3. (15 points) Consider the following nonlinear systems

$$\dot{x}_1 = -x_1^3 + x_2 + \theta\phi_1(x_1),$$
$$\dot{x}_2 = u + \theta\phi_2(x_1, x_2),$$

where $\phi_1(x_1)$ and $\phi_2(x_1, x_2)$ are Lipschitz continuous and satisfying $\phi_1(0) = 0$ and $\phi_2(0,0) = 0$. Design an adaptive backstepping controller $u(x_1, x_2, \hat{\theta})$ such that $\lim_{t\to\infty} x_1(t) = \lim_{t\to\infty} x_2(t) = 0$.

Solution: **Step 1:** *With x_2 viewed as the virtual control, we design the first stabilizing function α_1 as follows:*

$$\alpha_1 = -\hat{\theta}\phi_1(x_1). \tag{14}$$

The first Lyapunov function is now chosen as

$$V_1 = \frac{1}{2}x_1^2 + \frac{1}{2\gamma}\tilde{\theta}^2, \tag{15}$$

where $\tilde{\theta} \triangleq \hat{\theta} - \theta$ is the parameter error, and $\gamma > 0$ is the adaptation gain. With $z_1 \triangleq x_2 - \alpha_1$, the derivative of V_1 is

$$\dot{V}_1 = -x_1^4 + x_1 z_1 + \frac{\tilde{\theta}}{\gamma}(\dot{\hat{\theta}} - \gamma x_1\phi_1). \tag{16}$$

We postpone the choice of update law for $\hat{\theta}$ until the next step. The first error subsystem becomes

$$\dot{x}_1 = -x_1^3 + z_1 - \tilde{\theta}\phi_1. \tag{17}$$

Step 2: *The derivative of $z_1 = x_2 - \alpha_1$ is*

$$\begin{aligned}\dot{z}_1 &= u + \theta\phi_2 - \frac{\partial\alpha_1}{\partial x_1}\dot{x}_1 - \frac{\partial\alpha_1}{\partial\hat{\theta}}\dot{\hat{\theta}}\\ &= u + \theta\phi_2 - \frac{\partial\alpha_1}{\partial x_1}(x_2 - x_1^3) - \hat{\theta}\phi_1\frac{\partial\alpha_1}{\partial x_1} + \tilde{\theta}\phi_1\frac{\partial\alpha_1}{\partial x_1} - \frac{\partial\alpha_1}{\partial\hat{\theta}}\dot{\hat{\theta}}.\end{aligned}$$

To design the control u, we consider the augmented Lyapunov function

$$V_2 = V_1 + \frac{1}{2}z_1^2 \tag{18}$$

The derivative of V_2 is

$$\begin{aligned}\dot{V}_2 =& -x_1^4 + x_1 z_1 + \frac{\tilde{\theta}}{\gamma}(\dot{\hat{\theta}} - \gamma x_1\phi_1) + z_1\big(u + \theta\phi_2\\ &- \frac{\partial\alpha_1}{\partial x_1}(x_2 - x_1^3) - \hat{\theta}\phi_1\frac{\partial\alpha_1}{\partial x_1} + \tilde{\theta}\phi_1\frac{\partial\alpha_1}{\partial x_1} - \frac{\partial\alpha_1}{\partial\hat{\theta}}\dot{\hat{\theta}}\big)\\ =& -x_1^4 + \frac{\tilde{\theta}}{\gamma}(\dot{\hat{\theta}} - \gamma z_1\phi_2 - \gamma x_1\phi_1 + \gamma z_1\phi_1\frac{\partial\alpha_1}{\partial x_1}) + z_1\big(x_1 + u + \hat{\theta}\phi_2\\ &- \frac{\partial\alpha_1}{\partial x_1}(x_2 - x_1^3) - \hat{\theta}\phi_1\frac{\partial\alpha_1}{\partial x_1} - \frac{\partial\alpha_1}{\partial\hat{\theta}}\dot{\hat{\theta}}\big)\end{aligned}$$

