HW6 [redacted]

1. (30') Consider the LTI system

$$\begin{aligned}\dot{x} &= Ax + Bu\\ y &= Cx\end{aligned} \qquad (1)$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^q$ is the input, and $y \in \mathbb{R}^p$ is the output. Suppose that (A, C) is observable. Show that there exists a suitable matrix $L \in \mathbb{R}^{n\times p}$, such that all eigenvalues of $A - LC$ can be arbitrarily assigned.

解: $C \in \mathbb{R}^{p\times n}$.

若系统能观的，则可以通过可逆变换 $x = p\bar{x}$

$$\begin{cases}\dot{\bar{x}} = p^{-1}Ap\bar{x} + p^{-1}Bu \\ y = Cp\bar{x}\end{cases} \Leftrightarrow \begin{cases}\dot{\bar{x}} = \bar{A}\bar{x} + \bar{B}u \\ y = \bar{C}\bar{x}\end{cases}$$

其中 $\bar{A} = \begin{bmatrix} -\alpha_{n-1} & 1 & & \\ \vdots & & \ddots & \\ -\alpha_1 & & & 1 \\ -\alpha_0 & 0 & \cdots & 0 \end{bmatrix}$, $\bar{C} = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}$

目标 $\det(\bar{A}) = s^n + \bar{\alpha}_{n-1}s^{n-1} + \bar{\alpha}_{n-2}s^{n-2} + \cdots + \bar{\alpha}_0$

令 $\bar{L} = \begin{bmatrix} L_1 \\ L_2 \\ \vdots \\ L_p \end{bmatrix}$ ∴ $\bar{L}\bar{C} = \begin{bmatrix} L_1 & 0 & 0 \\ L_2 & \cdots & 0 \\ \vdots & & \\ L_n & 0 & 0 \end{bmatrix}$

∴ $\bar{A} - \bar{L}\bar{C} = \begin{bmatrix} -\alpha_{n-1} - L_1 & 1 & & \\ -\alpha_{n-2} - L_2 & & \ddots & \\ \vdots & & & 1 \\ -\alpha_0 - L_n & 0 & \cdots & 0 \end{bmatrix}$

∴ $\det(\bar{A} - \bar{L}\bar{C}) = s^n + (\alpha_{n-1} + L_1)s^{n-1} + (\alpha_{n-2} + L_2)s^{n-2} + \cdots (\alpha_0 + L_n)$

$$\Rightarrow \begin{cases} \alpha_{n-1} + L_1 = \bar{\alpha}_{n-1} \\ \alpha_{n-2} + L_2 = \bar{\alpha}_{n-2} \\ \vdots \\ \alpha_0 + L_n = \bar{\alpha}_0 \end{cases} \Rightarrow \begin{cases} L_1 = \bar{\alpha}_{n-1} - \alpha_{n-1} \\ L_2 = \bar{\alpha}_{n-2} - \alpha_{n-2} \\ \vdots \\ L_n = \bar{\alpha}_0 - \alpha_0 \end{cases}$$ ∴ 均有解

∴ 存在 $\bar{L}$

则 ∵ $x = p\bar{x}$

∴ $\dot{\bar{x}} = (\bar{A} - \bar{L}\bar{C})\bar{x} + \bar{B}u \Rightarrow \dot{\bar{x}} = (p^{-1}Ap - \bar{L}\bar{C})\bar{x}$

$p\dot{\bar{x}} = Ap\bar{x} - p\bar{L}Cp\bar{x}$ 即 $\dot{x} = Ax - LCx$

∴ $L = p^{-1}\bar{L}$

∴ 得证

2. (20') Please show that whether the following system can be stabilized by a state feedback control $u = kx$. If it does, find a suitable k.

$$\dot{x} = \begin{bmatrix} 4 & 2 \\ 0 & -2 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u \tag{2}$$

解：①先判断能控性

$P_c = [B \ AB] = \begin{bmatrix} 1 & 4 \\ 0 & 0 \end{bmatrix}$

特征值 $|\lambda I - A| = \begin{vmatrix} \lambda-4 & -2 \\ 0 & \lambda+2 \end{vmatrix} = (\lambda-4)(\lambda+2) \Rightarrow \lambda_1 = 4, \lambda_2 = -2$

进行能控性分解，取变换矩阵 $P = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$，$x = P\bar{x}$

$\dot{\bar{x}} = P^{-1}AP\bar{x} + P^{-1}Bu = \begin{bmatrix} 4 & 2 \\ 0 & -2 \end{bmatrix}\bar{x} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}u$

