一、【每小题 2 分，其中 10 小题，共计 20 分】

假设 a，b，c，d ，θ_i 是未知参数，υ 是噪声，写出下列系统的辨识模型

(1) $y(t)=\theta_1+\theta_2 t+e^t$

解答：

$$\begin{cases} -e^t+y(t)=\varphi^T(t)\theta \\ \varphi^T(t)=[1,t] \\ \theta=[\theta_1,\theta_2]^T \end{cases}$$

(2) $y(t)=\theta_1+\theta_2 t+e^t+2\cos(t)$

解答：

$$\begin{cases} -e^t-2\cos(t)+y(t)=\varphi^T(t)\theta \\ \varphi^T(t)=[1,t] \\ \theta=[\theta_1,\theta_2]^T \end{cases}$$

(3) $y(t)=\theta_1+\theta_2 t+\dfrac{1}{\theta_3}t^2+\upsilon(t)$

解答：

$$\begin{cases} y(t)=\varphi^T(t)\theta+\upsilon(t) \\ \varphi^T(t)=[1,t,t^2] \\ \theta=[\theta_1,\theta_2,\dfrac{1}{\theta_3}]^T \end{cases}$$

(4) $y(t)=\theta_1+\theta_2 t+\theta_3+e^t+\upsilon(t)$

解答：

$$\begin{cases} y(t)-e^t=\varphi^T(t)\theta+\upsilon(t) \\ \varphi^T(t)=[1,t] \\ \theta=[\theta_1+\theta_3,\theta_2]^T \end{cases}$$

(5) $y=ax^2+bx+c+d\ln|x|+d$

解答：

$$\begin{cases} y=\varphi^T(t)\theta \\ \varphi^T(t)=[x^2,x,1,\ln|x|+1] \\ \theta=[a,b,c,d]^T \end{cases}$$

(6) $y=ax^2+\dfrac{x}{b}+c+d\ln|x|+d$

解答：

$$\begin{cases} y = \varphi^T(t)\theta \\ \varphi^T(t) = [x^2, x, 1, \ln(|x|+1)] \\ \theta = [a, \frac{1}{b}, c, d]^T \end{cases}$$

(7) $y = ax^2 + \frac{x+1}{b} + e^c \cos(x/\pi)$

解答：

$$\begin{cases} y = \varphi^T(t)\theta \\ \varphi^T(t) = [x^2, x+1, \cos(x/\pi)] \\ \theta = [a, \frac{1}{b}, e^c]^T \end{cases}$$

(8) $y = ax_1 + bx_2 + ... + cx_n + \upsilon$

解答：

$$\begin{cases} y = \varphi^T(t)\theta + v(t) \\ \varphi^T(t) = [x_1, x_2, ..., x_n] \\ \theta = [a, b, ..., c]^T \end{cases}$$

(9) $y = ax_1 + bx_2 + ... + cx_n + d + \upsilon$

解答：

$$\begin{cases} y = \varphi^T(t)\theta + v(t) \\ \varphi^T(t) = [x_1, x_2, ..., x_n, 1] \\ \theta = [a, b, ..., c, d]^T \end{cases}$$

(10) $y = ax_1 + bx_2 + ... + cx_n + dx_1x_2...x_n + \upsilon$

解答：

$$\begin{cases} y = \varphi^T(t)\theta + v(t) \\ \varphi^T(t) = [x_1, x_2, ..., x_n, x_1x_2...x_n] \\ \theta = [a, b, ..., c, d]^T \end{cases}$$

(11) $y = ax_1 + be^{x_2} + ... + \pi c \sin(x_n) + \upsilon$

解答：

$$\begin{cases} y = \varphi^T(t)\theta + v(t) \\ \varphi^T(t) = [x_1, e^{x_2}, ..., \pi \sin(x_n)] \\ \theta = [a, b, ..., c]^T \end{cases}$$

