Homework Solutions

Nonlinear and Adaptive Control

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Outline

Outline

Problem 1

Consider the system defined by the following equations:

$$\dot{x}_1 = \frac{2}{3}x_2$$

$$\dot{x}_2 = -x_1 + x_2(1 - 3x_1^2 - 2x_2^2)$$

(a) Show that the points defined by (i) $x = (0, 0)$ and (ii) $1 - (3x_1^2 + 2x_2^2) = 0$ are invariant sets.

(b) Study the stability of the origin and the invariant set $1 - (3x_1^2 + 2x_2^2) = 0$, respectively, using LaSalle's Invariant Theorem.

Theorem (LaSalle)

For autonomous system

$$\dot{x} = f(x) \tag{1}$$

where $f : D \to \mathbb{R}^n$ is a continuous and differentiable function, $D \subset \mathbb{R}^n$ is a field containing the origin. Suppose that

- *$\Omega \subset D$ is a compact positive invariant set;*
- *$V : D \to \mathbb{R}$ is a continuous and differentiable function, and $\dot{V}(x) \leq 0, \forall x \in \Omega$;*
- *$E = \{x \in \Omega : \dot{V}(x) = 0\}$;*
- *M is the largest invariant set in E.*

Then every solution starting in Ω approaches M as $t \to \infty$.

Problem 1 (a)

Consider the system defined by the following equations:

$$\dot{x}_1 = \frac{2}{3}x_2$$

$$\dot{x}_2 = -x_1 + x_2(1 - 3x_1^2 - 2x_2^2)$$

(a) Show that the points defined by (i) $x = (0, 0)$ and (ii) $1 - (3x_1^2 + 2x_2^2) = 0$ are invariant sets.

Solution: For (i), $x = (0, 0)$ implies that the system equation becomes

$$\dot{x}_1 = 0$$

$$\dot{x}_2 = 0$$

Hence, if we have $x(0) = 0, t = t_0$, then $\forall t \geq t_0,\ x(t) \equiv 0.$ Therefore, $x = (0, 0)$ is an invariant set.

Problem 1 (a) (Cont.)

Consider the system defined by the following equations:

$$\dot{x}_1 = \frac{2}{3}x_2$$
$$\dot{x}_2 = -x_1 + x_2(1 - 3x_1^2 - 2x_2^2)$$

(a) Show that the points defined by (i) $x = (0,0)$ and (ii) $1 - (3x_1^2 + 2x_2^2) = 0$ are invariant sets.

Solution: For (ii), the system equation becomes

$$\dot{x}_1 = \frac{2}{3}x_2$$
$$\dot{x}_2 = -x_1$$

Consider function $f(x_1, x_2) = 3x_1^2 + 2x_2^2$, we have

$$\dot{f} = 6x_1\dot{x}_1 + 4x_2\dot{x}_2 = 0$$

integrating both sides yields

$$f(x_1, x_2) = 3x_1^2 + 2x_2^2 \equiv 1, \forall t \geq t_0$$

which implies that any trajectory starting at (x_1, x_2) that satisfies $1 - (3x_1^2 + 2x_2^2) = 0$ stays on this trajectory function. Therefore, $1 - (3x_1^2 + 2x_2^2) = 0$ is an invariant set.

Problem 1 (b)

Consider the system defined by the following equations:

$$\dot{x}_1 = \frac{2}{3}x_2$$
$$\dot{x}_2 = -x_1 + x_2(1 - 3x_1^2 - 2x_2^2)$$

(b) Study the stability of the origin and the invariant set $1 - (3x_1^2 + 2x_2^2) = 0$, respectively, using LaSalle's Invariant Theorem.

Solution: For the origin, consider $V(x) = \frac{3}{4}x_1^2 + \frac{1}{2}x_2^2$. Its derivative along the trajectory is

$$\begin{aligned}\dot{V}(x) &= \frac{3}{2}x_1\dot{x}_1 + x_2\dot{x}_2 \\ &= x_1x_2 + x_2(-x_1 + x_2(1 - 3x_1^2 - 2x_2^2)) \\ &= x_2^2(1 - 3x_1^2 - 2x_2^2)\end{aligned}$$

which is positive definite in the neighborhood of origin. Therefore, **the origin is not stable**.

Problem 1 (b) (Cont.)

Consider the system defined by the following equations:

$$\dot{x}_1 = \frac{2}{3}x_2$$

$$\dot{x}_2 = -x_1 + x_2(1 - 3x_1^2 - 2x_2^2)$$

(b) Study the stability of the origin and the invariant set $1 - (3x_1^2 + 2x_2^2) = 0$, respectively, using LaSalle's Invariant Theorem.

Solution: For the invariant set $S = \{(x_1, x_2) | 3x_1^2 + 2x_2^2 = 1\}$, consider $V(x) = \frac{1}{8}(1 - 3x_1^2 - 2x_2^2)^2$. Its derivative along the trajectory is

$$\begin{aligned}\dot{V}(x) &= \frac{1}{4}(3x_1^2 + 2x_2^2 - 1)(6x_1\dot{x}_1 + 4x_2\dot{x}_2)\\ &= \frac{1}{4}(3x_1^2 + 2x_2^2 - 1)(-4x_2^2(3x_1^2 + 2x_2^2 - 1))\\ &= -x_2^2(3x_1^2 + 2x_2^2 - 1)^2 \leq 0\end{aligned}$$

Note

V evaluates the "distance" from the limit cycle. Note that V need not to be positive definite.