In the last equation, all the terms containing $\tilde{\theta}$ have been grouped together. To eliminate them, the update law is chosen as

$$\dot{\hat{\theta}} = \gamma z_1\phi_2 + \gamma x_1\phi_1 - \gamma z_1\phi_1\frac{\partial\alpha_1}{\partial x_1}.$$

Then, the last bracketed term will be rendered equal to $-k_1 z_1^2$ with the control

$$u = -k_1 z_1 - x_1 - \hat{\theta}\phi_2 + \frac{\partial\alpha_1}{\partial x_1}(x_2 - x_1^3) + \hat{\theta}\phi_1\frac{\partial\alpha_1}{\partial x_1} + \frac{\partial\alpha_1}{\partial\hat{\theta}}\dot{\hat{\theta}}. \tag{19}$$

We then obtain

$$\dot{V}_2 = -x_1^4 - k_1 z_1^2. \tag{20}$$

From LaSalle-Yoshizawa theorem, we can conclude the result.

5. (20 points) The nonlinear dynamic equations for an m-link robot take the form

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + D\dot{q} + g(q) + d(q,\dot{q},t) = u$$

where $q \in \mathbb{R}^p$ is the vector of generalized coordinates representing the joint positions, $M(q) \in \mathbb{R}^{p\times p}$ is the symmetric positive-definite inertia matrix, $C(q,\dot{q})\dot{q} \in \mathbb{R}^p$ is the vector of Coriolis and centrifugal torques, $D\dot{q} \in \mathbb{R}^p$ is the vector of viscous damping with D being a constant matrix, $g(q) \in \mathbb{R}^p$ is the gravitational torque, $d(t)$ is the external disturbance, and $u \in \mathbb{R}^p$ is the control torque. The following assumptions hold

(A1) There exist positive constants $k_{\underline{m}}$ and $k_{\overline{m}}$ such that $0 < k_{\underline{m}}I_p \leq M(q) \leq k_{\overline{m}}I_p$. For $x, y, z \in \mathbb{R}^p$, $0 \leq \|C(x,y)z\| \leq k_C\|y\|\|z\|$.

(A2) $\dot{M}(q) - 2C(q,\dot{q})$ is skew symmetric and D is positive semidefinite.

(A3) There are parameter uncertainties in $M(q), C(q,\dot{q}), D$ and $g(q)$. For $x, y, z \in \mathbb{R}^p$, $M(q)x + C(q,\dot{q})y + Dz + g(q) = Y(q,\dot{q},x,y,z)\theta$, with θ being an unknown constant vector.

$\lambda > 0$

$s = \dot{\tilde{q}} + \lambda\tilde{q} = \dot{q} - \dot{q}_d + \lambda\tilde{q} = \dot{q} - (\dot{q}_d - \lambda\tilde{q})$

$\dot{q}_r$

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(a) Assume that $d(q,\dot{q},t) = 0$. Design a sliding mode controller such that $q(t)$ asymptotically tracks a reference trajectory $q_d(t)$, where $q_d(t), \dot{q}_d(t)$, and $\ddot{q}_d(t)$ are continuous and bounded.

(b) Assume that $\|d(q,\dot{q},t)\| \leq d_{\max}(\|q\|^2 + \|\dot{q}\|^2)$, with $d_{\max}$ being an unknown positive constant. Design a sliding mode controller with sign function such that $q(t)$ asymptotically tracks a reference trajectory $q_d(t)$, where $q_d(t)$, $\dot{q}_d(t)$, and $\ddot{q}_d(t)$ are continuous and bounded.

