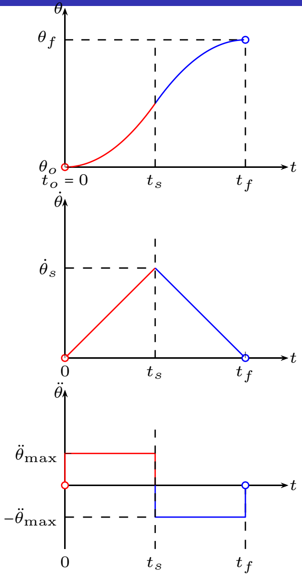
$$\Rightarrow t_b = \frac{t_d}{2} - \frac{\sqrt{\ddot{\theta}_b^2 t_d^2 - 4\ddot{\theta}_b(\theta_f - \theta_o)}}{2\ddot{\theta}_b} \quad (5.1.3)$$

◇ Constraints on $\ddot{\theta}_b$:

$$\ddot{\theta}_b \geq \frac{4(\theta_f - \theta_o)}{t_d^2} \quad (5.1.4)$$

◇ Observation:
- As $\ddot{\theta}_b \uparrow$, $t_b \downarrow$ and linear time $t_d - 2t_b \uparrow$; as $\ddot{\theta}_b \to \infty$, LFPB becomes linear interpolation.
- With equality in (5.1.4), linear portion of LFPB shrinks to zero.

Minimum Time Trajectory (Bang Bang)



◇ Given: θ_o, θ_f and $\ddot{\theta}_{\max}$

◇ Minimize: t_d

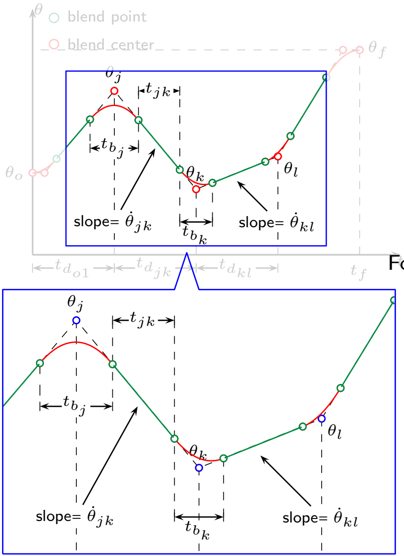
◇ Solution: Bang-bang trajectory

$$\ddot{\theta}(t) = \begin{cases} \ddot{\theta}_{\max} & 0 \le t \le t_s \\ -\ddot{\theta}_{\max} & t_s \le t \le t_d \end{cases}$$

where the switching time t_s is obtained from (5.1.3)

$$\boxed{t_s = \frac{t_d}{2} = \sqrt{\frac{\theta_f - \theta_o}{\ddot{\theta}_{\max}}}}$$

LFPB for a Path with Via Points ([3])



◇ Given: $t_o, \theta_o, t_f, \theta_f, \ddot\theta_b$, via points $\{\theta_i\}_1^m$ at time $\{t_i\}_1^m$ (time duration $t_{d_{jk}} \triangleq t_k - t_j$)

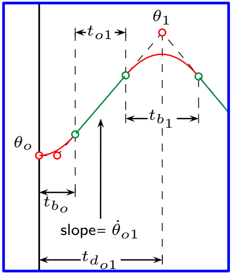
◇ Find: $\theta(t)$ interpolating θ_o, θ_f and approximating $\{\theta_i\}_1^m$.

◇ Solution: LFPB with via points

For via points $j, k, l = 1, \ldots, m$:

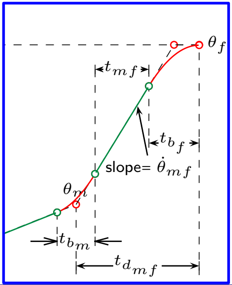
$$\begin{aligned} \dot\theta_{jk} &= \frac{\theta_k - \theta_j}{t_{d_{jk}}} \text{ (linear vel.)} \\ \ddot\theta_k &= \mathrm{Sgn}(\dot\theta_{kl} - \dot\theta_{jk}) \left|\ddot\theta_b\right| \text{ (Blend acc.)} \\ t_{b_k} &= \frac{\dot\theta_{kl} - \dot\theta_{jk}}{\ddot\theta_k} \text{ (Blend dur.)} \\ t_{jk} &= t_{d_{jk}} - \frac{1}{2} t_{b_j} - \frac{1}{2} t_{b_k} \text{ (Linear dur.)} \end{aligned} \tag{5.1.5}$$

LFPB for a Path with Via Points ([3])



◇ **First Segment:**

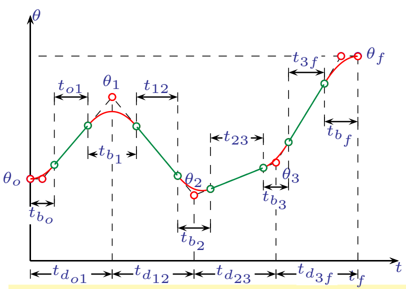
$$
\begin{aligned}
\ddot{\theta}_o &= \mathrm{Sgn}(\theta_1 - \theta_o)\left|\ddot{\theta}_b\right| \ \text{(Blend acc.)} \\
t_{b_o} &= t_{d_{o1}} - \sqrt{t_{d_{o1}}^2 - \frac{2(\theta_1 - \theta_o)}{\ddot{\theta}_o}} \ \text{(Blend dur.)} \\
\dot{\theta}_{o1} &= \frac{\theta_1 - \theta_o}{t_{d_{o1}} - \frac{1}{2}t_{b_o}} \ \text{(linear vel.)} \\
t_{o1} &= t_{d_{o1}} - t_{b_o} - \frac{1}{2}t_{b_1} \ \text{(linear dur.)}
\end{aligned}
\quad (5.1.6)
$$



◇ **Last Segment:**

$$
\begin{aligned}
\ddot{\theta}_f &= \mathrm{Sgn}(\theta_m - \theta_f)\left|\ddot{\theta}_b\right| \ \text{(Blend acc.)} \\
t_{b_f} &= t_{d_{mf}} - \sqrt{t_{d_{mf}}^2 + \frac{2(\theta_f - \theta_m)}{\ddot{\theta}_f}} \ \text{(Blend dur.)} \\
\dot{\theta}_{mf} &= \frac{\theta_f - \theta_m}{t_{d_{mf}} - \frac{1}{2}t_{b_f}} \ \text{(linear vel.)} \\
t_{mf} &= t_{d_{mf}} - t_{b_f} - \frac{1}{2}t_{b_m} \ \text{(linear dur.)}
\end{aligned}
\quad (5.1.7)
$$

Example: LFPB with 3 via points



Given:

$$\theta_o = 1, \theta_f = 2.5, \ddot{\theta}_b = 4, \theta_1 = 2, \theta_2 = .8, \theta_3 = 1.2$$

