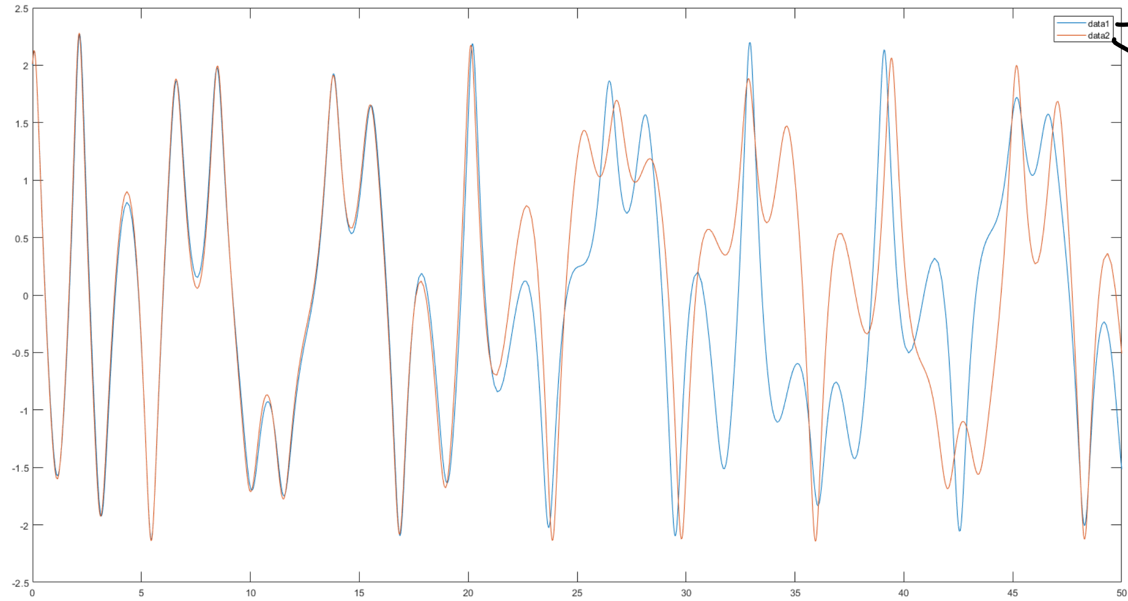
1.

(1) $\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -x_1^5 - 0.1x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 6\sin t \end{bmatrix}$

(2) They are similar when $0 \le t \le 10s$, but quite different when $t \ge 40s$



$X_0 = [2, 3]^T$

$X_0 = [2.01, 3.01]^T$

2.

$$u(t) = C \cdot \dot{V}_c + i_L$$

$$V_c = L \cdot \dot{i}_L + i_L \cdot R$$

if $x = [V_c, i_L]^T$

then we have

$$\begin{cases} \dot{V}_c = -\frac{1}{C} i_L + \frac{1}{C} u(t) \\ \dot{i}_L = -\frac{R}{L} i_L + \frac{1}{L} V_c \end{cases} \Rightarrow \dot{x} = \begin{bmatrix} 0 & -\frac{1}{C} \\ \frac{1}{L} & -\frac{R}{L} \end{bmatrix} x + \begin{bmatrix} \frac{1}{C} \\ 0 \end{bmatrix} u(t)$$

if $x = [V_c, V_o]^T$, since $i_L = \frac{V_o}{R}$, then we have

$$\begin{cases} \dot{V}_c = -\frac{1}{CR} V_o + \frac{1}{C} u(t) \\ \dot{V}_o = -\frac{R}{L} V_o + \frac{R}{L} V_c \end{cases} \Rightarrow \dot{x} = \begin{bmatrix} 0 & -\frac{1}{CR} \\ \frac{R}{L} & -\frac{R}{L} \end{bmatrix} x + \begin{bmatrix} \frac{1}{C} \\ 0 \end{bmatrix} u(t)$$