(12) $y(t)=ax_1(t)+bx_2(t)+...+cx_n(t)+dx_1(t)x_2(t)...x_n(t)+\upsilon(t)$

解答：

$$\begin{cases} y(t)=\varphi^T(t)\theta+v(t) \\ \varphi^T(t)=[x_1(t),x_2(t),...,x_n(t),x_1(t)x_2(t)...x_n(t)] \\ \theta=[a,b,...,c,d]^T \end{cases}$$

二、【每个 2 分，共计 20 分】

假设 θ_i 是未知参数，υ 是噪声，写出下列系统辨识模型

(1) $y(t)=\theta_1+\theta_2 t+\theta_3 e^t+1$

解答：

$$\begin{cases} -1+y(t)=\varphi^T(t)\theta \\ \varphi^T(t)=[1,t,e^t] \\ \theta=[\theta_1,\theta_2,\theta_3]^T \end{cases}$$

(2) $y(t)=\theta_1 u(t)+\theta_2 u^2(t)+...+\theta_m u^m(t)+\upsilon(t)$

解答：

$$\begin{cases} y(t)=\varphi^T(t)\theta+\upsilon(t) \\ \varphi^T(t)=[u(t),u^2(t),...,u^m(t)] \\ \theta=[\theta_1,\theta_2,...,\theta_m]^T \end{cases}$$

(3) $y(t)=\theta_1 u(t-1)+\theta_2 u^2(t-2)+...+\theta_n u^n(t-n)+\upsilon(t)$

解答：

$$\begin{cases} y(t)=\varphi^T(t)\theta+\upsilon(t) \\ \varphi^T(t)=[u(t-1),u^2(t-2),...,u^n(t-n)] \\ \theta=[\theta_1,\theta_2,...,\theta_n]^T \end{cases}$$

(4) $y(t)=\theta_1 y(t-1)+\theta_2 y(t-2)y(t-3)+\theta_3 u(t)+\theta_4 u(t-1)+\upsilon(t)$

解答：

$$\begin{cases} y(t)=\varphi^T(t)\theta+\upsilon(t) \\ \varphi^T(t)=[y(t-1),y(t-2)y(t-3),u(t),u(t-1)] \\ \theta=[\theta_1,\theta_2,\theta_3,\theta_4]^T \end{cases}$$

(5) $y(t)+\theta_1(t)y(t-1)+\theta_2 y(t-2)=\theta_3 u(t-1)+\theta_4 u(t-2)+\upsilon(t)$

解答：

$$\begin{cases} y(t) = -\theta_1(t)y(t-1) - \theta_2 y(t-2) + \theta_3 u(t-1) + \theta_4 u(t-2) + \upsilon(t) = \varphi^T(t)\theta + \upsilon(t) \\ \varphi^T(t) = [-y(t-1), -y(t-2), u(t-1), u(t-2)] \\ \theta = [\theta_1, \theta_2, \theta_3, \theta_4]^T \end{cases}$$

(6) $y(t) + \theta_1(t)y(t-1)y(t-2) = \theta_2(t)u(t-1) + \theta_3(t)u^2(t-2) + \upsilon(t)$

解答：

$$\begin{cases} y(t) = -\theta_1(t)y(t-1)y(t-2) + \theta_2 u(t-1) + \theta_3 u^2(t-2) + \upsilon(t) = \varphi^T(t)\theta + \upsilon(t) \\ \varphi^T(t) = [-y(t-1)(t-2), u(t-1), u^2(t-2)] \\ \theta = [\theta_1, \theta_2, \theta_3]^T \end{cases}$$

(7) $y(t) + \theta_1 \sin(t/\pi)y(t-1) = \theta_2 u(t-1) + \theta_3 \cos(t) + \upsilon(t)$

解答：

$$\begin{cases} y(t) = -\theta_1(t)\sin(t/\pi)y(t-1) + \theta_2 u(t-1) + \theta_3 \cos(t) + \upsilon(t) = \varphi^T(t)\theta + \upsilon(t) \\ \varphi^T(t) = [-\sin(t/\pi)y(t-1), u(t-1), \cos(t)] \\ \theta = [\theta_1, \theta_2, \theta_3]^T \end{cases}$$

(8) $y(t) + \theta_1(t)y(t-1)y(t-2) = \theta_2(t)u(t-1) + \theta_3(t)u^2(t-2) + \upsilon(t)$