Problem 1 (b) (Cont.)

Consider the system defined by the following equations:

$$\dot{x}_1 = \frac{2}{3}x_2$$
$$\dot{x}_2 = -x_1 + x_2(1 - 3x_1^2 - 2x_2^2)$$

(b) Study the stability of the origin and the invariant set $1 - (3x_1^2 + 2x_2^2) = 0$, respectively, using LaSalle's Invariant Theorem.

Solution:

$$V(x) = \frac{1}{8}(1 - 3x_1^2 - 2x_2^2)^2, \quad \dot{V}(x) = -x_2^2(3x_1^2 + 2x_2^2 - 1)^2 \leq 0$$

Consider $\Omega_C = \{x \in \mathbb{R}^2 | V(x) \leq c\}$. We know from the derivative above that it's an invariant set.
Then consider $E = \{x \in \Omega_C | \dot{V} = 0\}$, We have $E = S \cup \{x \in \Omega_C | x_2 = 0\}$.
Define M as the largest invariant set in E, i.e. $M = S \cup (0, 0)$.
Choose $c \in (0, \frac{1}{8})$, Ω_C includes the ellipse but not the origin. Then LaSalle's Theorem shows that every motion initiating in Ω_C converges to the limit cycle, and therefore S **is stable**.

Note

Choose $c = \frac{1}{8} - \varepsilon$, where $\varepsilon > 0$ is an arbitrarily small number, we can show that states initiating in any neighborhood of the origin will not approach the origin, which also implies that the origin is not stable.

Outline

Problem 2

It is known that a given dynamical system with the state $x = (x_1, x_2)$ has an equilibrium point at the origin. For this system, a function $V(\cdot)$ have been proposed, and its derivative $\dot{V}(\cdot)$ has been computed. Assuming that $V(\cdot)$ and $\dot{V}(\cdot)$ are given below you are asked to classify the origin, in each case, as (a) stable, (b) locally uniformly asymptotically stable, and/or (c) globally uniformly asymptotically stable. Explain you answer in each case.

(i) $V(x,t) = x_1^2 + x_2^2$, $\dot{V}(x,t) = -x_1^2$.

(ii) $V(x,t) = x_1^2 + x_2^2$, $\dot{V}(x,t) = -(x_1^2 + x_2^2)e^{-t}$.

(iii) $V(x,t) = x_1^2 + x_2^2$, $\dot{V}(x,t) = -(x_1^2 + x_2^2)e^{t}$.

(iv) $V(x,t) = (x_1^2 + x_2^2)e^{t}$, $\dot{V}(x,t) = -(x_1^2 + x_2^2)(1 + \sin^2 t)$.

(v) $V(x,t) = (x_1^2 + x_2^2)e^{-t}$, $\dot{V}(x,t) = -(x_1^2 + x_2^2)$.

(vi) $V(x,t) = (x_1^2 + x_2^2)(1 + e^{-t})$, $\dot{V}(x,t) = -x_1^2 e^{-t}$.

(vii) $V(x,t) = (x_1^2 + x_2^2)(1 + \cos^2 t)$, $\dot{V}(x,t) = -(x_1^2 + x_2^2)e^{-t}$.

(viii) $V(x,t) = (x_1^2 + x_2^2)(1 + \cos^2 t)$, $\dot{V}(x,t) = -(x_1^2 + x_2^2)(1 + e^{-t})$.

Problem 2

(i) $V(x,t) = x_1^2 + x_2^2$, $\dot{V}(x,t) = -x_1^2$.
Solution: Let $W_1(x) = x_1^2 + x_2^2$ and $W_2(x) = 2x_1^2 + 2x_2^2$, we have

$$W_1(x) = V(x,t) \leq W_2(x), \quad \dot{V}(x,t) \leq 0.$$

Thus, V is positive definite and decrescent, and $\dot{V}$ is negative semidefinite.
If $x_1 = 0, x_2 \neq 0$, then $\dot{V} = 0$, while any positive definite function has a positive value. Hence, we cannot find a proper positive definite function $W_3(x)$ such that $\dot{V}(x,t) \leq -W_3(x)$.
Therefore, the origin is **(uniformly) stable**.

(ii) $V(x,t) = x_1^2 + x_2^2$, $\dot{V}(x,t) = -(x_1^2 + x_2^2)e^{-t}$.
Solution: Let $W_1(x) = x_1^2 + x_2^2$ and $W_2(x) = 2x_1^2 + 2x_2^2$. Analogous to (i), V is positive definite and decrescent, and $\dot{V}$ is negative semidefinite.
Note that $\dot{V}(x,t)$ can become arbitrarily small when t is sufficiently large. Hence we cannot find a proper positive definite function $W_3(x)$ such that $\dot{V}(x,t) \leq -W_3(x)$.
Therefore, the origin is **(uniformly) stable**.