Apply (5.1.6):

$$\ddot{\theta}_o = 4, t_{b_o} = 1 - \sqrt{1^2 - \frac{2(2-1)}{4}} = 0.29,$$

$$\dot{\theta}_{o1} = \frac{2-1}{1 - \frac{1}{2}0.29} = 1.17$$

Apply (5.1.5):

$$\dot{\theta}_{12} = \frac{0.8-2}{1} = -1.2, \ddot{\theta}_1 = -4, t_{b_1} = \frac{-1.2-1.17}{-4} = 0.59, t_{o1} = 1 - 0.29 - \frac{1}{2}0.59 = 0.41$$

$$\dot{\theta}_{23} = \frac{1.2-0.8}{1} = 0.4, \ddot{\theta}_2 = 4, t_{b_2} = \frac{0.4+1.2}{4} = 0.4, t_{12} = 1 - \frac{1}{2}0.59 - \frac{1}{2}0.4 = 0.51$$

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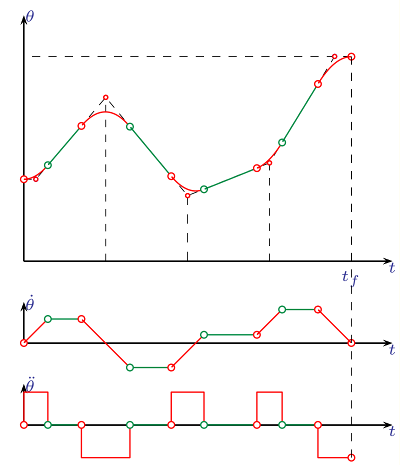
Example: LFPB with 3 via points



Apply (5.1.7):

$$\ddot\theta_f = -4, t_{b_f} = 1 - \sqrt{1^2 + \frac{2(2.5-1.2)}{-4}} = 0.41,$$

$$\dot\theta_{3f} = \frac{2.5-1.2}{1-\frac{1}{2}0.41} = 1.63$$

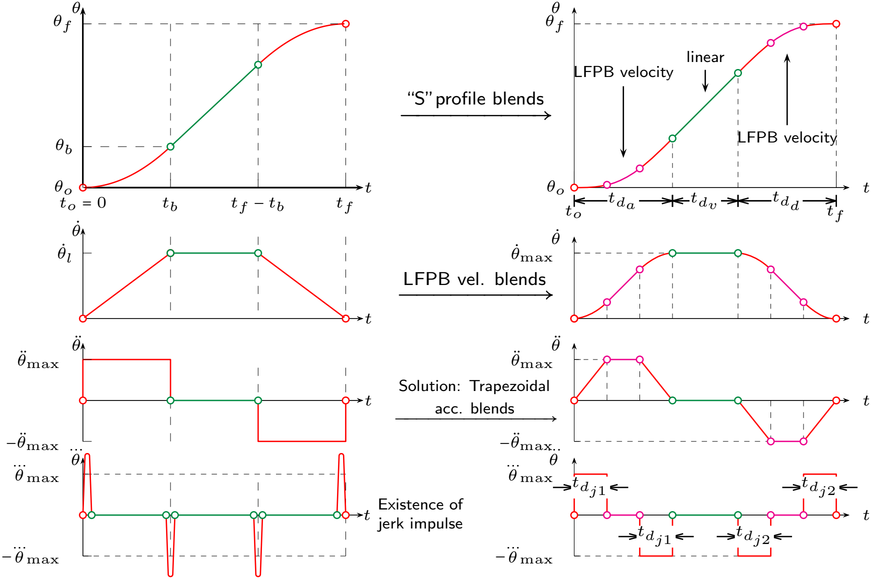
Apply (5.1.5):

$$\ddot\theta_3 = 4, t_{b_3} = \frac{1.63-0.4}{4} = 0.31,$$

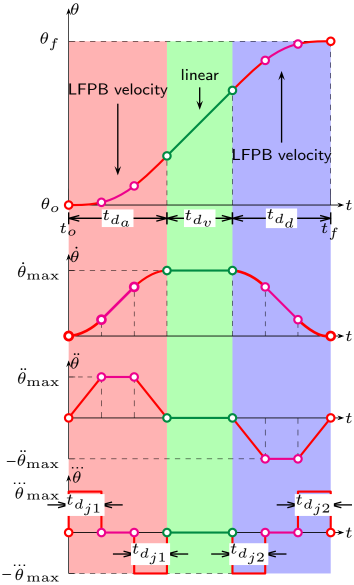
$$t_{3f} = 1 - \frac{1}{2}0.31 - 0.41 = 0.44,$$

$$t_{23} = 1 - \frac{1}{2}0.4 - \frac{1}{2}0.31 = 0.65$$

Disadvantage of LFPB



Linear function with Double S trajectory



♢ **Double "S"trajectory:**
Linear trajectory with LFPB velocity blends

♢ **Advantage over LFPB:** Bounded jerk

♢ **Input:**

$$\theta_o, \theta_f, \dot{\theta}_o, \dot{\theta}_f, \ddot{\theta}_o, \ddot{\theta}_f, \dot{\theta}_{\max}, \ddot{\theta}_{\max}, \dddot{\theta}_{\max}$$

♢ **Output (for details see [4]):**

t_{d_a}: Acceleration duration

t_{d_v}: Linear duration

t_{d_d}: Deceleration duration

$t_{d_{j1}}, t_{d_{j2}}$:
Jerk duration for acceleration and deceleration

♢ **Generalization:** Double S
with via points (similar to LFPB with via points)

Computation of the double S trajectory ($\theta_f > \theta_0$)

Notations:

$\dot{\theta}_{\lim}(\leq \dot{\theta}_{\max})$: maximal velocity
$\ddot{\theta}_{\lim_a}(\leq \ddot{\theta}_{\max})$: maximal acceleration in the acceleration phase
$\ddot{\theta}_{\lim_d}(\leq \ddot{\theta}_{\max})$: maximal acceleration in the deceleration phase

Acceleration phase:

$$\theta(t) = \begin{cases} \theta_o + \dot{\theta}_o t + \dddot{\theta}_{\max}\frac{t^3}{6} & t \in [0, t_{d_{j1}}] \\ \theta_o + \dot{\theta}_o t + \frac{\ddot{\theta}_{\lim_a}}{6}(3t^2 - 3t_{d_{j1}} t + t_{d_{j1}}^2) & t \in [t_{d_{j1}}, t_{d_a} - t_{d_{j1}}] \\ \theta_o + (\dot{\theta}_{\lim} + \dot{\theta}_o)\frac{t_{d_a}}{2} - \dot{\theta}_{\lim}(t_{d_a} - t) - \dddot{\theta}_{\max}\frac{(t_{d_a} - t)^3}{6} & t \in [t_{d_a} - t_{d_{j1}}, t_{d_a}] \end{cases}$$

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Computation of the double S trajectory ($\theta_f > \theta_0$)