解答：

$$\begin{cases} y(t) = -\theta_1(t)y(t-1)y(t-2) + \theta_2(t)u(t-1) + \theta_3(t)u^2(t-2) + \upsilon(t) = \varphi^T(t)\theta + \upsilon(t) \\ \varphi^T(t) = [-y(t-1)(t-2), u(t-1), u^2(t-2)] \\ \theta = [\theta_1(t), \theta_2(t), \theta_3(t)]^T \end{cases}$$

(9) $y(t) = au^2(t) + bu(t) + 2c + d\sin(t/\pi)$

解答：

$$\begin{cases} y(t) = \varphi^T(t)\theta \\ \varphi^T(t) = [u^2(t), u(t), 2, \sin(t/\pi)] \\ \theta = [a, b, c, d]^T \end{cases}$$

(10) $y(t) = a_1 y(t-1) + a_2 y^2(t-2) + \frac{1}{b_0}[u(t) + b_1 u(t-1)]$

解答：

$$\begin{cases} y(t) = \varphi^T(t)\theta \\ \varphi^T(t) = [y(t-1), y^2(t-2), u(t), u(t-1)] \\ \theta = [a_1, a_2, \frac{1}{b_0}, \frac{b_1}{b_0}]^T \end{cases}$$

三、【20 分】设三阶 AR 模型为 $y(t) + ay(t-1) + by(t-2) + cy(t-3) = \upsilon(t)$ 其中 $\{y(t)\}$ 是

已知观测序列，$\{\upsilon(t)\}$ 是零均值方差为 δ^2 的随机白噪声序列，其辨识模型为

$y(t)=\varphi^T(t)\theta+\upsilon(t)$

- 写出信息向量 $\phi(t)$ 和参数向量 υ 的表达式
- 写出 υ 的一次完成最小二乘估计式（数据长度为 L）
- 写出 υ 的递推最小二乘辨识算法

解答：

$$y(t)=-ay(t-1)-by(t-2)-cy(t-3)+\upsilon(t)$$

$$=\varphi^T(t)\theta+\upsilon(t)$$

其中：$\varphi^T(t)=[-y(t-1),-y(t-2),-y(t-3)]$

$$\theta=[a,b,c]^T$$

令 $Y_L=\begin{bmatrix} y(1) \\ y(2) \\ \vdots \\ y(L) \end{bmatrix}$ $\quad H_L=\begin{bmatrix} \varphi^T(1) \\ \varphi^T(2) \\ \vdots \\ \varphi^T(L) \end{bmatrix}$

则 θ 的 RL 算式如下：

$$\hat{\theta}_{RL}=(H^T{}_LH_L)^{-1}H^T{}_LY_L$$

RLS 算法如下：

$$\hat{\theta}(t)=\hat{\theta}(t-1)+L(t)[y(t)-\varphi^T(t)\hat{\theta}(t-1)]$$

$$L(t)=P(t-1)\varphi(t)[1+\varphi^T(t)P(t-1)\varphi(t)]^{-1}$$

$$P(t)=[I-L(t)\varphi^T(t)]P(t-1),P(0)=P_0I$$

$$\varphi(t)=[-y(t-1),-y(t-2),-y(t-3)]^T$$

$$\hat{\theta}(t)=[\hat{a},\hat{b},\hat{c}]^T$$

四、【20 分】设有限脉冲响应（FIR）模型为 $y(t)=b_1u(t-1)+b_2u(t-2)+b_3u(t-3)+4+v(t)$

其中{y(t)}是已知观测序列，{ $v(t)$ }是零均值方差为 δ^2 的随即白噪声序列，其辨识模型为

$y(t)=\varphi^T(t)\theta+\upsilon(t)$

- 写出信息向量 $\phi(t)$ 和参数向量 υ 的表达式

- 写出θ的一次完成最小二乘估计式（数据长度为L）
- 写出θ的递推最小二乘（RLS）辨识算法

解答：

$$Y(t)-\varphi=b_1u(t-1)+b_2u(t-2)+b_3u(t-3)+v(t)$$

$$=\varphi^T(t)\theta+v(t)$$

其中：$\varphi^T(t)=[u(t-1),u(t-2),u(t-3)]$

$$\theta=[b_1,b_2,b_3]^T$$

令 $Y_L=\begin{bmatrix}Y(1)-4\\Y(2)-4\\Y(3)-4\end{bmatrix}$ $\qquad H_L=\begin{bmatrix}\varphi^T(1)\\\varphi^T(2)\\\varphi^T(3)\end{bmatrix}$