可知 $\lambda = -2$ 为不能控模态，所以系统是可以稳定的

② 极点配置：

设 $k = [k_1 \ k_2]$，目标极点 $\lambda_1 = -1$，$\lambda_2 = -2$（不变）

$\therefore A + BK = \begin{bmatrix} 4 & 2 \\ 0 & -2 \end{bmatrix} + \begin{bmatrix} k_1 & k_2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} k_1+4 & 2 \\ 0 & k_2-2 \end{bmatrix}$

$|SI - A| = \begin{vmatrix} S-k_1-4 & -2 \\ 0 & S-k_2+2 \end{vmatrix} = (S-k_1-4)(S-k_2+2)$

$\begin{cases} k_1 + 4 = -1 \\ S - k_2 + 2 = S + 2 \end{cases} \Rightarrow \begin{cases} k_1 = -5 \\ k_2 = 0 \end{cases} \therefore K = [-5, 0]$

$\therefore$ 解毕

3. (20') Consider $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$. For any matrix $K \in \mathbb{R}^{m\times n}$, show that $(A - BK, B)$ is controllable if and only if (A, B) is controllable.

解：(1) 证 (A, B) 能控 $\Rightarrow$ $(A-BK, B)$ 能控

已知 (A,B) 能控，则 $[A-\lambda I, B]$ 对系统所有特征值都有行满秩

$P_c = [B \ AB \ A^2B \ \cdots A^{n-1}B]$

$\bar{P}_c = [B \ (A-BK)B \ (A-BK)^2B \ (A-BK)^3B \ \cdots (A-BK)^{n-1}B]$

$$= \underbrace{[B \ AB \ A^2B \cdots A^{n-1}B]}_{P_c} \underbrace{\begin{bmatrix} I & -KB & K(BK-A)B & \cdots & K(BK-A)^{n-2}B \\ 0 & I & -KB & & \vdots \\ 0 & 0 & I & & \vdots \\ \vdots & \vdots & 0 & & \vdots \\ 0 & 0 & \vdots & & I \end{bmatrix}}_{H} \quad —①$$

$\because rank(P_c) = n$, $rank(H) = n$

$\therefore \begin{cases} rank(\bar{P}_c) \geq rank(P_c) + rank(H) - n \\ rank(\bar{P}_c) \leq min\{rank(P_c), rank(H)\} \end{cases} \Rightarrow rank(\bar{P}_c) = n$

$\therefore$ 可得系统 $(A-BK, B)$ 是满秩即可控

(2) 证 $(A-BK, B)$ 能控 $\Rightarrow$ (A,B) 能控

由① 可知 H 可逆

$\therefore P_c = \bar{P}_c \cdot H^{-1}$

由 $\begin{cases} rank(\bar{P}_c) = n \\ rank(H^{-1}) = n \end{cases} \Rightarrow rank(P_c) = n$

$\therefore$ 得证

$\Rightarrow$ 综上得证

4. (30') Consider the LTI system

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned} \tag{3}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^q$ is the input, and $y \in \mathbb{R}^p$ is the output. Suppose (A, B, C) is controllable and observable. Let one of its state observers be in the following form

$$\dot{\hat{x}} = A\hat{x} + Bu + L(y - C\hat{x}) \tag{4}$$

Show that the observer (4) is a special case of the following observer

$$\begin{aligned} \dot{z} &= Fz + Gy + Hu \\ \hat{x} &= T^{-1}z \end{aligned} \tag{5}$$

i.e., $TA - FT = GC$, $H = TB$, and all eigenvalues of F have negative real parts.

解：$\dot{\hat{x}} = T^{-1}\dot{z} = T^{-1}(Fz + Gy + Hu)$

$= T^{-1}(FT\hat{x} + Gy + Hu)$

$= T^{-1}FT\hat{x} + T^{-1}Gy + T^{-1}Hu$

由题易知 $H = TB \Rightarrow T^{-1}H = B$

$TA = FT + GC \Rightarrow A = T^{-1}FT + T^{-1}GC$ —①

$\therefore \dot{\hat{x}} = (A - T^{-1}GC)\hat{x} + T^{-1}Gy + Bu$

$= A\hat{x} - T^{-1}GC\hat{x} + T^{-1}Gy + Bu$

$= A\hat{x} + Bu + T^{-1}G(y - C\hat{x})$

只要令 $T^{-1}G = L$ 即可得 $\dot{\hat{x}} = A\hat{x} + Bu + L(y - C\hat{x})$

并且当 $T^{-1}G = L$ 时，由①：$A = T^{-1}FT + LC$

即 $T^{-1}FT = A - LC$

可知 $A - LC$ 和 F 的特征值一致均为非实部的

故动态误差 $\to 0$，即可以构成状态观测器

$\therefore$ 可知 (4) 是 (5) 的一种形式