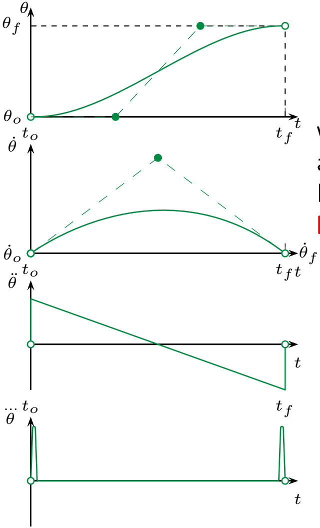
Constant velocity phase:

$$\theta(t) = \theta_o + (\dot{\theta}_{\lim} + \dot{\theta}_o)\frac{t_{d_a}}{2} + \dot{\theta}_{\lim}(t - t_{d_a}), t \in [t_{d_a}, t_{d_a} + t_{d_v}]$$

Deceleration phase: Define $t_d = t_{d_a} + t_{d_v} + t_{d_d}$:

$$\theta(t) = \begin{cases} \theta_f - (\dot{\theta}_{\lim} + \dot{\theta}_f)\frac{t_{d_d}}{2} + \dot{\theta}_{\lim}(t - t_d + t_{d_d}) - \dddot{\theta}_{\max}\frac{(t-t_d+t_{d_d})^3}{6} & t \in [t_{d_a} + t_{d_v}, t_{d_a} + t_{d_v} + t_{d_{j2}}] \\ \theta_f - (\dot{\theta}_{\lim} + \dot{\theta}_f)\frac{t_{d_a}}{2} + \dot{\theta}_{\lim}(t - t_d + t_{d_d}) + \frac{\ddot{\theta}_{\lim_d}}{6}\left(3(t - t_d + t_{d_d})^2 - 3t_{d_{j2}}(t - t_d - t_{d_d}) + t_{d_{j2}}^2\right) & t \in [t_{d_a} + t_{d_v} + t_{d_{j2}}, t_d - t_{d_{j2}}] \\ \theta_f - \dot{\theta}_f(t_d - t) - \dddot{\theta}_{\max}\frac{(t_d-t)^3}{6} & t \in [t_d - t_{d_{j2}}, t_d] \end{cases}$$

Cubic polynomial trajectory



$$\theta(t) = a_0 + a_1(t - t_o) + a_2(t - t_o)^2 + a_3(t - t_o)^3, \quad t \in [t_o, t_f], t_d \triangleq t_f - t_o$$

where 4 parameters a_0, a_1, a_2, a_3 are to be determined by boundary conditions.

Property:

bounded acceleration, **jerk impulse at both ends**.

$$\begin{cases} \theta(t_o) = a_0 = \theta_o \\ \dot\theta(t_o) = a_1 = \dot\theta_o \\ \theta(t_f) = \sum_{i=0}^{3} a_i t_d^i = \theta_f \\ \dot\theta(t_f) = \sum_{i=0}^{2} (i+1) a_{i+1} t_d^i = \dot\theta_f \end{cases} \Rightarrow \begin{cases} a_0 = \theta_o \\ a_1 = \dot\theta_o \\ a_2 = \dfrac{3h - (2\dot\theta_o + \dot\theta_f)t_d}{t_d^2} \\ a_3 = \dfrac{-2h + (\dot\theta_o + \dot\theta_f)t_d}{t_d^3} \end{cases}$$

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Multipoint Cubic interpolation

Given $\theta_o, \theta_f, \dot{\theta}_o, \dot{\theta}_f$ at t_o, t_f and via points $\{\theta_k\}_1^m$ at time $\{t_k\}_1^m$, solve $\boxed{a_{0k} + a_{1k}(t-t_k) + a_{2k}(t-t_k)^2 + a_{3k}(t-t_k)^3}$ for the unknowns $\{a_{0k}, a_{1k}, a_{2k}, a_{3k}\}_o^m$.

1. If via-point velocities $\{\dot{\theta}_k\}_1^m$ are directly assigned by user, solve the $m+1$ BVPs:

$$\begin{cases} a_{0k} = \theta_o, & a_{1k} = \dot{\theta}_o \\ a_{2k} = \dfrac{3h - (2\dot{\theta}_o + \dot{\theta}_f)t_d}{t_d^2}, & a_{3k} = \dfrac{-2h + (2\dot{\theta}_o + \dot{\theta}_f)t_d}{t_d^3} \end{cases}, k=o,1,\dots,m$$

2. If only $\dot{\theta}_o, \dot{\theta}_f$ are given:
   1. compute $\{\dot{\theta}_k\}_1^m$ using a heuristic method; or
   2. design $\{\dot{\theta}_k\}_1^m$ so as to achieve acceleration continuity

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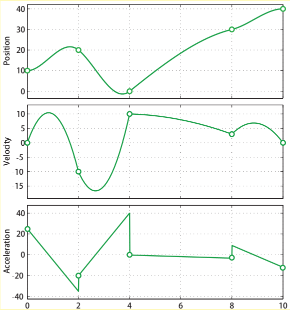
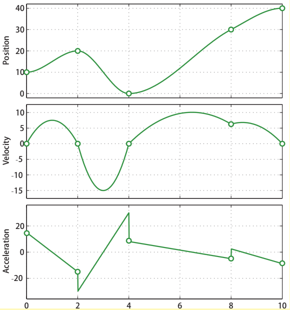
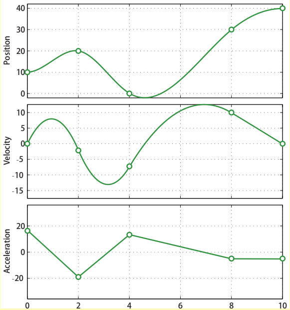
Example: Cubic interpolation with 3 via points



Approach 1



Approach 2

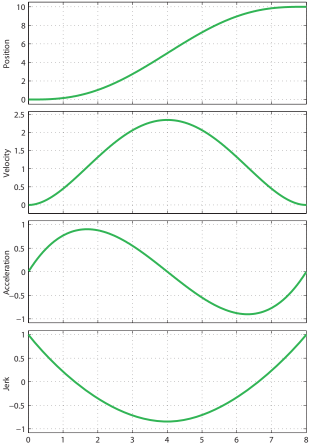


Approach 3

In Approach 1, via point velocities are arbitrarily assigned. This may lead to large and discontinuous accelerations. In Approach 2, $\dot{\theta}_k = 0$, if $\mathrm{Sign}(d_k) \neq \mathrm{Sign}(d_{k+1})$, and $\frac{1}{2}(d_k + d_{k+1})$, otherwise. Here $d_k = \frac{\theta_k - \theta_{k-1}}{t_{d_{k-1,k}}}$ is the slope from θ_{k-1} to θ_k. Note the discontinuity in acceleration. In Approach 3, we choose the polynomials so that acceleration is continuous.

Quintic polynomial trajectory

$$\theta(t) = \sum_{i=0}^{5} a_i (t - t_o)^i, t \in [t_o, t_f]$$



with 6 unknowns coefficients $a_i, i = 0, \ldots, 5$.

