HOMEWORK 1

5 points

1. Gaussian RV: uncorrelatedness implies independence

Show that if two random variables X and Y are jointly normal and are uncorrelated, then they are independent.

Hint: Two random variables X, Y are jointly normal if

$$\text{pdf}(X, Y) = f_{X,Y}(x, y) = \frac{1}{2\pi|\det(C_Z)|^{1/2}} \exp\left[-\frac{1}{2}(Z - E(Z))^T C_Z^{-1}(Z - E(Z))\right],$$

in which $Z = [X, Y]^T$.

Proof. As X and Y are uncorrelated, then we have

$$cov(X, Y) = 0.$$

The covariance matrix C_z can then be written as,

$$C_z = \begin{bmatrix} \sigma_x^2 & 0 \\ 0 & \sigma_y^2 \end{bmatrix}$$

2

Then the joint pdf of X, Y is,

$$f(x, y) = \frac{1}{2\pi\sigma_x\sigma_y} \exp\left[-\frac{(x - EX)^2}{2\sigma_x^2} \cdot -\frac{(y - EY)^2}{2\sigma_y^2}\right]$$

1

and we can obtain the marginal distribution as follows,

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma_x} e^{-\frac{(x-EX)^2}{\sigma_x^2}}, f_Y(y) = \frac{1}{\sqrt{2\pi}\sigma_y} e^{-\frac{(y-EY)^2}{\sigma_y^2}}$$

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2. Wide-sense stationary

Consider the signal plus noise $\boldsymbol{z}(n) = \boldsymbol{s} + \boldsymbol{v}(n)$, where $\boldsymbol{s}$ is a RV with $E[\boldsymbol{s}] = 1, E[\boldsymbol{s}^2] = 2$, and for each value of n, $\boldsymbol{v}(n) \sim \mathcal{N}(0, 1)$. It is known that $E[\boldsymbol{s}\boldsymbol{v}(i)] = 1$ for all i, and $\boldsymbol{v}(i)$ is independent of $\boldsymbol{v}(j)$ for all $i \neq j$.

(1) Compute the autocorrelation function $R_z(i, j)$ of $\boldsymbol{z}(n)$ for all integers i, j.
(2) Is $\boldsymbol{z}(n)$ WSS(Wide sense stationary)? If so, derive a mathematical expression for $R_z(k)$.

- Solution:

$$R_z(i,j) = E[\boldsymbol{z}(i)\boldsymbol{z}(j)] = E[(\boldsymbol{s}+\boldsymbol{v}(i))(\boldsymbol{s}+\boldsymbol{v}(j))] = \begin{cases} 4 & i \neq j \\ 5 & i = j \end{cases} \quad 2$$

Yes, $\boldsymbol{z}(n)$ is wide sense stationary, which is because

$$E[\boldsymbol{z}(n)] = 1 \quad 1$$

and

$$R_z(k) = \begin{cases} 4 & k \neq 0 \\ 5 & k = 0 \end{cases} \quad 2$$