Linear Systems Theory: HW4

（Due: Oct. 21, 2023）

1. (50’) Consider the following LTI system

$$\dot{x} = Ax + Bu$$
$$y = Cx$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^p$ is the input, $y \in \mathbb{R}^q$ is the output. Please show that this system is observable if and only if any of the following equivalent statements holds.

1) The $n \times n$ matrix $W_O(t) = \int_0^t e^{A^T\tau} C^T C e^{A\tau} d\tau$ is nonsingular for any $t > 0$.

2) The $nq \times n$ observability matrix

$$0 = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

has full column rank.

3) The $(n+q) \times n$ matrix $\begin{bmatrix} A - \lambda I \\ C \end{bmatrix}$ has full column rank at every eigenvalue λ of A.

4) If, in addition, all eigenvalues of A have negative real parts, then the unique solution of

$$A^T W_O + W_O A = -C^T C$$

is positive definite. The solution is called the observability Gramian and can be expressed as

$$W_O = \int_0^\infty e^{A^T\tau} C^T C e^{A\tau} d\tau$$

5) All columns of Ce^{At} are linearly independent on $[0, \infty)$.

2. (10’) Show that if the state equation

$$\dot{x} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} x + \begin{bmatrix} B_1 \\ 0 \end{bmatrix} u$$

is controllable, then the pair (A_{22}, A_{21}) is controllable.

3. (10’) Is the state equation

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix} x + \begin{bmatrix} 0 & 2 \\ 1 & 0 \\ 0 & 0 \end{bmatrix} u$$
$$y = \begin{bmatrix} 1 & 0 & -2 \end{bmatrix} x$$

controllable? Observable?

4. (15’) Consider the linear system

$$\dot{x} = Ax + Bu, \tag{4}$$

where $x \in \mathbb{R}^n$ is the state and $u \in \mathbb{R}^m$ is the input. Show that this system is controllable if and only if the matrix

$$W_c(t) = \int_0^t e^{A\tau} B B^T e^{A^T\tau} d\tau = \int_0^t e^{A(t-\tau)} B B^T e^{A^T(t-\tau)} d\tau \tag{5}$$

is positive definite for any $t > 0$.

5. (15') If the matrix $A \in \mathbb{R}^{n \times n}$ is Hurwitz, then for any matrix $Q \in \mathbb{R}^{n \times n}$, there exists a unique solution of , and the solution is $P = \int_0^\infty e^{A^T t} Q e^{At} \, dt$.