Final Examination-Standard Solutions

September 6, 2017

1. (20 points) Answer the following questions:

   (a) What are the main contents of this course?

   (b) What are the reasons to use nonlinear control technologies?

   Solution:

   (a) The main contents of this course include: the introduction of nonlinear systems, phase plane analysis of second-order systems, Lyapunov stability theory for autonomous systems and non-autonomous systems, nonlinear control design methods including backstepping and sliding model control, and control applications for robotic manipulators.

   (b) The reasons to use nonlinear control technologies include: improvement of existing control systems, analysis of hard nonlinearities, dealing with model uncertainties, and design simplicity.

2. (20 points) Consider the following second-order nonlinear systems

$$\dot{x}_1 = x_2 - x_1(x_1^2 + x_2^2 - 1),$$
$$\dot{x}_2 = -x_1 - x_2(x_1^2 + x_2^2 - 1).$$

   (a) Find the equilibrium points of the system and determine the type of each isolated one.

   (b) Show that there exists a limit cycle and determine its stability.

   Solution:

   (a) To find the equilibrium points, let

$$0 = x_2 - x_1(x_1^2 + x_2^2 - 1), \qquad (1)$$
$$0 = -x_1 - x_2(x_1^2 + x_2^2 - 1). \qquad (2)$$

   After some manipulation, we can get

$$x_1^2 + x_2^2 = 0.$$

Therefore, the equilibrium point is $x_1 = 0, x_2 = 0$. To determine the type of the equilibrium point, we can compute the Jacobian matrix

$$\frac{\partial f}{\partial x} = \begin{pmatrix} -3x_1^2 - x_2^2 + 1 & 1 - 2x_1x_2 \\ -1 - 2x_1x_2 & -x_1^2 - 3x_2^2 + 1 \end{pmatrix}$$

Clearly,

$$\frac{\partial f}{\partial x}\mid_{x_1=0, x_2=0} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

whose eigenvalues are $\lambda_{1,2} = 1 \pm i$. The equilibrium point is an unstable focus.

(b) Assume that $r = \sqrt{x_1^2 + x_2^2}$, $\theta = \arctan \frac{x_2}{x_1}$. We have

$$\dot{r} = r^{-1}(x_1\dot{x}_1 + x_2\dot{x}_2) = -r^2(r^2 - 1)$$
$$\dot{\theta} = \frac{1}{1 + (\frac{x_2}{x_1})^2} \cdot \frac{\dot{x}_2x_1 - x_2\dot{x}_1}{x_1^2} = -1.$$

Let $\dot{r} = 1$. We have $r = 0$ and $r = 1$. Clearly, $r = 1$ represents the equilibrium point and $r = 1$ represents a limit cycle. Note that when $0 < r < 1$, $\dot{r} > 0$, which means that r will increase. On the other hand, when $r > 1$, $\dot{r} < 0$, which means that r will decrease. Therefore, $r = 1$ is a stable limit cycle.

3. (10 points) Consider the following nonlinear systems

$$\dot{x}_1 = x_1^2 - x_1^3 + x_2,$$
$$\dot{x}_2 = x_3,$$
$$\dot{x}_3 = x_2 \sin x_1 + e^{-x_3}u.$$

Design a backstepping controller $u(x_1, x_2, x_3)$ such that the origin $(x_1, x_2, x_3) = (0, 0, 0)$ is asymptotically stable.