(iii) $V(x,t) = x_1^2 + x_2^2$, $\dot{V}(x,t) = -(x_1^2 + x_2^2)e^{t}$.
Solution: Let $W_1(x) = x_1^2 + x_2^2$, $W_2(x) = 2x_1^2 + 2x_2^2$, and $W_3(x) = x_1^2 + x_2^2$, we have

$$W_1(x) = V(x,t) \leq W_2(x), \quad \dot{V}(x,t) \leq -W_3(x).$$

Additionally, $W_1(x)$ is radially unbounded. Therefore, the origin is **globally uniformly asymptotically stable**.

Problem 2

(iv) $V(x,t) = (x_1^2 + x_2^2)e^t$, $\dot{V}(x,t) = -(x_1^2 + x_2^2)(1 + \sin^2 t)$.
Solution: Choose $W_1(x) = x_1^2 + x_2^2$, We see that $V(x,t) \geq W_1(x)$, which implies that V is positive definite. Choose $W_3(x) = x_1^2 + x_2^2$, we also have $\dot{V}(x,t) \leq -W_3(x)$, which implies that $\dot{V}$ is negative definite. However, V is not decrescent since as $t \to \infty$, $V(x,t) \to \infty$.
We then consider the function

$$V'(x,t) = V(x,t)e^{-t} = x_1^2 + x_2^2$$

Thus we have $W_1(x) \triangleq x_1^2 + x_2^2 = V'(x,t) \leq 2(x_1^2 + x_2^2) \triangleq W_2(x)$. And the derivative should be

$$\begin{aligned}
\dot{V}'(x,t) &= \dot{V}(x,t)e^{-t} - V(x,t)e^{-t} \\
&= -(x_1^2 + x_2^2)(1 + \sin^2 t)e^{-t} - (x_1^2 + x_2^2) \\
&\leq -(x_1^2 + x_2^2) \triangleq -W_3(x).
\end{aligned}$$

With $W_1(x)$ radially unbounded, we conclude that the origin is **globally uniformly asymptotically stable**.

Note

The Lyapunov stability theorem is only sufficient!

Problem 2

(v) $V(x,t) = (x_1^2 + x_2^2)e^{-t}$, $\dot{V}(x,t) = -(x_1^2 + x_2^2)$.
Solution: We cannot find a positive definite function $W_1(x)$ such that $V(x,t) \geq W_1(x)$, which shows $V(x,t)$ is positive definite.
Consider a new Lyapunov function $V'(x,t) = V(x,t)e^t = x_1^2 + x_2^2$. Analogous to (iv), it's positive definite and decrescent, and its derivative should be

$$\begin{aligned}\dot{V}'(x,t) &= \dot{V}(x,t)e^t + V(x,t)e^t \\ &= (x_1^2 + x_2^2)(1 - e^t) \leq 0\end{aligned}$$

Therefore, the origin is **uniformly stable**.

Note

The Lyapunov stability theorem is only sufficient!

(vi) $V(x,t) = (x_1^2 + x_2^2)(1 + e^{-t})$, $\dot{V}(x,t) = -x_1^2 e^{-t}$.
Solution: Let $W_1(x) = x_1^2 + x_2^2$ and $W_2(x) = 2x_1^2 + 2x_2^2$, and then we have

$$W_1(x) \leq V(x,t) \leq W_2(x), \quad \dot{V}(x,t) \leq 0.$$

Analogous to (i), we cannot find a proper positive definite function $W_3(x)$ such that $\dot{V}(x,t) \leq -W_3(x)$. Therefore, the origin is **(uniformly) stable**.

Problem 2

(vii) $V(x,t) = (x_1^2 + x_2^2)(1 + \cos^2 t)$, $\dot{V}(x,t) = -(x_1^2 + x_2^2)e^{-t}$.
Let $W_1(x) = x_1^2 + x_2^2$ and $W_2(x) = 2x_1^2 + 2x_2^2$, and then we have

$$W_1(x) \le V(x,t) \le W_2(x), \quad \dot{V}(x,t) \le 0.$$

Analogous to (ii), we cannot find a proper positive definite function $W_3(x)$ such that $\dot{V}(x,t) \le -W_3(x)$. Therefore, the origin is **(uniformly) stable**.

(viii) $V(x,t) = (x_1^2 + x_2^2)(1 + \cos^2 t)$, $\dot{V}(x,t) = -(x_1^2 + x_2^2)(1 + e^{-t})$.
Let $W_1(x) = x_1^2 + x_2^2$, $W_2(x) = 2x_1^2 + 2x_2^2$, $W_3(x) = x_1^2 + x_2^2$ and then we have

$$W_1(x) \le V(x,t) \le W_2(x), \quad \dot{V}(x,t) \le -W_3(x).$$

With $W_1(x)$ radially unbounded, we conclude that the origin is **globally uniformly asymptotically stable**.