Let $x_1 = [V_c, i_L]^T$ $x_2 = [V_c, V_o]^T$, since $V_o = i_L R$, then

$$\begin{bmatrix} V_c \\ V_o \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & R \end{bmatrix} \begin{bmatrix} V_c \\ i_L \end{bmatrix} \Rightarrow x_2 = T x_1$$

$$\dot{x}_2 = \begin{bmatrix} 0 & -\frac{1}{CR} \\ \frac{R}{L} & -\frac{R}{L} \end{bmatrix} x_2 + \begin{bmatrix} \frac{1}{C} \\ 0 \end{bmatrix} u(t)$$

$$T\dot{x}_1 = \begin{bmatrix} 0 & -\frac{1}{CR} \\ \frac{R}{L} & -\frac{R}{L} \end{bmatrix} T x_1 + \begin{bmatrix} \frac{1}{C} \\ 0 \end{bmatrix} u(t)$$

$$\dot{x}_1 = T^{-1} \begin{bmatrix} 0 & -\frac{1}{CR} \\ \frac{R}{L} & -\frac{R}{L} \end{bmatrix} T x_1 + T^{-1} \begin{bmatrix} \frac{1}{C} \\ 0 \end{bmatrix} u(t)$$

$$\dot{x}_1 = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{R} \end{bmatrix} \begin{bmatrix} 0 & -\frac{1}{CR} \\ \frac{R}{L} & -\frac{R}{L} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & R \end{bmatrix} x_1 + \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{R} \end{bmatrix} \begin{bmatrix} \frac{1}{C} \\ 0 \end{bmatrix} u(t)$$

$$= \begin{bmatrix} 0 & -\frac{1}{C} \\ \frac{1}{L} & -\frac{R}{L} \end{bmatrix} x + \begin{bmatrix} \frac{1}{C} \\ 0 \end{bmatrix} u(t)$$

So these two models are similar realizations.

3. proof:

$$A q_i = \lambda_i q_i$$

$$\therefore \quad AQ = Q \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$$

$$\therefore \quad AQQ^{-1} = Q \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} Q^{-1} = A$$

$$Q^{-1} A = Q^{-1} Q \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} Q^{-1}$$

$$\because P = Q^{-1}$$

$$\therefore \quad PA = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} P \Rightarrow \begin{bmatrix} p_1^T \\ \vdots \\ p_n^T \end{bmatrix} A = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} \begin{bmatrix} p_1^T \\ \vdots \\ p_n^T \end{bmatrix}$$

$$\therefore \quad p_i^T A = \lambda_i p_i^T$$

So p_i is a left eigenvector of A associated with λ_i.

4. (1) let $A=[\alpha_1\ \alpha_2\ \cdots\ \alpha_n]$ α_i is the column vector of A

a solution x exists in $Ax=y$ means :

$$\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n = y$$

which means $y \in \text{span}\{\alpha_i\}$

That is to say: y lies in the range space of A

So, the proposition is proved.

(2) If $\text{rank} A = m$, then $\{\alpha_i\}$ have m independent vectors.

these m vectors form a set of bases of R^m

$$\mathbb{R}^m = \text{Span}\{\alpha_i\}_{i=1,\cdots,m}$$

Since $y \in \mathbb{R}^m$, then $y = x_1\alpha_1 + x_2\alpha_2 + \cdots + x_m\alpha_m$

So x exists in $Ax=y$ for every y

Similarly, if x exists in $Ax=y$ for every y, then all column vectors of A $\{\alpha_i\}_{i=1,\cdots,n}$ can span $\mathbb{R}^m$:

$$\mathbb{R}^m = \text{span}\{\alpha_i\}_{i=1,\cdots,n}$$

Then $\text{rank} A = \text{rank}\{\alpha_i\}_{i=1,\cdots,n} = m$

The proposition is proved.

5. $f_A(\lambda) = |\lambda I - A|$ B is a similar matric with A

let $A = P^{-1}BP$

$$f_A(\lambda) = |\lambda I - P^{-1}BP| = |P^{-1}(\lambda I - B)P| = |P^{-1}||\lambda I - B||P|$$

$$= |\lambda I - B| = f_B(\lambda)$$

So similar matrics have the same characteristic polynomials.

1.

$$A = \begin{bmatrix} 2 & -1 \\ -3 & 3 \\ -1 & 2 \end{bmatrix} \xrightarrow{\text{elementary transform}} \begin{bmatrix} 2 & -1 \\ 0 & \frac{3}{2} \\ 0 & 0 \end{bmatrix}$$

$\therefore \text{Rank}A = 2$

when $y = [-1, 0, -1]^T$,

$$[A \ y] = \begin{bmatrix} 2 & -1 & -1 \\ -3 & 3 & 0 \\ -1 & 2 & -1 \end{bmatrix} \xrightarrow[\text{transform}]{\text{elementary}} \begin{bmatrix} 2 & -1 & -1 \\ 0 & \frac{3}{2} & -\frac{3}{2} \\ 0 & 0 & 0 \end{bmatrix}$$

$\text{Rank}[A \ y] = \text{Rank}A = 2$

And the columns of A is full ranks.