(a) Define the tracking errors as follows

$$\tilde{q} = q - q_d, \qquad \dot{\tilde{q}} = \dot{q} - \dot{q}_d.$$

Introduce the following auxiliary variables

$$\dot{q}_r = \dot{q}_d - \lambda\tilde{q}$$
$$s = \dot{q} - \dot{q}_r = \dot{\tilde{q}} + \lambda\tilde{q}$$

with $\lambda > 0$. We then have the following dynamic equation with $d(q,\dot{q},t) = 0$

$$\begin{aligned}M(q)\dot{s} + C(q,\dot{q})s + Ds &= u - M(q)\ddot{q}_r - C(q,\dot{q})\dot{q}_r - D\dot{q}_r - g(q)\\ &= u - Y(q,\dot{q},\ddot{q}_r,\dot{q}_r)\theta.\end{aligned}$$

We then design the following sliding mode controller

$$u = -Ks + Y(q,\dot{q},\ddot{q}_r,\dot{q}_r)\hat{\theta}. \tag{36}$$

where K is positive definite. Therefore, the closed-loop system can be written as

$$M(q)\dot{s} + C(q,\dot{q})s + Ds = -Ks + Y(q,\dot{q},\ddot{q}_r,\dot{q}_r)\tilde{\theta}, \tag{37}$$

with $\tilde{\theta} = \hat{\theta} - \theta$. Consider the following Lyapunov function

$$V = \frac{1}{2}s^T M(q)s + \frac{1}{2}\tilde{\theta}^T\tilde{\theta}.$$

Its derivative is

$$\begin{aligned}\dot{V} &= s^T M(q)\dot{s} + \frac{1}{2}s^T\dot{M}(q)s + \tilde{\theta}^T\dot{\hat{\theta}}\\ &= -s^T(K + D)s + \tilde{\theta}^T(\dot{\hat{\theta}} + Y^T s).\end{aligned}$$

With the following adaptive updating law

$$\dot{\hat{\theta}} = -Y^T s,$$

we have $\dot{V} = -s^T(K + D)s$. By noticing that K is positive definite and D is positive semidefinite, we can conclude the result.

(b) When $\|d(q,\dot{q},t)\| \leq d_{\max}(\|q\|^2 + \|\dot{q}\|^2)$ the dynamics is

$$M(q)\dot{s} + C(q,\dot{q})s + Ds = u - Y(q,\dot{q},\ddot{q}_r,\dot{q}_r)\theta + d(q,\dot{q},t). \tag{38}$$

We then design the following control input

$$u = -Ks + Y\hat{\theta} - sgn(s)\hat{d}(\|q\|^2 + \|\dot{q}\|^2). \tag{39}$$

Consider the following Lyapunov function candidate

$$V = \frac{1}{2}s^T M(q)s + \frac{1}{2}\tilde{\theta}^T\tilde{\theta} + \frac{1}{2\gamma}(\hat{d}(t) - d_{max})^2. \tag{40}$$

Its derivative can be written as

$$\begin{aligned}\dot{V} =& s^T M(q)\dot{s} + \frac{1}{2}s^T\dot{M}(q)s + \tilde{\theta}^T\dot{\hat{\theta}} + \frac{1}{\gamma}(\hat{d}(t) - d_{max})\dot{\hat{d}}(t)\\ =& -s^T(K + D)s + \tilde{\theta}^T(\dot{\hat{\theta}} + Y^T s) - \|s\|_1\hat{d}(\|q\|^2 + \|\dot{q}\|^2)\\ &- s^T d + \frac{1}{\gamma}(\hat{d}(t) - d_{\max})\dot{\hat{d}}\\ \leq& -s^T(K + D)s + \tilde{\theta}^T(\dot{\hat{\theta}} + Y^T s) - \|s\|_1\hat{d}(\|q\|^2 + \|\dot{q}\|^2)\\ &+ \|s\|_1 d_{\max}(\|q\|^2 + \|\dot{q}\|^2) + \frac{1}{\gamma}(\hat{d}(t) - d_{\max})\dot{\hat{d}}\end{aligned}$$

Design

$$\dot{\hat{\theta}} = -Y^T s, \tag{41}$$

$$\dot{\hat{d}}(t) = \gamma \|s\|_1 \hat{d}(\|q\|^2 + \|\dot{q}\|^2) \tag{42}$$

We then have $\dot{V} \leq -s^T(K+D)s$. By noticing that K is positive definite and D is positive semidefinite, we can conclude the result.