Outline

Problem 3

A pendulum with time-varying friction is represented by

$$\dot{x}_1 = x_2,$$
$$\dot{x}_2 = -\sin x_1 - g(t)x_2.$$

Suppose that $g(t)$ is continuously differentiable and satisfies

$$0 < a < \alpha \leq g(t) \leq \beta < \infty \quad \text{and} \quad \dot{g}(t) \leq \gamma < 2$$

for all $t \geq 0$. Consider the Lyapunov function candidate

$$V(t,x) = \frac{1}{2}(a\sin x_1 + x_2)^2 + [1 + ag(t) - a^2](1 - \cos x_1)$$

(a) Show that $V(t,x)$ is positive definite and decrescent.

(b) Show that

$$\dot{V} \leq -(\alpha - a)x_2^2 - a(2-\gamma)(1 - \cos x_1) + O(\|x\|^3),$$

where $O(\|x\|^3)$ is a term bounded by $k\|x\|^3$ in some neighborhood of the origin.

(c) Show that the origin is uniformly asymptotically stable.

Problem 3 (a)

A pendulum with time-varying friction is represented by

$$\dot{x}_1 = x_2,$$
$$\dot{x}_2 = -\sin x_1 - g(t)x_2.$$

Suppose that $g(t)$ is continuously differentiable and satisfies

$$0 < a < \alpha \leq g(t) \leq \beta < \infty \quad \text{and} \quad \dot{g}(t) \leq \gamma < 2$$

for all $t \geq 0$. Consider the Lyapunov function candidate

$$V(t,x) = \frac{1}{2}(a \sin x_1 + x_2)^2 + [1 + ag(t) - a^2](1 - \cos x_1)$$

(a) Show that $V(t,x)$ is positive definite and decrescent.

Proof:

$$V(t,x) > \frac{1}{2}(a \sin x_1 + x_2)^2 + 2\sin^2 \frac{x_1}{2}[1 + a^2 - a^2]$$
$$= \frac{1}{2}(a \sin x_1 + x_2)^2 + 2\sin^2 \frac{x_1}{2} = W_1(x)$$

Let $D = \{(x_1, x_2) | x_1 \in [-\pi, \pi], x_2 \in \mathbb{R}\}$, then $W_1(x) \geq 0$, and $x_1 = x_2 = 0 \iff W_1(x) = 0$. Therefore, $W_1(x)$ is positive definite and $V(t,x)$ is positive definite.

Problem 3 (a) (Cont.)

A pendulum with time-varying friction is represented by

$$\dot{x}_1 = x_2,$$
$$\dot{x}_2 = -\sin x_1 - g(t)x_2.$$

Suppose that $g(t)$ is continuously differentiable and satisfies

$$0 < a < \alpha \le g(t) \le \beta < \infty \quad \text{and} \quad \dot{g}(t) \le \gamma < 2$$

for all $t \ge 0$. Consider the Lyapunov function candidate

$$V(t,x) = \frac{1}{2}(a\sin x_1 + x_2)^2 + [1 + ag(t) - a^2](1 - \cos x_1)$$

(a) Show that $V(t,x)$ is positive definite and decrescent.

Proof:

$$V(t,x) \le \frac{1}{2}(a\sin x_1 + x_2)^2 + (1 + a\beta - a^2)(1 - \cos x_1) = W_2(x)$$

Since $1 + a\beta - a^2 > 0$, $W_2(x) \ge 0$; and we have $x_1 = x_2 = 0 \iff W_2(x) = 0$. Hence, $W_2(x)$ is positive definite. Thus $V(t,x)$ is decrescent.

Problem 3 (b)

(b) Show that

$$\dot{V} \leq -(\alpha - a)x_2^2 - a(2-\gamma)(1-\cos x_1) + O(\|x\|^3),$$

where $O(\|x\|^3)$ is a term bounded by $k\|x\|^3$ in some neighborhood of the origin.

Proof:

$$\dot{V}(t,x) = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x)$$

$$V(t,x) = \frac{1}{2}(a\sin x_1 + x_2)^2 + [1 + ag(t) - a^2](1-\cos x_1)$$

$$= a\dot{g}(t)(1-\cos x_1) + [(a\sin x_1 + x_2)(a\cos x_1) + (1 + ag(t) - a^2)\sin x_1]\dot{x}_1 + (a\sin x_1 + x_2)\dot{x}_2$$

$$\dot{x}_1 = x_2, \dot{x}_2 = -\sin x_1 - g(t)x_2$$

$$= a\dot{g}(t)(1-\cos x_1) + [(a\sin x_1 + x_2)(a\cos x_1) + (1 + ag(t) - a^2)\sin x_1]x_2$$
$$+ (a\sin x_1 + x_2)(-\sin x_1 - g(t)x_2)$$

$$= a\dot{g}(t)(1-\cos x_1) + (a\sin x_1 + x_2)(a\cos x_1)x_2 + \cancel{x_2\sin x_1} + \cancel{ag(t)x_2\sin x_1} - a^2 x_2 \sin x_1$$
$$- a\sin^2 x_1 - \cancel{ag(t)x_2\sin x_1} - \cancel{x_2\sin x_1} - g(t)x_2^2$$

$$= a\dot{g}(t)(1-\cos x_1) + a^2 x_2 \sin x_1 \cos x_1 + a x_2^2 \cos x_1 - a^2 x_2 \sin x_1 - a\sin^2 x_1 - g(t)x_2^2$$

Problem 3 (b) (Cont.)