So there exists an unique solution x s.t. $Ax = y$

when $y = [1, 1, 1]^T$

$$[A \ y] = \begin{bmatrix} 2 & -1 & 1 \\ -3 & 3 & 1 \\ -1 & 2 & 1 \end{bmatrix} \xrightarrow[\text{transform}]{\text{elementary}} \begin{bmatrix} 2 & -1 & 1 \\ 0 & \frac{3}{2} & \frac{5}{2} \\ 0 & 0 & \frac{1}{2} \end{bmatrix} = 3$$

$\text{Rank}[A \ y] \neq \text{Rank}A$

So there doesn't exist a solution x s.t. $Ax = y$

2. (1)

$$|\lambda I - A| = \begin{vmatrix} \lambda-1 & -4 & -10 \\ 0 & \lambda-2 & 0 \\ 0 & 0 & \lambda-3 \end{vmatrix} = (\lambda-1)(\lambda-2)(\lambda-3) = 0$$

$\therefore \lambda_1 = 1 \quad \lambda_2 = 2 \quad \lambda_3 = 3$

$$\text{Jordan}(A) = \begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{bmatrix} = \begin{bmatrix} 1 & & \\ & 2 & \\ & & 3 \end{bmatrix}$$

(2) $|\lambda I - A| = \begin{vmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ 2 & 4 & \lambda+3 \end{vmatrix} = \lambda[\lambda(\lambda+3)+4] - (-1)\cdot 2$

$= \lambda^3 + 3\lambda^2 + 4\lambda + 2$

$= (\lambda+1)(\lambda^2+2\lambda+2) = 0$

$\lambda_1 = -1 \quad \lambda_2 = -1+i \quad \lambda_3 = -1-i.$

$\text{Jordan}(A) = \begin{bmatrix} -1 & & \\ & -1+i & \\ & & -1-i \end{bmatrix}$

3.

$Ax = \lambda x$

Taylor expansion for $f(x)$, we have:

$f(x) = \sum_{i=0} a_i x^i \qquad a_i = f^{(i)}(0) \cdot \frac{1}{i!}$

So $f(\lambda) = \sum_{i=0} a_i \lambda^i \quad f(A) = \sum_{i=0} a_i A^i$

$f(A)\cdot x = (\sum_{i=0} a_i A^i)\, x = \sum_{i=0} a_i A^i x$

$\because Ax = \lambda x \quad \therefore A^i x = A^{i-1}(\lambda x) = \lambda(A^{i-1}x) = \lambda^i x$

$\therefore f(A)\cdot x = \sum_{i=0} a_i \lambda^i x = (\sum_{i=0} a_i \lambda^i)\, x = f(\lambda)\cdot x$

So $f(\lambda)$ is an eigenvalue of $f(A)$ with same eigenvector x.

4.

$$|\lambda I - A| = \begin{vmatrix} \lambda-1 & -4 & -10 \\ & \lambda-2 & \\ & & \lambda-3 \end{vmatrix} = (\lambda-1)(\lambda-2)(\lambda-3)=0$$

$\lambda_1 = 1 \quad \lambda_2 = 2 \quad \lambda_3 = 3$

eigen vector: $x_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad x_2 = \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix} \quad x_3 = \begin{bmatrix} 5 \\ 0 \\ 1 \end{bmatrix}$

$$T = [x_1 \ x_2 \ x_3] = \begin{bmatrix} 1 & 4 & 5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \Lambda = \begin{bmatrix} 1 & & \\ & 2 & \\ & & 3 \end{bmatrix} = T^{-1}AT$$

$$\therefore e^{At} = T e^{\Lambda t} T^{-1} = T \begin{bmatrix} e^t & & \\ & e^{2t} & \\ & & e^{3t} \end{bmatrix} T^{-1}$$

