Linear System Theory: HW3

（Due: Oct. 17, 2022）

1. (20’) Consider the linear time invariant system (1).

$$\dot{x} = \begin{bmatrix} -1 & 5 \\ 0 & 2 \end{bmatrix} x + \begin{bmatrix} 2 \\ 0 \end{bmatrix} u$$
$$y = \begin{bmatrix} -2 & 4 \end{bmatrix} x - 2u \qquad (1)$$

   (1). Is it BIBO stable?

   (2). Is the state equation marginally stable or asymptotically stable?

2. (10’) For system $\dot{x} = f(x), x \in \mathbb{R}^n$ with $f(0) = 0$. Define stability and instability using $\varepsilon - \delta$ language.

3. (10’) Consider the system $\dot{x} = f(x), \quad x \in \mathbb{R}^n$ with $f(0) = 0$. Show that if the equilibrium point $x^* = 0$ is exponentially stable, then it is asymptotically stable.

4. (20’) Consider the system

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -\frac{x_1}{(1+x_1^2)^2} - \frac{x_2}{(1+x_1^2+x_2^2)^2}$$

   (1). Find its equilibrium point.

   (2). Show that $V(x) = x_2^2 + \dfrac{x_1^2}{1+x_1^2}$ is a Lyapunov function of this system.

5. (20’) For $A_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{bmatrix}$, $A_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix}$.
   Is the system $\dot{x} = A_1 x / \ \dot{x} = A_2 x$ asymptotically stable? What about marginally stable?

6. (20’) Consider the CT-LTI system $\dot{x} = Ax, x \in R^n$. Suppose there exists a positive constant μ and positive definite matrices P, Q for which the following Lyapunov equation

$$A'P + PA + 2\mu P = -Q$$

   hold. Prove that all eigenvalues of A have real parts less than $-\mu$.

   (Hint: prove $A + \mu I$ is a stability matrix).