(b) Show that

$$\dot{V} \leq -(\alpha - a)x_2^2 - a(2-\gamma)(1-\cos x_1) + O(\|x\|^3),$$

where $O(\|x\|^3)$ is a term bounded by $k\|x\|^3$ in some neighborhood of the origin.

Proof:

$$\begin{aligned}
\dot{V}(t,x) &= a\dot{g}(t)(1-\cos x_1) + a^2 x_2 \sin x_1 \cos x_1 + a x_2^2 \cos x_1 - a^2 x_2 \sin x_1 - a\sin^2 x_1 - g(t)x_2^2 \\
&= (a\cos x_1 - g(t))x_2^2 + a\dot{g}(t)(1-\cos x_1) + a^2 x_2 \sin x_1(\cos x_1 - 1) - a(1-\cos^2 x_1) \\
&= -(g(t) - a\cos x_1)x_2^2 - a(2-\dot{g}(t))(1-\cos x_1) + a^2 x_2 \sin x_1(\cos x_1 - 1) \\
&\quad - a(-2 + 1 + \cos x_1)(1-\cos x_1)
\end{aligned}$$

$0 < a < \alpha \leq g(t) \leq \beta < \infty$ and $\dot{g}(t) \leq \gamma < 2$

$$\leq -(\alpha - a)x_2^2 - a(2-\gamma)(1-\cos x_1) + a^2 x_2 \sin x_1(\cos x_1 - 1) + a(1-\cos x_1)^2$$

$\cos x_1 = 1 - \frac{1}{2}x_1^2 + O(\|x\|^4), \sin x_1 = x_1 + O(\|x\|^3)$

$$\begin{aligned}
&= -(\alpha - a)x_2^2 - a(2-\gamma)(1-\cos x_1) + a^2 x_2(O(\|x_1\|))(O(\|x_1\|^2)) + a(O(\|x_1\|^2))^2 \\
&= -(\alpha - a)x_2^2 - a(2-\gamma)(1-\cos x_1) + O(\|x\|^3)
\end{aligned}$$

Problem 3 (c)

(c) Show that the origin is uniformly asymptotically stable.

Proof:

$$\begin{aligned}
\dot{V} &\le -(\alpha - a)x_2^2 - a(2-\gamma)(1-\cos x_1) + O(\|x\|^3) \\
&\le -(\alpha - a)x_2^2 - a(2-\gamma)\left(\frac{1}{2}x_1^2 + O(\|x\|^4)\right) + O(\|x\|^3) \\
&= -(\alpha - a)x_2^2 - \frac{a(2-\gamma)}{2}x_1^2 + O(\|x\|^3)
\end{aligned}$$

Let $k = \min\{\alpha - a, \frac{a(2-\gamma)}{2}\}$, and we have

$$\dot{V} \le -k\|x\|^2 + O(\|x\|^3) = -\|x\|^2\left(k - \frac{O(\|x\|^3)}{\|x\|^2}\right)$$

Define $W_3(x) = \|x\|^2\left(k - \frac{O(\|x\|^3)}{\|x\|^2}\right)$. Since $\frac{O(\|x\|^3)}{\|x\|^2} \to 0$ as $\|x\| \to 0$, we have

$$\forall \varepsilon > 0, \exists \delta > 0, \text{s.t. } \|x\| \le \delta, \left\|\frac{O(\|x\|^3)}{\|x\|^2}\right\| \le \varepsilon$$

Choose $\varepsilon = \frac{1}{2}k$, then $W_3(x)$ is positive definite in the neighborhood corresponding to this ε. From (a), $V(t,x)$ is positive definite and decresent. Therefore, the origin is (locally) uniformly asymptotically stable.

Outline

Problem 4

We denote by $|x|$ the absolute value of x if x is scalar and the euclidean norm of x is x is a vector. For functions of time, the L_2 norm is given by

$$\|x\|_p = \left(\int_0^\infty |x(\tau)|^p \mathrm{d}\tau\right)^{\frac{1}{p}},$$

for $p \in [1, \infty]$, while

$$\|x\|_\infty = \sup_{t \geq 0} |x(t)|.$$

We say that $x \in \mathbb{L}_p$ when $\|x\|_p < \infty$.

(a) Write down the Barbalat's lemma, the Lyapunov-like lemma, and the Lashalle-Yoshizawa theorem.

(b) Use Barbalat's lemma to prove the Lyapunov-like lemma, Lashalle-Yoshizawa theorem, and the following corollary.
Corollary: If $x \in \mathbb{L}_2 \bigcap \mathbb{L}_\infty$ and $\dot{x} \in \mathbb{L}_\infty$, then $\lim_{t \to \infty} x(t) = 0$.

Problem 4 (Cont.)