则θ的 RL 算法如下：

$$\theta_{RL}=(H_L^TH_L)^{-1}H_L^TY_L$$

θ的 RLS 算法：

$$\hat{\theta}(t)=\hat{\theta}(t-1)+L(t)[y(t)-\varphi^T(t)\hat{\theta}(t-1)]$$

$$L(t)=p(t-1)\theta(t)[1+\varphi^T(t)p(t-1)\varphi(t)]^{-1}$$

$$p(t)=[I-L(t)\varphi^T(t)]p(t-1),p(0)=p_0I$$

$$\varphi(t)=[u(t-1),u(t-2),u(t-3)]^T$$

$$\hat{\theta}(t)=[\hat{b}_1,\hat{b}_2,\hat{b}_3]^T$$

五、【10 分】设三阶 MA 模型为$y(t)=v(t)+d_1v(t-1)+d_2v(t-2)+d_3v(t-3)$.其中，$\{y(t)\}$是已知观测序列，$\{v(t)\}$是零均值方差为$\sigma^2$的随机白噪声序列，其便是模型为

$$y(t)=\varphi^T(t)\theta+v(t)$$

- 写出信息向量$\varphi(t)$和参数向量θ的表达式
- 写出θ的递推增广最小二乘(RELS)辨识算法.

解答：

$$y(t)=\varphi^T(t)\theta+v(t)$$

其中，$\theta^T(t)=[v(t-1),v(t-2),v(t-3)]$

$$\theta = [d_1, d_2, d_3]^T$$

θ的$R-RELS$算法如下：

$$\hat{\theta}(t) = \hat{\theta}(t-1) + L(t)[y(t) - \hat{\varphi}(t)\hat{\theta}(t-1)]$$

$$\dot{L}(t) = p(t)\hat{\varphi}(t) = \frac{p(t-1)\hat{\varphi}(t)}{1+\hat{\varphi}^T(t)p(t-1)\hat{\varphi}(t)}$$

$$p(t) = [1 - L(t)\hat{\varphi}^T(t)]p(t-1) \qquad p(0) = p_0 I$$

$$\hat{\varphi}^T(t) = [\hat{v}(t-1), \hat{v}(t-2), \hat{v}(t-3)]^T$$

$$v(t) = \varphi(t) - \hat{\varphi}^T(t)\hat{\theta}(t)$$

$$\hat{\theta}(t) = [\hat{d}_1 \hat{d}_2 \hat{d}_3]^T$$

六、证明题【每小题 2 分，其中 5 题，计 10 分】设

$$p^{-1}(t) = p^{-1}(t-1) + \varphi(t)\varphi^T(t), \|\varphi(t)\|^2 \geq 0, \varphi(t) \in R^n$$

$p(0) = I_n, I_n$为n阶单位矩阵，证明以下格式

（1） $p(t)\varphi(t) \; = \dfrac{p(t-1)\varphi(t)}{1+\varphi^T(t)p(t-1)\varphi(t)}$

（2） $\varphi^T(t)p(t)\varphi(t) \leq 1$

(3) $p(t-1)\varphi(t) = \dfrac{p(t)\varphi(t)}{1-\varphi^T(t)p(t)\varphi(t)}$

(4) $\varphi^T(t)p^2(t)\varphi(t) \leq \varphi^T(t)p(t)p(t-1)\varphi(t)$

(5) $\sum_{t=1}^{\infty}\varphi^T(t)p(t)p(t-1)\varphi(t) \prec \infty$

(6) $\sum_{t=1}^{\infty}\varphi^T(t)p^2(t)\varphi(t) \prec \infty$

解答：

(1) $p^{-1}(t) = p^{-1}(t-1) + \varphi(t)\varphi^T(t)$ ①

对①式用矩阵求逆引理，则

$$p(t) = p(t-1) - p(t-1)\varphi(t)[I + \varphi^T(t)p(t-1)\varphi(t)]^{-1}\varphi^T(t)p(t-1)$$

