HOMEWORK 2

1. Casino example

Assume the transition matrix is

$$A = \begin{bmatrix} & F & L \\ F & 0.95 & 0.05 \\ L & 0.1 & 0.9 \end{bmatrix}$$

and the emission probability matrix is

$$B = \begin{bmatrix} & 1 & 2 & 3 & 4 & 5 & 6 \\ F & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ L & \frac{1}{10} & \frac{1}{10} & \frac{1}{10} & \frac{1}{10} & \frac{1}{10} & \frac{1}{2} \end{bmatrix}$$

in which "F" denotes fair die and "L" represents loaded die. Denote Y as the number of dies, and X the status of the die, i.e., $X = 0$ means loaded and $X = 1$ indicates fair. If we have the observation $Y = \{6, 6, 6\}$, use maximum likelihood and maximum *a posteriori* estimate to estimate the status of the die.

2. Different estimates

(1) Suppose that $\boldsymbol{z} = \boldsymbol{s} + \boldsymbol{v}$, where $\boldsymbol{s}$ and $\boldsymbol{v}$ are independent, jointly distributed RVs with $\boldsymbol{s} \sim \mathcal{N}(\eta, \sigma^2)$ and $\boldsymbol{v} \sim \mathcal{N}(0, V^2)$.
   (a) Derive an expression for $E[\boldsymbol{s} | \boldsymbol{z} = z]$.
   (b) Derive an expression for $E[\boldsymbol{s}^2 | \boldsymbol{z} = z]$.

(2) Suppose that $\boldsymbol{z} = \boldsymbol{s} + \boldsymbol{v}$, where $\boldsymbol{s}$ and $\boldsymbol{v}$ are independent, jointly distributed RVs with $\boldsymbol{s} \sim \mathcal{N}(\eta_s, \sigma_s^2)$ and $\boldsymbol{v} \sim \mathcal{N}(0, \sigma_v^2)$. Assume we have measurements $\boldsymbol{z}(1), \dots, \boldsymbol{z}(n)$,
   (a) Derive the maximum likelihood estimate for $\boldsymbol{s}$;
   (b) Derive the maximum *a posteriori* estimate for $\boldsymbol{s}$;
   (c) Derive the minimum mean square estimate for $\boldsymbol{s}$;
   (d) Derive the linear minimum mean square estimate for $\boldsymbol{s}$;
   (e) Derive the least squares estimate for $\boldsymbol{s}$ provided measurements $z(1), \dots, z(n)$;
   (f) Suppose at each time k ($k \in \{1, \dots, n\}$), there is a new measurement $z(k)$, derive the recursive least squares estimate for $\boldsymbol{s}$. (Assume $\hat{s}_0 = E(\boldsymbol{s})$, the initial error covariance is P_0);
   (g) Compare all these 6 kinds of estimates.