Barbalat's lemma and Lyapunov-like lemma

Lemma (Barbalat)

If a differentiable function $f(t)$ *has a finite limit as* $t \to \infty$, *and if* $\dot{f}(t)$ *is uniformly continuous, then* $\lim_{t\to\infty} \dot{f}(t) = 0$.

Lemma (Lyapunov-like)

If a scalar function $V(t,x)$ *satisfies the following conditions:*

- $V(t,x)$ *is lower bounded,*
- $\dot{V}(t,x)$ *is negative semi-definite,*
- $\dot{V}(t,x)$ *is uniformly continuous,*

then $\lim_{t\to\infty} \dot{V}(t,x) = 0$.

Proof of Lyapunov-like lemma:

From the first two conditions of $V(t,x)$, we know that $V(t,x)$ is non-increasing and bounded below. Hence, it converges to some finite value V_∞ as $t \to \infty$, i.e., $\lim_{t\to\infty} V(t,x) = V_0$.

With the third condition, by Barbalat's lemma, we have $\lim_{t\to\infty} \dot{V}(t,x) = 0$.

Problem 4 (Cont.)

Theorem (LaSalle-Yoshizawa)

Let $x = 0$ be an equilibrium point of $\dot{x} = f(t, x)$ and suppose that $f(t, x)$ is piecewise continuous in t and locally Lipschitz in x and uniformly in t. Let $V(t, x)$ be a continuously differentiable function such that $\forall t \geq 0,\ x \in \mathbb{R}^n$

$$\alpha_1(\|x\|) \leq V(t, x) \leq \alpha_2(\|x\|)$$

$$\dot{V} = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t, x) \leq -W(x) \leq 0$$

where $\alpha_1(\cdot)$ and $\alpha_2(\cdot)$ are class $\mathcal{K}_\infty$ functions and $W(x)$ is a continuous function. Then the solutions of $\dot{x} = f(t, x)$ satisfy

$$\lim_{t \to \infty} W(x(t)) = 0$$

In addition, if $W(x)$ is positive definite, $x = 0$ is globally uniformly asymptotically stable.

Basic idea:

Barbalat's lemma	f has finite limit	$\dot{f}$ is uniformly continuous	$\Longrightarrow$	$\dot{f} \to 0$
	$\Updownarrow$	$\Updownarrow$		
LaSalle-Yoshizawa Theorem	$\int_0^t W(x(\tau))\mathrm{d}\tau$ has finite limit	W is uniformly continuous w.r.t. t	$\Longrightarrow$	$W(x(t)) \to 0$
	$\Uparrow$	$\Uparrow$		
	$\int_0^t W(x(\tau))\mathrm{d}\tau$ is non-decreasing and upper bounded	W is uniformly continuous w.r.t. x; x is uniformly continuous w.r.t. t		

Problem 4 (Cont.)

LaSalle-Yoshizawa Theorem

Basic idea:

Barbalat's lemma	f has finite limit	$\dot{f}$ is uniformly continuous	$\Longrightarrow$	$\dot{f} \to 0$
	$\Updownarrow$	$\Updownarrow$		
LaSalle-Yoshizawa Theorem	$\int_0^t W(x(\tau))\mathrm{d}\tau$ has finite limit	W is uniformly continuous w.r.t. t	$\Longrightarrow$	$W(x(t)) \to 0$
	$\Uparrow$	$\Uparrow$		
	$\int_0^t W(x(\tau))\mathrm{d}\tau$ is non-decreasing and upper bounded	W is uniformly continuous w.r.t. x; x is uniformly continuous w.r.t. t		

Since $\dot{V} \leq -W(x) \leq 0$, integrating both sides yields $V(t) - V(0) \leq -\int_0^t W(x(\tau))\mathrm{d}\tau$, i.e.,

$$\int_0^t W(x(\tau))\mathrm{d}\tau \leq V(0,x) - V(t,x) \leq V(0,x).$$

This implies $\int_0^t W(x(\tau))\mathrm{d}\tau$ has an upper bound.

Besides, $W(x) \geq 0$, which means $\int_0^t W(x(\tau))\mathrm{d}\tau$ is non-decreasing.

Thus, $\int_0^t W(x(\tau))\mathrm{d}\tau$ has a finite limit.

Problem 4 (Cont.)

LaSalle-Yoshizawa Theorem

Basic idea:

Barbalat's lemma	f has finite limit	$\dot{f}$ is uniformly continuous	$\Longrightarrow$	$\dot{f} \to 0$
	$\Updownarrow$	$\Updownarrow$		
LaSalle-Yoshizawa Theorem	$\int_0^t W(x(\tau))\mathrm{d}\tau$ has finite limit	W is uniformly continuous w.r.t. t	$\Longrightarrow$	$W(x(t)) \to 0$
	$\Uparrow$	$\Uparrow$		
	$\int_0^t W(x(\tau))\mathrm{d}\tau$ is non-decreasing and upper bounded	W is uniformly continuous w.r.t. x x is uniformly continuous w.r.t. t		

Since $\dot{V} \le 0$, and $\alpha_1(\|x\|) \le V(t,x) \le \alpha_2(\|x\|)$, then we have

$$\alpha_1(\|x\|) \le V(t,x) \le V(0,x),$$

where $\alpha_1(\|x\|)$ belongs to class $\mathcal{K}_\infty$, i.e., $\alpha_1(\|x\|)$ is strictly increasing. Thus, we have

$$\|x(t)\| \le \alpha_1^{-1}(V(0,x(0))) \triangleq R.$$

Therefore, the domain of $W(x)$ is bounded and closed, i.e., the domain is a compact set. From the fact that *every continuous function on a compact set is uniformly continuous*, $W(x)$ is uniformly continuous in x on $\|x(t)\| \le R$.