Properties:

◇ Smooth and bounded jerk

◇ Acc. continuity in composite curves.

Boundary conditions:

$$\theta(t_o) = \theta_o, \quad \theta(t_f) = \theta_f$$
$$\dot{\theta}(t_o) = \dot{\theta}_o, \quad \dot{\theta}(t_f) = \dot{\theta}_f$$
$$\ddot{\theta}(t_o) = \ddot{\theta}_o, \quad \ddot{\theta}(t_f) = \ddot{\theta}_f$$

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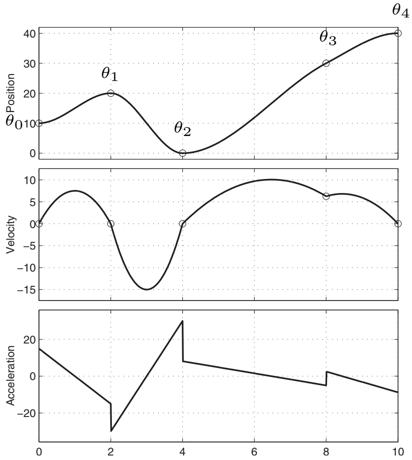
Quintic polynomial trajectory

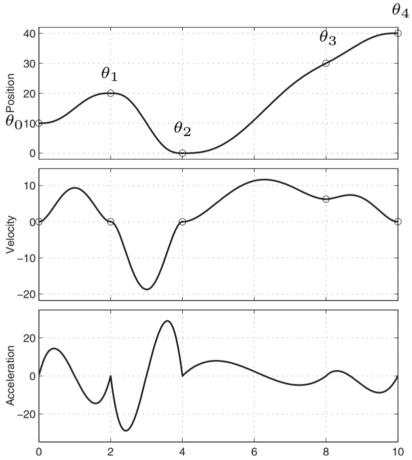
Define $t_d \triangleq t_f - t_o, h \triangleq \theta_f - \theta_o$, then:

$$
\begin{aligned}
a_0 &= \theta_o \\
a_1 &= \dot{\theta}_o \\
a_2 &= \frac{1}{2}\ddot{\theta}_o \\
a_3 &= \frac{1}{2t_d^3}[20h - (8\dot{\theta}_f + 12\dot{\theta}_o)t_d - (3\ddot{\theta}_o - \ddot{\theta}_f)t_d^2] \\
a_4 &= \frac{1}{2t_d^4}[-30h - (14\dot{\theta}_f + 16\dot{\theta}_o)t_d - (3\ddot{\theta}_o - 2\ddot{\theta}_f)t_d^2] \\
a_5 &= \frac{1}{2t_d^5}[12h - 6(\dot{\theta}_f + \dot{\theta}_o)t_d - (\ddot{\theta}_f - \ddot{\theta}_o)t_d^2]
\end{aligned}
$$

Comparison of Cubic and Quintic Composites



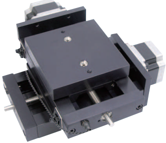
Composition of cubic polynomials: acceleration discontinuity.



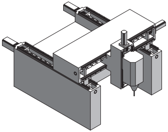
Composition of quintic polynomials: continuity in acceleration.

Trajectory Generation in Task Space

1. **Euclidean space:**

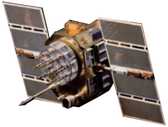


xy table ($\mathbb{R}^2$)



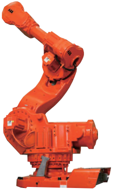
3-axis machine ($\mathbb{R}^3$)

2. **Subgroups (of $SE(3)$):**



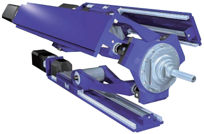
Satellite ($SO(3)$)

pick-and-place (X)

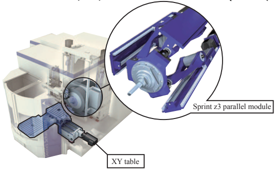


6 dof robot ($SE(3)$)

3. **Submanifolds of $SE(3)$:**



tooling module ($SE(3)/PL(z)$)



five-axis machining ($SE(3)/R(o,z)$)

Trajectory Generation in $\mathbb{R}^n$

◇ **A trajectory in** $\mathbb{R}^3$ $p : [t_o, t_f] \mapsto \mathbb{R}^n$

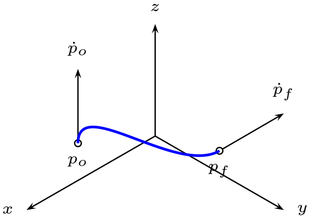
$$\text{e.g. } p(t) = \begin{bmatrix} a_{01} \\ \vdots \\ a_{0n} \end{bmatrix} + \begin{bmatrix} a_{11} \\ \vdots \\ a_{1n} \end{bmatrix}(t - t_o) + \cdots + \begin{bmatrix} a_{m1} \\ \vdots \\ a_{mn} \end{bmatrix}(t - t_o)^m, t \in [t_o, t_f]$$

◇ **A cubic example:**

Given $p_o, p_f, \dot{p}_o, \dot{p}_f, t_o, t_f, t_d = t_f - t_d, \vec{h} = p_f - p_o$, generate:

$$\vec{a}_o + \vec{a}_1(t - t_o) + \vec{a}_2(t - t_o)^2 + \vec{a}_3(t - t_o)^3, t \in [0, 1], \vec{a}_i \in \mathbb{R}^n$$

$$\Rightarrow \begin{cases} \vec{a}_0 = p_o \\ \vec{a}_1 = \dot{p}_o \\ \vec{a}_2 = \dfrac{3\vec{h} - (2\dot{p}_o + \dot{p}_f)t_d}{t_d^2} \\ \vec{a}_3 = \dfrac{-2\vec{h} + (\dot{p}_o + \dot{p}_f)t_d}{t_d^3} \end{cases}$$



For more information, see [4].

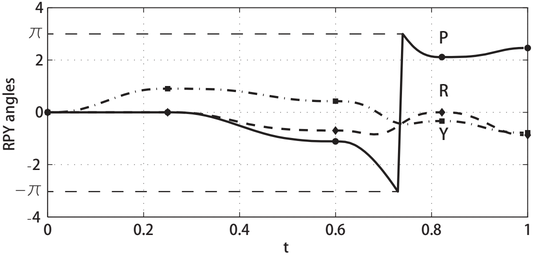
Trajectory Generation in $SO(3)$

A naive approach:

Generate a trajectory using Euler angles, e.g., roll-pitch-yaw (RPY) angles or ZYZ angles.