Solution:

We start with the scalar systems

$$\dot{x}_1 = x_1^2 - x_1^3 + x_2 \quad (3)$$

with x_2 being viewed as the input and proceed to design a feedback control $x_2 = \phi_1(x_1)$ to stabilize the origin $x_1 = 0$. Clearly, with

$$\phi_1(x_1) = -x_1^2 - x_1$$

We have

$$\dot{x}_1 = -x_1^3 - x_1. \quad (4)$$

Consider a Lyapunov function candidate

$$V_1(x_1) = \frac{1}{2}x_1^2.$$

Its derivative along (4) *is*

$$\dot{V}_1(x_1) = x_1\dot{x}_1 = -x_1^2 - x_1^4.$$

Hence, the origin $x_1 = 0$ *of* $\dot{x}_1 = -x_1^3 - x_1$ *is globally asymptotically stable.*
To backstep, we use the change of variables

$$y_1 = x_2 - \phi_1(x_1) = x_2 + x_1^2 + x_1 \tag{5}$$

to transform the system (3) *into the form*

$$\dot{x}_1 = x_1^2 - x_1^3 + x_2 = x_1^2 - x_1^3 + y_1 + \phi_1(x_1) = -x_1^3 - x_1 + y_1, \tag{6}$$
$$\dot{y}_1 = \dot{x}_2 + 2x_1\dot{x}_1 + \dot{x}_1 = x_3 + (2x_1 + 1)(x_1^2 - x_1^3 + x_2). \tag{7}$$

Consider a Lyapunov function candidate

$$V_2(x_1, y_1) = \frac{1}{2}x_1^2 + \frac{1}{2}y_1^2.$$

Its derivative along (6) *and* (7) *is*

$$\begin{aligned}\dot{V}_2 &= x_1\dot{x}_1 + y_1\dot{y}_1 \\ &= -x_1^2 - x_1^4 + y_1[x_3 + x_1 + (2x_1 + 1)(x_1^2 - x_1^3 + x_2)].\end{aligned}$$

Clearly, with

$$x_3 = -x_1 - (2x_1 + 1)(x_1^2 - x_1^3 + x_2) - k_1y_1 \triangleq \phi_2(x_1, x_2), \quad k_1 > 0,$$

We have

$$\dot{V}_2 = -x_1^2 - x_1^4 - k_1y_1^2.$$

To backstep, we use the change of variables

$$y_2 = x_3 - \phi_2(x_1, x_2) \tag{8}$$

to transform the system into the form

$$\dot{x}_1 = x_1^2 - x_1^3 + x_2, \tag{9}$$
$$\dot{x}_2 = y_2 + \phi_2(x_1, x_2), \tag{10}$$
$$\begin{aligned}\dot{y}_2 &= \dot{x}_3 - \dot{\phi}_2(x_1, x_2) \\ &= x_2 \sin x_1 + e^{-x_3}u - \frac{\partial\phi_2(x_1, x_2)}{\partial x_1}(x_1^2 - x_1^3 + x_2) \\ &\quad - \frac{\partial\phi_2(x_1, x_2)}{\partial x_2}[y_2 + \phi_2(x_1, x_2)].\end{aligned} \tag{11}$$

Consider the combined Lyapunov function candidate,

$$V_c = V_2 + \frac{1}{2}y_2^2. \tag{12}$$

Its derivative along (9)-(11) is

$$\begin{aligned}\dot{V}_c &= \dot{V}_2 + y_2\dot{y}_2 \\ &= -x_1^2 - x_1^4 - k_1y_1^2 + y_1y_2 + y_2[x_2\sin x_1 + e^{-x_3}u \\ &\quad - \frac{\partial\phi_2(x_1,x_2)}{\partial x_1}(x_1^2 - x_1^3 + x_2) - \frac{\partial\phi_2(x_1,x_2)}{\partial x_2}(y_2 + \phi_2(x_1,x_2))].\end{aligned}$$

Then the control input can be designed as

$$\begin{aligned}u = e^{-x_3}\{ &- k_2y_2 - x_2\sin x_1 - y_1 + \frac{\partial\phi_2(x_1,x_2)}{\partial x_1}(x_1^2 - x_1^3 + x_2) \\ &+ \frac{\partial\phi_2(x_1,x_2)}{\partial x_2}(y_2 + \phi_2(x_1,x_2))\}\end{aligned} \tag{13}$$

Then we have

$$\dot{V}_c = -x_1^2 - x_1^4 - k_1y_1^2 - k_2y_2^2.$$

It follows from LaSalle-Yoshizawa Theorem that $\lim_{t\to\infty} x_1(t) = \lim_{t\to\infty} y_1(t) = \lim_{t\to\infty} y_2(t) = 0$. *From* (5), *we can get* $\lim_{t\to\infty} x_2(t) = 0$. *From* (8), *we can get* $\lim_{t\to\infty} x_3(t) = 0$.