Notice that $f(t,x)$ is locally Lipschitz in x and uniformly in t, then we have, $\forall t_2 > t_1$,

$$\|x(t_2) - x(t_1)\| = \left\| \int_{t_1}^{t_2} f(\tau, x(\tau))\, \mathrm{d}\tau \right\| \le L_R \int_{t_1}^{t_2} \|x(\tau)\|\, \mathrm{d}\tau \le L_R R |t_2 - t_1|,$$

this implies $x(t)$ is uniformly continuous in t. Since $W(x)$ is uniformly continuous in x, thus, $W(x(t))$ is uniformly continuous in t. From Barbalat's lemma, we have $\lim_{t\to\infty} W(x(t)) = 0$.

Corollary

If $x \in \mathbb{L}_2 \bigcap \mathbb{L}_\infty$ *and* $\dot{x} \in \mathbb{L}_\infty$*, then* $\lim_{t\to\infty} x(t) = 0$.

Basic idea: (The idea presented below is used for an extended version of the corollary above: If $x \in \mathbb{L}_2 \bigcap \mathbb{L}_\infty$ and $\dot{x} \in \mathbb{L}_\infty$, then $\lim_{t\to\infty} x(t) = 0$.)

$$
\begin{array}{ccccc}
\text{Barbalat's lemma} & f \text{ has finite limit} & \dot{f} \text{ is uniformly continuous} & \Longrightarrow & \dot{f} \to 0 \\
 & \Updownarrow & \Updownarrow & & \\
\text{Corollary} & \int_0^t |x(\tau)|^p \mathrm{d}\tau \text{ has finite limit} & x^p \text{ is uniformly continuous in } t & \Longrightarrow & x^p \to 0 \Longrightarrow x \to 0 \\
 & \Uparrow & \Uparrow & & \\
 & x \in \mathbb{L}_p & \frac{\mathrm{d}x^p}{\mathrm{d}t} = px^{p-1}\dot{x} \text{ is bounded} & \Leftarrow & x, \dot{x} \in \mathbf{L}_\infty
\end{array}
$$

The proof can be immediately constructed from the guidance above.

Outline

Problem 5

Consider the following multi-dimensional system

$$\dot{x} = Ax + B(u + \Theta^T \Phi(x))$$

where $x \in \mathbb{R}^n$ is the state, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$ are known matrices, $u \in \mathbb{R}^m$ is the control input, $\Phi(x) \in \mathbb{R}^k$ is a bounded function, and $\Theta \in \mathbb{R}^{k\times m}$ is an unknown constant matrix. Assume that (A, B) is controllable.

(a) Design an adaptive control law to stabilize the system.

(b) Design an adaptive control law with adaptive σ-modification to stabilize the system.

Problem 5 (a)

$$\dot{x} = Ax + B(u + \Theta^T \Phi(x))$$

(a) Design an adaptive control law to stabilize the system.

Solution:

If Θ is known, we can design the following control law:

$$u = Kx - \Theta^T \Phi(x)$$

where $K \in \mathbb{R}^{n \times m}$, then the system dynamic is as follow:

$$\dot{x} = Ax + B(u + \Theta^T \Phi(x)) = (A + BK)x$$

Let $A + BK \triangleq A^*$, we can find a proper K to make A^* Hurwitz, since (A, B) is controllable.

Since Θ is unknown, we modify the control law as follow:

$$u = Kx - \hat{\Theta}^T \Phi(x)$$

Then the system dynamic is as follow:

$$\dot{x} = Ax + B(u + \Theta^T \Phi(x)) = (A + BK)x + B(\Theta^T \Phi(x) - \hat{\Theta}^T \Phi(x)),$$

Problem 5 (a) (Cont.)