Problems:

1. Parametrization singularity!



e.g., RPY angles, defined on $[-\pi, \pi]^3$ encounter a parametrization singularity

2. Derivatives of the Euler angles have no physical meaning!

Trajectory Generation in $SO(3)$

A more meaningful approach:

1. Choose physically meaningful coordinates;
2. Add via-points to avoid parametrization singularity;
3. Generate trajectory and use inverse kinematics to obtain joint trajectory

Candidate coordinates:

- Unit quaternion:

$$Q(R) = (\cos\frac{\theta}{2}, \omega \sin\frac{\theta}{2}), \hat{\omega} = \frac{R - R^T}{2\sin\theta}, \theta = \arccos\frac{\mathrm{Tr}R - 1}{2}$$

- Canonical coordinate:

$$\hat{r}(R) = \log R = \hat{\omega}\theta, \hat{\omega} = \frac{R - R^T}{2\sin\theta}, \theta = \arccos\frac{\mathrm{Tr}R - 1}{2}$$

A cubic trajectory on $SO(3)$

Given R_0, R_1 and $\omega_0 = R^T(0)\dot{R}(0)$, $\omega_1 = R^T(1)\dot{R}(1)$, consider a *minimum angular acceleration curve*:

$$R(t) = R_0 e^{\hat{r}(t)}, t \in [0,1]$$

that minimizes $\int_0^1 \dot{\omega}^T \dot{\omega} \mathrm{d}t$.

□ **Exact solution [5]:**

$$\omega^{(3)} + \omega \times \ddot{\omega} = 0 \quad (5.1.8)$$

which is hard to solve.

□ **Approximate Solution [6]:**

$$r(0) = 0, r(1) = \log(R_0^T R_1)^\vee, \omega = A(r)\dot{r},$$

$$A(r) = I + \frac{\cos\|r\| - 1}{\|r\|^2}\hat{r} + \frac{\|r\| - \sin\|r\|}{\|r\|^3}\hat{r}^2 \quad r \neq 0, A(0) = I$$

(Continues next slide)

Example: A cubic traj. on $SO(3)$ (ctned)

◊ Approximation of $\dot{\omega}$:

$$\dot{\omega} \approx \ddot{r}$$

$$(5.1.8) : \omega^{(3)} + \omega \times \ddot{\omega} \approx \omega^{(3)} = r^{(4)} = 0$$

which shows that r is a cubic curve:

$$r(t) = at^3 + bt^2 + ct, t \in [0, 1]$$

◊ Approximate solution:

$$\begin{aligned} \dot{r}(0) &= c = \omega_0 \\ r(1) &= a + b + c = \log(R_0^T R_1)^\vee \\ \dot{r}(1) &= 3a + 2b + c = A^{-1}(r(1))\omega_1 \end{aligned}$$

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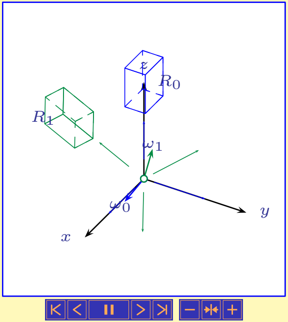
Example: A cubic traj. on $SO(3)$ (ctned)



$$\log(R_0) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \log(R_1) = \begin{bmatrix} 0.6 \\ 0.4 \\ 0.4 \end{bmatrix},$$

$$\omega_0 = c = \begin{bmatrix} 0.5 \\ 0.1 \\ 0.1 \end{bmatrix}, \omega_1 = \begin{bmatrix} 0.2 \\ 0.2 \\ 0.5 \end{bmatrix},$$

$$a + b + c = \log(R_0^T R_1)^\vee = \begin{bmatrix} 0.6 \\ 0.4 \\ 0.4 \end{bmatrix},$$

$$3a + 2b + c = A^{-1}(r(1))\omega_1 = \begin{bmatrix} 0.2688 \\ 0.0920 \\ 0.5048 \end{bmatrix}$$

$$\Rightarrow a = \begin{bmatrix} -0.4312 \\ -0.6080 \\ -0.1952 \end{bmatrix}, b = \begin{bmatrix} 0.5312 \\ 0.9080 \\ 0.4952 \end{bmatrix}$$

Trajectory Generation on $SE(3)$

◇ Candidate approaches:

1. Observe that $SE(3) \cong \mathbb{R}^3 \rtimes SO(3)$, we can interpolate position ($\mathbb{R}^3$) and orientation ($SO(3)$) separately.
2. Canonical coordinate ([5]):

$$\xi \in \mathbb{R}^6 \mapsto e^{\hat{\xi}} \in SE(3)$$

3. Frenet frame following ([4]):

$$g(t) = \begin{bmatrix} R(t) & p(t) \\ 0 & 1 \end{bmatrix}, R(t) = [T \ N \ B], T = \frac{\dot{p}(t)}{\|\dot{p}(t)\|}$$

$$\begin{bmatrix} \dot{T} \\ \dot{N} \\ \dot{B} \end{bmatrix} = \begin{bmatrix} 0 & k & 0 \\ -k & 0 & -\tau \\ 0 & \tau & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix},$$

Manipulator control problems

Recall the manipulator dynamics equation:

$$\underbrace{M(\theta)\ddot{\theta}}_{\text{Inertia force}} + \underbrace{C(\theta,\dot{\theta})\dot{\theta}}_{\text{Coriolis \& Centrifugal force}} + \underbrace{N(\theta)}_{\text{gravity}} = \underbrace{\tau}_{\text{Joint torque}} - \left(\underbrace{A^T(\theta)\cdot F}_{\text{Interaction force}} \right)$$

Problem 1: Position control ($A^T(\theta)\cdot F = 0$)
Given the dynamics equation of a manipulator:

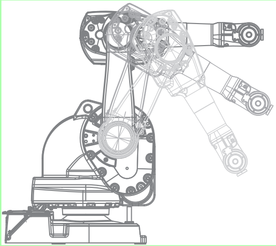
$$M(\theta)\ddot{\theta} + C(\theta,\dot{\theta})\dot{\theta} + N(\theta) = \tau$$

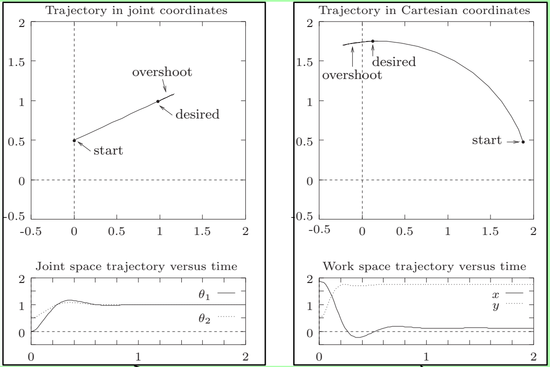
and a fixed position (**regulation**) θ_d/x_d or a generated trajectory (**tracking**) $\theta_d(t)/x_d(t)$ in manipulator joint space Θ (or task space Q), design the joint torque inputs τ such that the manipulator is regulated to the desired position or tracks the desired trajectory:

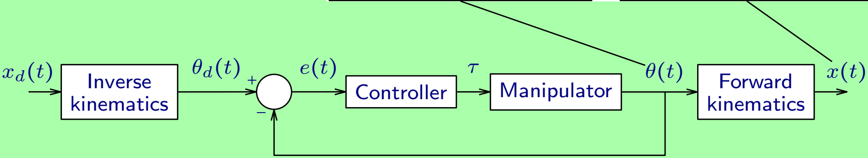
$$e(t) \triangleq \theta_d - \theta(t) \to 0 \text{ or } e_x(t) \triangleq x_d - x(t) \to 0 \text{ asymptotically.}$$