$$= \begin{bmatrix} 1 & 4 & 5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} e^t & & \\ & e^{2t} & \\ & & e^{3t} \end{bmatrix} \begin{bmatrix} 1 & -4 & -5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} e^t & 4e^{2t} & 5e^{3t} \\ 0 & e^{2t} & 0 \\ 0 & 0 & e^{3t} \end{bmatrix} \begin{bmatrix} 1 & -4 & -5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} e^t & 4(e^{2t}-e^t) & 5(e^{3t}-e^t) \\ 0 & e^{2t} & 0 \\ 0 & 0 & e^{3t} \end{bmatrix}$$

$$A^k = T\Lambda^k T^{-1} = \begin{bmatrix} 1 & 4 & 5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 2^k & \\ & & 3^k \end{bmatrix} \begin{bmatrix} 1 & -4 & -5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 4\cdot 2^k & 5\cdot 3^k \\ 0 & 2^k & 0 \\ 0 & 0 & 3^k \end{bmatrix} \begin{bmatrix} 1 & -4 & -5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 4\cdot(2^k-1) & 5(3^k-1) \\ & 2^k & \\ & & 3^k \end{bmatrix}$$

$$\therefore A^{103} = \begin{bmatrix} 1 & 4\times(2^{103}-1) & 5\times(3^{103}-1) \\ & 2^{103} & \\ & & 3^{103} \end{bmatrix}$$

5. $A q_i = \lambda_i q_i \qquad p_i A = \lambda_i p_i$

since all eigenvalues of A are distinct, then

$Q = [q_1 \; q_2 \cdots q_n] \qquad P = [p_1^T \; p_2^T \; \cdots \; p_n^T]^T$

$Q^{-1} = P \qquad A = Q \begin{bmatrix} \lambda_1 & & & & \\ & \ddots & & & \\ & & \lambda_i & & \\ & & & \ddots & \\ & & & & \lambda_n \end{bmatrix} Q^{-1} = Q \Lambda P$

$$(sI - A)^{-1} = (Q(sI - \Lambda)Q^{-1})^{-1} = Q(sI-\Lambda)^{-1} Q^{-1}$$

$$= Q \begin{bmatrix} s-\lambda_1 & & & & \\ & \ddots & & & \\ & & s-\lambda_i & & \\ & & & \ddots & \\ & & & & s-\lambda_n \end{bmatrix}^{-1} Q^{-1}$$

$$= Q \begin{bmatrix} \frac{1}{s-\lambda_1} & & & & \\ & \ddots & & & \\ & & \frac{1}{s-\lambda_i} & & \\ & & & \ddots & \\ & & & & \frac{1}{s-\lambda_n} \end{bmatrix} P$$

$$= \sum_{i=1}^{n} \frac{1}{s-\lambda_i} q_i p_i$$

6.

$$G(s) = C(sI-A)^{-1}B = [2 \;\; 3] \begin{bmatrix} s & -1 \\ 2 & s+2 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= \frac{1}{s(s+2)+2} [2 \;\; 3] \begin{bmatrix} s+2 & 1 \\ -2 & s \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= \frac{5s}{s^2+2s+2}$$

$Y(s) = G(s) \cdot U(s) = \dfrac{5}{s^2+2s+2} \qquad y(t) = \mathcal{L}^{-1}\left(\dfrac{5}{s^2+2s+2}\right) = 5e^{-t} \sin t$

HW3

1. (1) The transfer function :

$$G(s) = C(sI-A)^{-1}B + D = \begin{bmatrix} -2 & 4 \end{bmatrix} \begin{bmatrix} s+1 & -5 \\ 0 & s-2 \end{bmatrix}^{-1} \begin{bmatrix} 2 \\ 0 \end{bmatrix} - 2$$

$$= \frac{-2s-6}{s+1}$$

the pole is $s=-1$, has negative real part.

So the system is BIBO stable.

(2) $A = \begin{bmatrix} -1 & 5 \\ 0 & 2 \end{bmatrix}$ $\quad \|\lambda I - A\| = 0 \Rightarrow (\lambda+1)(\lambda-2) = 0$

$\lambda_1 = -1 \quad \lambda_2 = 2$

Since $\lambda_2 > 0$ the system is unstable.

2. Definition: $x=0$ is the equilibrium state of $\dot{x} = f(x)$, if for all $\varepsilon > 0$ and t_0, there exists $\delta(\varepsilon, t_0) > 0$ such that $\|x(t_0)\|_2 < \delta$ gives $\|x(t)\|_2 < \varepsilon$ for all $t \geq t_0$, then the equilibrium state is stable.