4. (10 points) We denote by $\|x\|$ the absolute value of x if x is a scalar and the Euclidean norm of x if x is a vector. For functions of time, the L_p norm is given by

$$\|x\|_p = \left(\int_0^\infty \|x(\tau)\|^p d\tau\right)^{\frac{1}{p}},$$

for $p \in [1, \infty)$, while

$$\|x\|_\infty = \sup_{t\geq 0} \|x(t)\|.$$

We say that $x \in \mathbb{L}_p$ when $\|x\|_p < \infty$. Write down Barbalat's lemma and use it to prove the following corollary.

Corollary 0.1 *If* $x \in \mathbb{L}_2 \bigcap \mathbb{L}_\infty$ *and* $\dot{x} \in \mathbb{L}_\infty$, *then* $\lim_{t\to\infty} x(t) = 0$.

Solution: *Barbalt's Lemma: If the differentiable function* $f(t)$ *has a finite limit as* $t \to \infty$, *and if* $\dot{f}$ *is uniformly continuous, then* $\lim_{t\to\infty} \dot{f}(t) = 0$.

Proof: Define

$$f(t) = \int_0^t \|x(\tau)\|^2 d\tau$$

Since $x \in \mathbb{L}_2$, we can get $f(t)$ has a finite limit as $t \to \infty$. Furthermore,

$$\dot{f}(t) = \|x(t)\|^2, \qquad \ddot{f}(t) = 2x^T(t)\dot{x}(t)$$

From the fact that $x, \dot{x} \in \mathbb{L}_\infty$, we can get $\ddot{f}(t) \in \mathbb{L}_\infty$, which implies that $\dot{f}$ is uniformly continuous. Therefore, we can get from Barbalat's lemma that $\lim_{t\to\infty} \dot{f}(t) = 0$, that is, $\lim_{t\to\infty} x(t) = 0$.

5. (10 points) Consider the following first-order system

$$\dot{x} = u + d_1(t) + d_2(t)x + d_3(t)x^2,$$

where $d_1(t), d_2(t)$, and $d_3(t)$ are time-varying continuous functions satisfying

$$\max_{t\geq 0}\{\|d_1(t)\|, \|d_2(t)\|, \|d_3(t)\|\} \leq d_{\max},$$

for some positive constant $d_{\max}$. Design a controller using the sliding mode technology such that $\lim_{t\to\infty} x(t) = 0$. *Solution: Consider the following Lyapunov function candidate*

$$V = \frac{1}{2}x^2 \tag{14}$$

whose derivative can be written as

$$\begin{aligned} \dot{V} = & x\dot{x} \\ = & x(u + d_1 + d_2x + d_3x^2) \\ \leq & xu + \|d_1x\| + \|d_2x^2\| + \|d_3x^3\| \\ \leq & xu + d_{\max}\|x\|(1 + \|x\| + x^2). \end{aligned}$$

To make $\dot{V}$ negative definite, we design the controller as follows

$$u = -k_1x - d_{\max}sgn(x)(1 + \|x\| + x^2) \tag{15}$$

where $k_1 > 0$ is a constant. We then have

$$\begin{aligned} \dot{V} \leq & x(-k_1x - d_{\max}sgn(x)(1 + \|x\| + x^2)) + d_{\max}\|x\|(1 + \|x\| + x^2) \\ \leq & -k_1x^2 - d_{\max}\|x\|(1 + \|x\| + x^2) + d_{\max}\|x\|(1 + \|x\| + x^2) \\ = & -k_1x^2 \end{aligned}$$

Clearly, $\dot{V}$ is negative definite, and we can conclude that $\lim_{t\to\infty} x(t) = 0$.