Manipulator control problems

Problem 1.A: Position Control in Joint Space

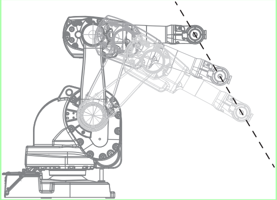


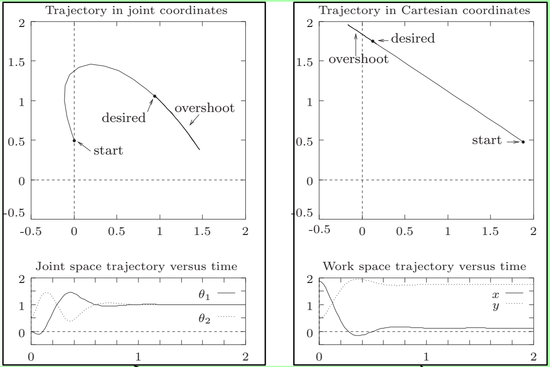


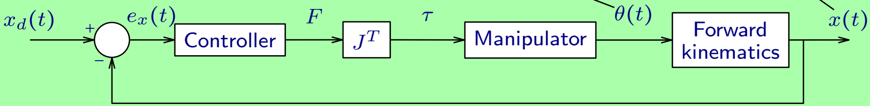


Manipulator control problems

Problem 1.B: Position Control in Task Space







Manipulator control problems

Problem 2: Force Control

Given a desired generalized force $F_d(t)$, find the control law τ so that:

$$e(t) \triangleq F(t) - F_d(t) \to 0$$

asymptotically.

Problem 3: Hybrid Position/Force Control

Given a desired constrained joint motion $\theta_d(t)$ satisfying $A(\theta_d)\dot{\theta}_d = 0$, and a desired constraint force $f_d = A^T F_d$, find the control law τ so that:

$$\theta(t) - \theta_d(t) \to 0$$
$$F(t) - F_d(t) \to 0$$

asymptotically.

Review: closed loop control

Define $e \triangleq \theta_d - \theta \in \mathbb{R}^k$, and consider the following second order linear differential equation:

$$\ddot{e} + K_v \dot{e} + K_p e = 0$$

e converges to 0 asymptotically for an arbitrary choice of positive definite gain matrices $K_v \in \mathbb{R}^{k \times k}$ and $K_p \in \mathbb{R}^{k \times k}$.

Given a second order differential equation:

$$\ddot{e} + f\dot{e} + ge = 0 \qquad (*)$$

define the state space variable $x = (e, \dot{e})$, then $(*)$ is equivalent to:

$$\frac{d}{dt}\begin{bmatrix} e \\ \dot{e} \end{bmatrix} = \begin{bmatrix} 0 & I \\ -g & -f \end{bmatrix}\begin{bmatrix} e \\ \dot{e} \end{bmatrix} \text{ or } \dot{x} = \begin{bmatrix} 0 & I \\ -g & -f \end{bmatrix} x$$

Review: Lyapunov method and Lasalle's Principle

Proposition 1: Lyapunov stability

Consider the following first order nonlinear differential equation:

$$\dot{x} = f(x)$$

If there exists a **Lyapunov function** $V : U \subset \mathbb{R}^n \mapsto \mathbb{R}_+$ which is positive definite:

$$V(x) \geq 0, \forall x \in U, V(x) = 0 \text{ iff } x = 0$$

and $\dot{V} = \frac{\partial f}{\partial x} \cdot f$ is negative definite on U, then any $x(t), x(0) \in U$ converges to 0 asymptotically, or we say 0 is asymptotically stable.

Proposition 2: Lasalle's Principle

Given $\dot{x} = f(x)$. Let $V : \mathbb{R}^n \mapsto \mathbb{R}$ be a locally positive definite function such that on the compact set $\Omega_c \triangleq \{x \in \mathbb{R}^n | V(x) \leq c\}$, we have $\dot{V}(x) \leq 0$. Define

$$S = \{x \in \Omega_c | \dot{V}(x) = 0\}$$

As $t \mapsto \infty$, the trajectory tends to the largest invariant set inside S. In particular, if S contains no invariant sets other than $x = 0$, then 0 is asymptotically stable.

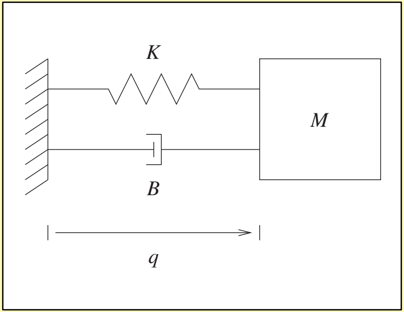
Example: Linear harmonic oscillator

Dynamics equation:

$$M\ddot{q} + B\dot{q} + Kq = 0$$

State space form:

$$\Rightarrow \frac{\mathrm{d}}{\mathrm{d}t}\underbrace{\begin{bmatrix} q \\ \dot{q} \end{bmatrix}}_{x} := \underbrace{\begin{bmatrix} \dot{q} \\ -\frac{K}{M}q - \frac{B}{M}\dot{q} \end{bmatrix}}_{f(x)}$$



Note that the jacobian A

$$A = \begin{bmatrix} 0 & 1 \\ -\frac{K}{M} & -\frac{B}{M} \end{bmatrix}, \mathrm{Re}(\lambda(A)) = \frac{-B \pm \sqrt{B^2 - 4KM}}{2M} < 0$$

$\Rightarrow$ The system always exponentially stable (by Lyapunov indirect method)

(see next page)

Example: Linear harmonic oscillator

Choose *Lyapunov function* to be the system energy:

$$V(x) = \frac{1}{2}M\dot{q}^2 + \frac{1}{2}Kq^2, \dot{V} = M\dot{q}\ddot{q} + Kq\dot{q} = -B\dot{q}^2 \leq 0$$

Apply Lasalle's principle:

$$S = \{x \in \Omega_c | \dot{V}(x) = 0\} \Rightarrow \dot{q} = 0 \Rightarrow \ddot{q} = 0 \Rightarrow q = 0 \Rightarrow$$

$x = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$: only quilibrium point inside S.

Thus $x(t) \to \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ asymptotically.