On the contrary, if there exists a $\varepsilon > 0$, for any $\delta > 0$, s.t. $\|x(t_0)\|_2 < \delta$ always exists $\bar{t}$ s.t. $\|x(\bar{t})\|_2 > \varepsilon$, the equilibrium state is unstable.

3. $x^* = 0$ is exponential stability. then

for all $\varepsilon > 0$ there exists $\delta(\varepsilon, t_0)$, $c, \lambda > 0$ such that

$\|x(t_0)\|_2 < \delta$, $\|x(t)\| \leq c e^{-\lambda(t-t_0)}$

$t \to \infty$, $ce^{-\lambda(t-t_0)} \to 0$ so $\|x(t)\| \to 0$

So $x^* = 0$ is asymptotically stable.

4.

(1) $\begin{cases} \dot{x}_1 = 0 \\ \dot{x}_2 = 0 \end{cases} \Rightarrow \begin{cases} x_2 = 0 \\ -\dfrac{x_1}{(1+x_1^2)^2} - \dfrac{x_2}{(1+x_1^2+x_2^2)^2} = 0 \end{cases}$

$\therefore$ $x_1^* = 0$ $x_2^* = 0$

So the equilibrium point is $X^* = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

(2) Proof:

$$V(x) = x_2^2 + \frac{x_1^2}{1+x_1^2} > 0 \text{ for } \forall x \neq 0$$

$$\dot{V}(x) = 2x_2\dot{x}_2 + \frac{2x_1(1+x_1^2)\dot{x}_1 - x_1^2 2x_1\dot{x}_1}{(1+x_1)^2}$$

$$= 2x_2\dot{x}_2 + \frac{2x_1\dot{x}_1}{(1+x_1^2)^2}$$

$\because$ $\dot{x}_1 = x_2$

$$\dot{x}_2 = -\frac{x_1}{(1+x_1^2)^2} - \frac{x_2}{(1+x_1^2+x_2^2)^2}$$

$\therefore$ $$\dot{V}(x) = \frac{2x_1x_2}{(1+x_1^2)^2} - 2x_2\left(\frac{x_1}{(1+x_1^2)^2} + \frac{x_2}{(1+x_1^2+x_2^2)^2}\right)$$

$$= \frac{-2x_2^2}{(1+x_1^2+x_2^2)^2} \leq 0$$

So $V(x)$ is positive definite and $\dot{V}(x)$ is semi-negative definite, then $V(x)$ is a lyapvnov function of this system

5.

$$|\lambda I - A_1| = \begin{vmatrix} \lambda & -1 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda+2 \end{vmatrix} = (\lambda+2)\lambda^2 = 0$$

$\lambda_1 = -2 \quad \lambda_2 = \lambda_3 = 0$

$$\text{rank}(A_1 - \lambda_2 I) = \text{rank}\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{bmatrix} = 2$$

$$\text{nullity}(A - \lambda_2 I) = 3 - 2 = 1$$

So $\dot{x} = A_1 x$ is neither asymptotically stable nor marginaly stable.

$$|\lambda I - A_2| = \begin{vmatrix} \lambda & 0 & 0 \\ 0 & \lambda+1 & -1 \\ 0 & 0 & \lambda+1 \end{vmatrix} = \lambda(\lambda+1)^2 = 0$$

$\lambda_1 = 0 \quad \lambda_2 = \lambda_3 = -1$

$$\text{rank}(A_2 - \lambda_1 I) = \text{rank}\begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix} = 2$$

$$\text{nullity}(A_2 - \lambda_1 I) = 3 - 2 = 1$$

So $\dot{x} = A_2 x$ is marginally stable, but is not asymptotically stable.

6. If we want to proove all the eigenvalue of A has real part less than μ, then we can proove that $A + \mu I$ is stability matrix.

$$(A+\mu I)^T P + P(A+\mu I) = A^T P + (\mu I)^T P + PA + P \cdot \mu I$$
$$= A^T P + PA + 2\mu P = -Q$$

P and Q are P.D.

According to Lyapunov equation theorem, $A + \mu I$ is stability matrix.
Proposition is proved.