6. (30 points) The nonlinear dynamic equations for an m-link robot take the form

$$M(q)\ddot{q} + C(q, \dot{q})\dot{q} + D\dot{q} + g(q) = u$$

where $q \in \mathbb{R}^p$ is the vector of generalized coordinates representing the joint positions, $M(q) \in \mathbb{R}^{p\times p}$ is the symmetric positive-definite inertia matrix, $C(q,\dot{q})\dot{q} \in \mathbb{R}^p$ is the vector of Coriolis and centrifugal torques, $D\dot{q} \in \mathbb{R}^p$ is the vector of viscous damping, $g(q) = [\partial P(q)/\partial q]^T \in \mathbb{R}^p$ is the gravitational torque, $P(q)$ is a positive definite function of q representing the total potential energy of the links due to gravity, and $u \in \mathbb{R}^p$ is the control torque. The following assumptions hold

(A1) There exist positive constants $k_{\underline{m}}$ and $k_{\overline{m}}$ such that $0 < k_{\underline{m}} I_p \leq M(q) \leq k_{\overline{m}} I_p$.

(A2) $\dot{M}(q) - 2C(q, \dot{q})$ is skew symmetric.

(A3) D is a positive semidefinite symmetric matrix.

(A4) $g(q) = 0$ has an isolated root at $q = 0$.

(a) With $u = 0$, use the total energy $V(q, \dot{q}) = \frac{1}{2}\dot{q}^T M(q)\dot{q} + P(q)$ as a Lyapunov function candidate to show that the origin $(q = 0, \dot{q} = 0)$ is stable.

(b) With $u = -K_v\dot{q}$, where K_v is a positive diagonal matrix, show that the origin is asymptotically stable.

(c) With $u = g(q) - K_p(q - q_d) - K_v\dot{q}$, where K_p and K_v are positive diagonal matrices, and q_d is a constant desired position, show that the point $(q = q_d, \dot{q} = 0)$ is asymptotically stable.

(d) Design a controller such that $q(t)$ asymptotically tracks a reference trajectory $q_d(t)$, where $q_d(t), \dot{q}_d(t)$, and $\ddot{q}_d(t)$ are continuous and bounded.

Solution:

(a) *Since $u = 0$ and $g(q) = 0$, $q = 0, \dot{q} = 0$ is an equilibrium point of the system. The derivative of V can be written as*

$$\begin{aligned}
\dot{V} &= \frac{1}{2}\dot{q}^T \dot{M}(q)\dot{q} + \dot{q}^T M(q)\ddot{q} + \frac{\partial P(q)}{\partial q}\dot{q} \\
&= \frac{1}{2}\dot{q}^T \dot{M}(q)\dot{q} + \dot{q}^T(-C(q,\dot{q})\dot{q} - D\dot{q} - g(q)) + g^T(q)\dot{q} \\
&= \frac{1}{2}\dot{q}^T(\dot{M}(q) - 2C(q,\dot{q}))\dot{q} - \dot{q}^T D\dot{q} \\
&= -\dot{q}^T D\dot{q}
\end{aligned}$$

Since D is positive semidefinite, we can get $\dot{V} \leq 0$. which implies that the origin is stable.

(b) With $u = -K_v\dot{q}$, the derivation of the Lyapunov function candidate is

$$\begin{aligned}\dot{V} &= \frac{1}{2}\dot{q}^T\dot{M}(q)\dot{q} + \dot{q}^T M(q)\ddot{q} + \frac{\partial P(q)}{\partial q}\dot{q}\\ &= \frac{1}{2}\dot{q}^T\dot{M}(q)\dot{q} + \dot{q}^T(-C(q,\dot{q})\dot{q} - D\dot{q} - g(q) - K_v\dot{q}) + g^T(q)\dot{q}\\ &= -\dot{q}^T D\dot{q} - \dot{q}^T K_v\dot{q}\end{aligned}$$

Since D is positive semidefinite and K_v is positive definite, we can get $\dot{V} \leq 0$. Note that the set $\{(q,\dot{q})|\dot{V} = 0\} = \{(q,\dot{q})|\dot{q} = 0\}$. When $\dot{q} = 0$, $\ddot{q} = 0$. We then can get $g(q) = 0$, which implies that $q = 0$. From LaShall's Theorem, we can conclude that the origin is asymptotically stable.