◊

Example: Nonlinear spring mass system with damper

State space equation:

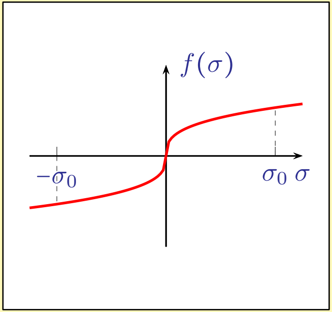
$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -f(x_2) - g(x_1)$$

Passivity of f and g:

$$\sigma f(\sigma) \geq 0, \forall \sigma \in [-\sigma_0, \sigma_0]$$
$$\sigma g(\sigma) \geq 0, \forall \sigma \in [-\sigma_0, \sigma_0]$$

Lyapunov function:

$$V(x) = \frac{x_2^2}{2} + \int_0^{x_1} g(\sigma)\mathrm{d}\sigma, \dot{V}(x) = -x_2 f(x_2)$$



(see next page)

Example: Nonlinear spring mass system with damper

Let

$$c \triangleq \min\left(V(-\sigma_0, 0), V(\sigma_0, 0)\right)$$
$$\dot{V}(x) \leq 0, \forall x \in \Omega_c \triangleq \{x | V(x) \leq c\}$$

$$\dot{V}(x) = 0 \Rightarrow x_2(t) = 0 \Rightarrow x_1(t) = x_{10} \Rightarrow$$
$$\dot{x}_2(t) = 0 = -f(0) - g(x_{10}) \Rightarrow$$
$$g(x_{10}) = 0 \Rightarrow x_{10} = 0$$

$$x = \begin{bmatrix} 0 \\ 0 \end{bmatrix}: \text{ largest invariant set inside } \Omega_c$$

Thus $x(t) \to \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ asymptotically.

♢

Position control (regulation) of robot manipulator

Question: How to design joint torque input τ such that the closed loop system has the equation of motion:

$$\ddot{e} + K_v \dot{e} + K_p e = 0$$

where $e = \theta_d - \theta, \dot{\theta}_d = 0$?

Solution: (computed torque)

$$\begin{aligned} &\ddot{e} + K_v \dot{e} + K_p e = 0 \\ &\Rightarrow M(\ddot{e} + K_v \dot{e} + K_p e) = 0 \\ &\Rightarrow M(\ddot{\theta}_d - \ddot{\theta}) + M(K_v \dot{e} + K_p e) = 0 \\ &\Rightarrow \tau = \underbrace{M(K_v \dot{e} + K_p e)}_{\text{feedback}} + \underbrace{C\dot{\theta} + N}_{\text{feedforward}} = M\ddot{\theta} + C\dot{\theta} + N \end{aligned}$$

Disadvantages:

1. $M(\theta), C(\theta, \dot{\theta}), N(\theta)$ have to be computed in real time.
2. M, C and N are almost impossible to be precisely identified in practice.

PD control in joint space

Question: Is it possible to use the simplified controller $\tau = K_v\dot{e} + K_p e$ so that $\theta(t) \mapsto \theta_d$?The answer is yes.

Proposition 3: PD control in joint space

If $\dot{\theta}_d = 0$ and $K_v, K_p > 0$, then under the control law:

$$\tau = K_v\dot{e} + K_p e$$

$\theta(t) \mapsto \theta_d$ globally (i.e., for all $\theta(0)$).

Proof :

Assume w.l.o.g that $\theta_d = 0$, and the closed loop equation motion is:

$$M(\theta)\ddot{\theta} + C(\theta,\dot{\theta})\dot{\theta} + K_v\dot{\theta} + K_p\theta = 0$$

We can choose the following Lyapunov function

$$V(\theta,\dot{\theta}) = \frac{1}{2}\dot{\theta}^T M(\theta)\dot{\theta} + \frac{1}{2}\theta^T K_p\theta$$

It is positive definite if and only if $K_p > 0$. Moreover,

$$\dot{V}(\theta,\dot{\theta}) = \dot{\theta}^T M\ddot{\theta} + \frac{1}{2}\dot{\theta}^T\dot{M}\dot{\theta} + \dot{\theta}^T K_p\theta = -\dot{\theta}^T K_v\dot{\theta} + \frac{1}{2}\dot{\theta}^T(\dot{M} - 2C)\dot{\theta}$$

$$= -\dot{\theta}^T K_v\dot{\theta} \text{ (since } \dot{M} - 2C \text{ is skew-symmetric)}$$

is negative definite if and only if $K_v > 0$. □

Augmented PD control in joint space

Proposition 4: Augmented PD control in joint space

If $K_v, K_p > 0$, then under the control law:

$$\tau = M(\theta)\ddot{\theta}_d + C(\theta,\dot{\theta})\dot{\theta}_d + N(\theta,\dot{\theta}) + K_v\dot{e} + K_p e$$

$\theta(t) \mapsto \theta_d(t)$ for $\|\theta(t)\| < \varepsilon$.

Proof :

The closed loop equation motion is:

$$M(\theta)\ddot{e} + C(\theta,\dot{\theta})\dot{e} + K_v\dot{\theta} + K_p\theta = 0$$

Define the following Lyapunov function

$$V(e,\dot{e},t) = \frac{1}{2}\dot{e}^T M(\theta)\dot{e} + \frac{1}{2}e^T K_p e + \varepsilon e^T M(\theta)\dot{e}$$

which is positive definite for ε sufficiently small. Then

$$\begin{aligned}\dot{V} &= \dot{e}^T M\ddot{e} + \frac{1}{2}\dot{e}^T\dot{M}\dot{e} + \dot{e}^T K_p e + \varepsilon\dot{e}^T M\dot{e} + \varepsilon e^T(M\ddot{e} + \dot{M}\dot{e}) \\ &= -\dot{e}^T(K_v - \varepsilon M)\dot{e} + \frac{1}{2}\dot{e}^T(\dot{M} - 2C)\dot{e} + \varepsilon e^T(-K_p e - K_v\dot{e} - C\dot{e} + \dot{M}\dot{e}) \\ &= -\dot{e}^T(K_v - \varepsilon M)\dot{e} - \varepsilon e^T K_p e + \varepsilon e^T(-K_v + \frac{1}{2}\dot{M})\dot{e}\end{aligned}$$

is negative definite if ε is sufficiently small. □

Workspace dynamics

Given the generalized coordinates $x \in \mathbb{R}^n$ of the manipulator workspace and the map $f : \theta \mapsto x$,

$$
\begin{aligned}
\dot{x} =& J(\theta)\dot{\theta}, J(\theta) = \frac{\partial f}{\partial \theta} \\
\Rightarrow& \dot{\theta} = J^{-1}\dot{x}, \ddot{\theta} = J^{-1}\ddot{x} + \frac{d}{dt}(J^{-1})\dot{x} \\
\Rightarrow& J^{-T} M J^{-1} \ddot{x} + \left(J^{-T} C J^{-1} + J^{-T} M \frac{d}{dt}(J^{-1}) \right) \dot{x} + J^{-T} N = J^{-T}\tau
\end{aligned}
$$

Denote:

$$
\begin{aligned}
\tilde{M} &= J^{-T} M J^{-1} \\
\tilde{C} &= J^{-T} \left(C J^{-1} + M \frac{d}{dt}(J^{-1}) \right) \\
\tilde{N} &= J^{-T} N \\
F &= J^{-T}\tau
\end{aligned}
$$

Then

$$
\tilde{M}(\theta)\ddot{x} + \tilde{C}(\theta, \dot{\theta})\dot{x} + \tilde{N}(\theta, \dot{\theta}) = F
$$

Structural properties of workspace dynamics

Property 1:

1. $\tilde{M}(\theta)$ is symmetric and positive definite.
2. $\dot{\tilde{M}} - 2\tilde{C}$ is a skew-symmetric matrix.