Define $\tilde{\Theta} \triangleq \hat{\Theta} - \Theta$, then the system can be written as

$$\dot{x} = A^* x - B\tilde{\Theta}^T \Phi(x).$$

Consider the following Lyapunov function candidate

$$V = x^T P x + tr(\tilde{\Theta}^T \Gamma^{-1} \tilde{\Theta})$$

where $\Gamma \in \mathbb{R}^{k\times k}, P \in \mathbb{R}^{n\times n}$ are positive definite and symmetric, and P satisfies

$$PA^* + A^{*T} P = -Q$$

where $Q \in \mathbb{R}^{n\times n}$ is positive definite.
The derivative of V is shown below:

$$\begin{aligned}
\dot{V} &= \dot{x}^T P x + x^T P \dot{x} + 2tr(\tilde{\Theta}^T \Gamma^{-1} \hat{\Theta}) \\
&= (A^* x - B\tilde{\Theta}^T \Phi(x))^T P x + x^T P (A^* x - B\tilde{\Theta}^T \Phi(x)) + 2tr(\tilde{\Theta}^T \Gamma^{-1} \dot{\hat{\Theta}}) \\
&= x^T A^{*T} P x - \Phi^T(x) \tilde{\Theta} B^T P x + x^T P A^* x - x^T P B \tilde{\Theta}^T \Phi(x) + 2tr(\tilde{\Theta}^T \Gamma^{-1} \dot{\hat{\Theta}}) \\
&= -x^T Q x - 2tr(\tilde{\Theta}^T \Phi(x) x^T P B) + 2tr(\tilde{\Theta}^T \Gamma^{-1} \dot{\hat{\Theta}}) \\
&= -x^T Q x + 2tr(\tilde{\Theta}^T (\Gamma^{-1} \dot{\hat{\Theta}} - \Phi(x) x^T P B))
\end{aligned}$$

Problem 5 (a) (Cont.)

where the fourth equality follows from

$$\begin{aligned}\Phi^T(x)\tilde{\Theta}B^T Px &= tr(\Phi^T(x)\tilde{\Theta}B^T Px) && [scalar]\\ &= tr(x^T PB\tilde{\Theta}^T\Phi(x)) = x^T PB\tilde{\Theta}^T\Phi(x) && [tr(A^T) = tr(A), scalar]\\ &= tr(\tilde{\Theta}^T\Phi(x)x^T PB) && [tr(AB) = tr(BA)]\end{aligned}$$

Let $\dot{\hat{\Theta}} = \Gamma\Phi(x)x^T PB$, then we have

$$\dot{V} = -x^T Qx \leq 0,$$

which implies $\forall t > 0,\ V(t) \leq V(0)$, i.e., $x,\ \tilde{\Theta} \in \mathbb{L}_\infty$. From LaSalle-Yoshizawa Theorem, $\lim_{t\to\infty} x^T Qx = 0$, i.e., $\lim_{t\to\infty} x(t) = 0$.

Problem 5 (b)

(b) Design an adaptive control law with adaptive σ-modification to stabilize the system.

Solution: Consider the adaptive control law with adaptive σ-modification:

$$\dot{\hat{\Theta}} = \Gamma(\Phi(x)x^T PB - \Sigma(\hat{\Theta} - \hat{\Theta}_1)),$$
$$\dot{\hat{\Theta}}_1 = \Delta(\hat{\Theta} - \hat{\Theta}_1),$$

where Σ and Δ are constant positive definite matrices, $\hat{\Theta}_1$ is the estimation of $\hat{\Theta}$.
Define $\tilde{\Theta}_1 = \hat{\Theta}_1 - \Theta$. Consider the following Lyapunov function candidate

$$V = x^T Px + tr(\tilde{\Theta}^T \Gamma^{-1} \tilde{\Theta}) + tr(\tilde{\Theta}_1^T \Sigma \Delta^{-1} \tilde{\Theta}_1).$$

Then its derivative is

$$\begin{aligned}
\dot{V} &= \dot{x}^T Px + x^T P\dot{x} + 2tr(\tilde{\Theta}^T \Gamma^{-1} \dot{\hat{\Theta}}) + 2tr(\tilde{\Theta}_1^T \Sigma \Delta^{-1} \dot{\hat{\Theta}}_1) \\
&= -x^T Qx + 2tr(\tilde{\Theta}^T (\Gamma^{-1} \dot{\hat{\Theta}} - \Phi(x)x^T PB)) + 2tr(\tilde{\Theta}_1^T \Sigma \Delta^{-1} \dot{\hat{\Theta}}_1) \\
&= -x^T Qx - 2tr(\tilde{\Theta}^T \Sigma(\hat{\Theta} - \hat{\Theta}_1)) + 2tr(\tilde{\Theta}_1^T \Sigma(\hat{\Theta} - \hat{\Theta}_1)) \\
&= -x^T Qx - 2tr((\tilde{\Theta} - \tilde{\Theta}_1)^T \Sigma(\hat{\Theta} - \hat{\Theta}_1)) \\
&= -x^T Qx - 2tr((\tilde{\Theta} - \tilde{\Theta}_1)^T \Sigma(\tilde{\Theta} - \tilde{\Theta}_1)) \leq 0,
\end{aligned}$$

which implies $\forall t > 0,\ V(t) \leq V(0)$, i.e., $x,\ \tilde{\Theta},\ \tilde{\Theta}_1 \in \mathbb{L}_\infty$. Thus $tr((\tilde{\Theta} - \tilde{\Theta}_1)^T \Sigma(\tilde{\Theta} - \tilde{\Theta}_1))$ is bounded.
From LaSalle-Yoshizawa Theorem, $\lim_{t\to\infty} x^T Qx = 0$, i.e., $\lim_{t\to\infty} x(t) = 0$.

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