(c) With $u = g(q) - K_p(q - q_d) - K_v\dot{q}$, the closed-loop system can be written as

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + D\dot{q} = -K_p(q - q_d) - K_v\dot{q}$$

Consider the following Lyapunov function candidate

$$V_1 = \frac{1}{2}\dot{q}^T M(q)\dot{q} + P(q) + \frac{1}{2}(q - q_d)^T K_p(q - q_d)$$

Its derivative is

$$\begin{aligned}\dot{V}_1 &= \frac{1}{2}\dot{q}^T\dot{M}(q)\dot{q} + \dot{q}^T M(q)\ddot{q} + \frac{\partial P(q)}{\partial q}\dot{q} + \dot{q}^T K_p(q - q_d)\\ &= \frac{1}{2}\dot{q}^T\dot{M}(q)\dot{q} + \dot{q}^T(-C(q,\dot{q})\dot{q} - D\dot{q} - g(q) - K_v\dot{q} - K_p(q - q_d))\\ &\quad + g^T(q)\dot{q} + \dot{q}^T K_p(q - q_d)\\ &= -\dot{q}^T D\dot{q} - \dot{q}^T K_v\dot{q}\end{aligned}$$

From the analysis in (c), we can conclude that $\dot{V}_1 \leq 0$ and $\{(q,\dot{q})|\dot{V} = 0\} = \{(q,\dot{q})|\dot{q} = 0\}$. When $\dot{q} = 0$, we have $\ddot{q} = 0$ and $q = q_d$. From LaShall's Theorem, we can conclude that the equilibrium point $(q = q_d, \dot{q} = 0)$ is asymptotically stable.

(d) Define the following tracking errors and auxiliary variables

$$\tilde{q} = q(t) - q_d(t), \quad \dot{\tilde{q}} = \dot{q}(t) - \dot{q}_d(t)$$

$$\dot{q}_r = -\lambda\tilde{q}, \quad s = \dot{\tilde{q}} - \dot{q}_r = \dot{\tilde{q}} + \lambda\tilde{q}$$

with $\lambda > 0$ being a constant. Then the dynamic equation can be written as

$$M(q)\dot{s} + C(q,\dot{q})s + Ds = u - M(q)\ddot{q}_r - C(q,\dot{q})\dot{q}_r - D\dot{q}_r - g(q)$$

Design the following control input

$$u = -Ks + M(q)\ddot{q}_r + C(q,\dot{q})\dot{q}_r + D\dot{q}_r + g(q)$$

where $K > 0$ *is a positive definite matrix. Then the closed-loop system can be written as*

$$M(q)\dot{s} + C(q,\dot{q})s + Ds - ks$$

Consider the following Lyapunov function candidate

$$V = \frac{1}{2}s^T M(q)s$$

Its derivative can be written as

$$\begin{aligned}\dot{V} =& \frac{1}{2}s^T \dot{M}(q)s + s^T M(q)\dot{s} \\ =& - s^T(K+D)s\end{aligned}$$

Since D *is positive semidefinite and* K *is positive definite, we can conclude that* $\dot{V}$ *is negative definite. Therefore, we can obtain* $\lim_{t\to\infty} s(t) = 0$. *Note that the following system is input-to-state stable with the input* s *and the state* $\tilde{q}$

$$\dot{\tilde{q}} = -\lambda\tilde{q} + s$$

Therefore, we can get $\lim_{t\to\infty} \tilde{q}(t) = \lim_{t\to\infty} \dot{\tilde{q}}(t) = 0.$