Proof :

$\tilde{M}$ is symmetric:

$$\tilde{M}^T = (J^{-T} M J^{-1})^T = J^{-T} M J^{-1} = \tilde{M}$$

and positive definite:

$$\dot{x}^T \tilde{M} \dot{x} = \dot{\theta}^T M \dot{\theta} \geq 0, \dot{x}^T \tilde{M} \dot{x} = 0 \Leftrightarrow \dot{\theta} = 0 \Leftrightarrow \dot{x} = 0$$

$\dot{\tilde{M}} - 2\tilde{C}$ is skew symmetric:

$$\dot{\tilde{M}} - 2\tilde{C} = J^{-T} \dot{M} J^{-1} + (\dot{J^{-T}}) M J^{-1} + J^{-T} M (\dot{J^{-1}}) - 2J^{-T} C J^{-1} - 2J^{-T} M (\dot{J^{-1}})$$

$$= J^{-T} \underbrace{(\dot{M} - 2C)}_{\text{skew}} J^{-1} + \underbrace{\left((\dot{J^{-T}}) M J^{-1}\right) - \left((\dot{J^{-T}}) M J^{-1}\right)^T}_{\text{skew}}$$

workspace control

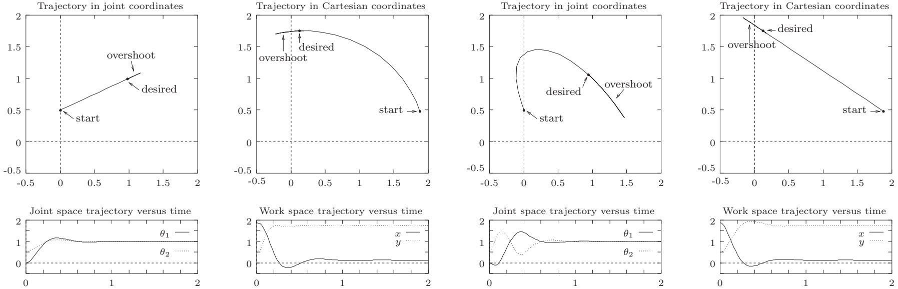
PD control in workspace:

$$\tau = J^T(K_v\dot{e}_x + K_p e_x), e_x \triangleq x_d - x$$

Augmented PD control in workspace:

$$\tau = J^T(\tilde{M}(\theta)\ddot{x}_d + \tilde{C}(\theta,\dot{\theta})\dot{x}_d + \tilde{N}(\theta,\dot{\theta}) + K_v\dot{e}_x + K_p e_x)$$

Comparison of joint space control and workspace control:



Adaptive computed torque control

Property 2: The equation of motion is linear in the inertia parameters:

$$M(\theta)\ddot{\theta} + C(\theta, \dot{\theta})\dot{\theta} + N(\theta) = Y(\theta, \dot{\theta}, \ddot{\theta})\pi$$

where $Y(\theta, \dot{\theta}, \ddot{\theta})$ is called the regressor matrix and π is a constant vector, comprised of link masses, moments of inertia, etc.

Estimated equation of motion:

$$\hat{M}(\theta)\ddot{\theta} + \hat{C}(\theta, \dot{\theta})\dot{\theta} + \hat{N}(\theta) = Y(\theta, \dot{\theta}, \ddot{\theta})\hat{\pi}$$

consider the following control law:

$$\begin{aligned} \tau &= \hat{M}(\theta)(\ddot{\theta}_d + K_v\dot{e} + K_p e) + \hat{C}(\theta, \dot{\theta})\dot{\theta} + \hat{N}(\theta, \dot{\theta}) \\ &= Y(\theta, \dot{\theta}, \ddot{\theta})\hat{\pi} + \hat{M}(\theta)(\ddot{e} + K_v\dot{e} + K_p e) \end{aligned}$$

(see next page)

Adaptive computed torque control

The closed loop system:

$$Y(\theta, \dot{\theta}, \ddot{\theta})(\pi - \hat{\pi}) = \hat{M}(\theta)(\ddot{e} + K_v \dot{e} + K_p e)$$

Define $x^T = (e^T, \dot{e}^T), \tilde{\pi} = \pi - \hat{\pi}$, then we have:

$$\dot{x} = Ax + B\hat{M}^{-1}(\theta)Y(\theta, \dot{\theta}, \ddot{\theta})\tilde{\pi},$$
$$A = \begin{bmatrix} 0 & I \\ -K_p & -K_v \end{bmatrix}, B = \begin{bmatrix} 0 \\ I \end{bmatrix}$$

Choose the following Lyapunov function:

$$V(x, \tilde{\pi}) = \frac{1}{2}x^T P x + \frac{1}{2}\tilde{\pi}^T \Gamma \tilde{\pi} \text{ s.t. } P > 0, \Gamma > 0$$

then:

$$\dot{V} = x^T P \dot{x} + \tilde{\pi}^T \Gamma \dot{\tilde{\pi}}$$

(see next page)

Adaptive computed torque control

$$\begin{aligned}\dot{V} &= x^T P(Ax + B\hat{M}^{-1}(\theta)Y(\theta,\dot{\theta},\ddot{\theta})\tilde{\pi}) + \tilde{\pi}^T\Gamma\dot{\tilde{\pi}} \\ &= -x^T Qx + \tilde{\pi}^T(\Gamma\dot{\tilde{\pi}} + Y^T(\theta,\dot{\theta},\ddot{\theta})\hat{M}^{-1}(\theta)B^T Px)\end{aligned}$$

where $Q = -(PA + A^T P)/2 > 0$. If the following adaptive law:

$$\dot{\tilde{\pi}} = -\dot{\hat{\pi}} = -\Gamma^{-1}Y^T(\theta,\dot{\theta},\ddot{\theta})\hat{M}^{-1}(\theta)B^T Px$$

is adopted,

$$\dot{V} = -x^T Qx \leq 0$$

By Lasalle's principle, 0 is asymptotically stable.

Proposition 5: Adaptive computed torque control

$$\begin{aligned}\tau &= Y(\theta,\dot{\theta},\ddot{\theta})\hat{\pi} + \hat{M}(\theta)(\ddot{e} + K_v\dot{e} + K_p e), K_v > 0, K_p > 0 \\ \dot{\hat{\pi}} &= \Gamma^{-1}Y^T(\theta,\dot{\theta},\ddot{\theta})\hat{M}^{-1}(\theta)B^T P\begin{bmatrix} e \\ \dot{e}\end{bmatrix}, P > 0, \Gamma > 0\end{aligned}$$

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