对上式两边乘$\varphi(t)$，可得

$$p(t)\varphi(t) = p(t-1)\varphi(t) - \frac{p(t-1)\varphi(t)\varphi^T(t)p(t-1)\varphi(t)}{1+\varphi^T(t)p(t-1)\varphi(t)}$$

$$= \frac{p(t-1)\varphi(t)}{1+\varphi^T(t)p(t-1)\varphi(t)}$$

（2）∵ $p(t)\varphi(t) = \dfrac{p(t-1)\varphi(t)}{1+\varphi^T(t)p(t-1)\varphi(t)}$ ②

对②式左乘 $\varphi^T(t)$，可得

$$\varphi^T(t)p(t)\varphi(t) = \frac{\varphi^T(t)p(t-1)\varphi(t)}{1+\varphi^T(t)p(t-1)\varphi(t)}$$

∵ $p(t-1)\geq 0$ ∴ $\varphi^T(t)p(t)\varphi(t)\leq 1$

（3）对①右乘 p(t),可得

$$\dot{I} = p^{-1}(t-1)p(t) + \varphi(t)\varphi^T(t)p(t) \quad ③$$

面对③左乘 $p(t-1)$，右乘 $\varphi(t)$，则有

$$p(t-1)\varphi(t) = p(t)\varphi(t) + p(t-1)\varphi(t)\varphi^T(t)p(t)\varphi(t) \quad ④$$

移向合并，可得

$$p(t-1)\varphi(t) = \frac{p(t)\varphi(t)}{1-\varphi^T(t)p(t)\varphi(t)}$$

④对②式左乘 φ^T(t)p(t)，得

$$\varphi^T(t)p^T(t)\varphi(t) = \frac{\varphi^T(t)p(t)p(t-1)\varphi(t)}{1+\varphi^T(t)p(t-1)\varphi(t)}$$

∵ $p(t-1)\geq 0$ ∴ $\varphi^T(t)p(t-1)\varphi(t)\geq 0$

$$\therefore \varphi^T(t)p^T(t)\varphi(t) \leq \varphi^T(t)p(t)p(t-1)\varphi(t)$$

$$p(t-1) = p(t) + p(t-1)\varphi^T(t)\varphi(t)p(t)$$

$$\therefore \sum_{t=1}^{\infty} p(t-1)\varphi(t)\varphi^T(t)p(t) = \sum_{i=1}^{\infty}\Delta p(t) = p(0) - p(\infty) \quad ⑤$$

$$\because p^{-1}(t) = p^{-1}(t-1) + \varphi(t)\varphi^T(t)$$

$$=p^{-1}(0)+\sum_{t=1}^{\infty}\varphi(t)\varphi^{T}(t)$$

$\therefore p^{-1}(t)\geq p^{-1}(0)$

$\therefore$when $t\to\infty$，则 $p(0)\geq p(\infty)$

对⑤式两队取迹，得

$$tr[\sum_{i=1}^{\infty}p(t-1)\varphi(t)\varphi^{T}(t)p(t)]=tr[\sum_{i=1}^{\infty}\varphi^{T}(t)p(t)p(t-1)\varphi(t)]$$

$$=tr[p(0)-p(\infty)]\prec\infty$$

⑥$\because \varphi^{T}(t)p^{T}(t)\varphi(t)\leq\varphi^{T}(t)p(t)p(t-1)\varphi(t)$

$$\therefore\sum_{t=1}^{\infty}\varphi^{T}(t)p^{T}(t)\varphi(t)\leq\sum_{t=1}^{\infty}\varphi^{T}(t)p(t)p(t-1)\varphi(t)]\prec\infty$$