HW4

1. $\dot{x} = Ax + Bu$ $\qquad y(t) = Ce^{At}x(0) + C\int_0^t e^{A(t-\tau)} B u(t)\, d\tau$

$y = Cx$

let input $u \equiv 0$, then $y(t) = Ce^{At}x(0)$

Prooving this system is observable is equal to prooving the solution of $y(t) = Ce^{At}x(0)$ is unique.

1) Proof:

If $W_o(t)$ is nonsingular,

$$W_o(t)\, x(0) = \int_0^t e^{A^T\tau} C^T C e^{A\tau} x(0)\, d\tau$$

$$= \int_0^t e^{A^T\tau} C^T \bar{y}(\tau)\, d\tau$$

$\bar{y}(t)$ is zero-input response.

$$\therefore x(0) = W_o^{-1}(t) \int_0^t A^{T\tau} C^T \bar{y}(\tau) d\tau$$

$x(0)$ is unique, so the system is observable.

If $W_o(t)$ is singular,

$$W_o(t) = \int_0^t (Ce^{A\tau})^T Ce^{A\tau}\, d\tau$$

then $W_o(t)$ is positive semi definite.

Then there exists a non-zero vector v such that

$$v^T W_o(t) v = 0 \Rightarrow \int_0^t \| Ce^{A\tau} v \|^2 d\tau = 0$$

$$\Rightarrow Ce^{At} v \equiv 0$$

Then for $y(t) = 0$, $x_1(0) = v$ and $x_2(0) = 0$ are both solutions.

So this system is unobservable.

Proof complete.

2) Proof:

$$e^{At} = I + At + \frac{1}{2}(At)^2 + \cdots$$

Cayley-Hamilton theorem tell us $A^n, A^{n+1} \cdots$ are linear combination of $I, A, A^2, \cdots, A^{n-1}$.

So, e^{At} is linear combination of $I, A, \cdots, A^{n-1}$

Ce^{At} is linear combination of $C, CA, \cdots, CA^{n-1}$

Suppose observability matrix O doesn't have full column rank:

then there exists an nonzero vector v such that

$$Ov = 0 \quad \text{or}$$

$$CA^k v = 0 \qquad k = 0, 1, \cdots, n-1$$

then $Ce^{At} v = 0$ This is contradicts the nonsingularity assumption of $W_o(t)$. Thus statements 1) $\Rightarrow$ 2)

Now we show 2) $\Rightarrow$ 1)

Suppose O has full column rank but $W_o(t)$ is singular, then there exists a nonzero vector v such that

$$Ce^{At} v = 0$$

let $t=0$, then $Cv = 0$. Differentiating $Ce^{At}v = 0$ and then let $t = 0$, we have $CAv = 0$. Continue this progress, we have

$$[C \ CA \cdots CA^{n-1}]^T v = 0$$

This contradicts the hypothesis that O has full column rank.

So, 2) and 1) are equivalence.

3) Proof:

2) $\rightarrow$ 3) suppose O has full column rank but there exists an eigenvalue λ_1 and nonzero vector q such that

$$\begin{bmatrix} A - \lambda_1 I \\ C \end{bmatrix} q = 0$$

Then $\quad Aq = \lambda_1 q \quad Cq = 0$

$$A^2 q = (Aq) q = \lambda_1 q^2$$

$$\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} q = \begin{bmatrix} Cq \\ \lambda_1 Cq \\ \vdots \\ \lambda_1^{n-1} Cq \end{bmatrix} = 0$$

This contradicts the hypothesis that O has full column rank.

3) $\longrightarrow$ 2)

If $\operatorname{rank} O < n$, suppose $\operatorname{rank} O = n-m$, then we can transform A to $\bar{A}$ such that

$$\bar{A} = P^{-1}AP = \begin{bmatrix} A_1 & A_2 \\ 0 & A_3 \end{bmatrix} \quad A_1 \text{ is } m\times m \quad \bar{C} = [0, \bar{C_0}]$$

λ_1 and q_1 is eigenvalue and vector of A_1

$$A_1 q_1 = \lambda_1 q_1$$

$$\begin{bmatrix} \bar{A} - \lambda_1 I \\ C \end{bmatrix} q = \begin{bmatrix} A_1 - \lambda_1 I & A_2 \\ 0 & A_3 - \lambda_1 I \\ 0 & \bar{C_0} \end{bmatrix} \begin{bmatrix} q_1 \\ 0 \end{bmatrix} = 0$$

So $\operatorname{rank} \begin{bmatrix} \bar{A} - \lambda I \\ \bar{C} \end{bmatrix} < n$, $\operatorname{rank} \begin{bmatrix} A - \lambda I \\ C \end{bmatrix} < n$

4) If all eigenvalue of A have negative real parts, then A is stable. Then the unique solution is

$$W_o = \int_0^{\infty} e^{A^T t} C^T C e^{At} d\tau$$

W_o is positive definite if and only if W_o is nonsigular.

So 1) $\Longleftrightarrow$ 4)

5) All columns of Ce^{At} are linearly independent means: there doesn't exist nonzero vector v such that

$$Ce^{At} v = 0$$

So 1) <=> 5)

2. Let $X = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$ $\begin{bmatrix} \dot{X}_1 \\ \dot{X}_2 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ 0 \end{bmatrix} u$

$$\dot{X}_1 = A_{11} X_1 + A_{12} X_2 + B_1 u$$

$$\dot{X}_2 = A_{21} X_1 + A_{22} X_2$$

If the system is controllable, then X_1, X_2 are Controllable.

Obviously, X_1 is controlable.

For X_2, we can view X_1 as input, then

$$\dot{X}_2 = A_{22} X_2 + A_{21} X_1$$

(A_{22}, A_{21}) is controllable $\longleftrightarrow$ X_2 is controllable

So, the property is provved.

3. $[B \ AB \ A^2B] = \begin{bmatrix} 0 & 2 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 2 & 0 & -2 & 0 \end{bmatrix} = \mathcal{C}$

$\text{rank}\, \mathcal{C} = 3$ The equation is controllable.

$$\begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -2 \\ 0 & -3 & 2 \\ 0 & 4 & -5 \end{bmatrix} = \mathcal{O}$$

$\text{rank}\, \mathcal{O} = 3$ The equation is observable.

4.

$W_c(t) = \int_0^t (e^{At}B)(e^{At}B)^T dt$ is positive semi definite.

If $W_c(t)$ is not positive definite. then there exists a nonzero vector v

$$v^T W_c(t) v = 0 \Rightarrow B^T e^{A^T t} v = 0 \quad \text{or} \quad v' e^{At} B = 0$$

If the system is controllable, the for $x(0) = e^{-At} v$ Let $x(t_1) = 0$

$$0 = v + \int_0^{t_1} e^{A(t_1 - \tau)} B u(\tau) dt$$

$$\therefore \quad 0 = v'v + \int_0^{t_1} v' e^{A(t_1 - \tau)} B u(\tau) d\tau$$

$\|v\|_2 = 0 \Rightarrow v = 0$ contradict $v \neq 0$. So If system is controllable, $W_c(t)$ is postive definite.

For positive defmtie $W_c(t)$, if $W_c(t)$ is singular, then $\exists$ nonzero V such that $W_c(t) V = 0$ the $v' W_c(t) V = 0$ contradict $v' W_c(t) v > 0$

So P.D $W_c(t)$ must be nonsingular. Then the system is controllable.

5\. First, we prove P is the solution

Substitute $P=\int_0^\infty e^{A^Tt}Qe^{At}dt$ to the equation:

$$A^TP+PA=\int_0^\infty A^Te^{A^Tt}Qe^{At}dt+\int_0^\infty e^{A^Tt}Qe^{At}A\,dt$$

$$=\int_0^\infty \frac{d}{dt}(e^{A^Tt}Qe^{At})dt$$

Since A is Hurwitz, so A is asymptotically stable. $e^{At}\big|_{t\to\infty}=0$

So $A^TP+PA=-Q$

Next we prove the solution is unique.

suppose there are two solutions

$$A^TP_1+P_1A=-Q \qquad A^TP_2+P_2A=-Q$$

$$A^T(P_1-P_2)+(P_1-P_2)A=0$$

$$e^{A^Tt}[A^T(P_1-P_2)+(P_1-P_2)A]e^{At}=\frac{d}{dt}[e^{A^Tt}(P_1-P_2)e^{At}]=0$$

$$e^{A^Tt}(P_1-P_2)e^{At}\Big|_0^\infty=0$$

$$\therefore\quad 0-(P_1-P_2)=0 \quad\Rightarrow P_1=P_2$$