(Fall Semester of 2024) Final Examination of Nonlinear and Adaptive Control

January 2, 2025

P.S.: This paper was memorized and typeset after the examination ended, and there's no cheating behaviour during the examination.

1. (10 points) Please write down the main content of this course.

2. (15 points) Consider the system defined by the following equations:

$$\begin{aligned}\dot{x}_1 &= 2x_2 \\ \dot{x}_2 &= -3x_1 + x_2(1 - 3x_1^2 - 2x_2^2)\end{aligned}$$

   (a) Show that the points defined by (i) $x = (0, 0)$ and (ii) $1-(3x_1^2+2x_2^2) = 0$ are invariant sets.

   (b) Using the linearization at $x = (0, 0)$ to study the stability of the origin.

   (c) Study the stability of the origin and the invariant set $1 - (3x_1^2 + 2x_2^2) = 0$, respectively, using LaSalle's Invariant Theorem.

3. (15 points) Euler equations for a rotating rigid spacecraft are given by

$$\begin{aligned}J_1\dot{\omega}_1 &= (J_2 - J_3)\,\omega_2\omega_3 + u_1, \\ J_2\dot{\omega}_2 &= (J_3 - J_1)\,\omega_3\omega_1 + u_2, \\ J_3\dot{\omega}_3 &= (J_1 - J_2)\,\omega_1\omega_2 + u_3,\end{aligned}$$

   where ω_1 to ω_3 are the components of the angular velocity vector ω along the principal axes, u_1 to u_3 are the torque inputs applied about the principal axes, and J_1 to J_3 are the principal moments of inertia.

   (a) Show that with $u_1 = u_2 = u_3 = 0$ the origin $\omega = 0$ is stable. Is it asymptotically stable?

   (b) Suppose the torque inputs apply the feedback control $u_i = -k_i\omega_i$, where k_1 to k_3 are positive constants. Show that the origin of the closed-loop system is globally asymptotically stable.

4. (10 points) Consider the following nonlinear systems

$$\dot{x}_1 = -x_1^5 - x_2 + \theta\phi_1(x_1),$$
$$\dot{x}_2 = u + \theta\phi_2(x_1, x_2),$$

where $\phi_1(x_1)$ and $\phi_2(x_1, x_2)$ are Lipschitz continuous and satisfying $\phi_1(0) = 0$ and $\phi_2(0, 0) = 0$. Design an adaptive backstepping controller $u(x_1, x_2, \hat{\theta})$ such that $\lim_{t\to\infty} x_1(t) = \lim_{t\to\infty} x_2(t) = 0$.

5. (10 points) Consider a linear system with nonlinear matched uncertainties in the form

$$\dot{x} = Ax + B\left(u + \Theta^T\Phi(x)\right) + \varepsilon(t),$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the control input, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $\Lambda \in \mathbb{R}^{m\times m}$, $\Theta \in \mathbb{R}^{N\times m}$ are constant matrices, and $\varepsilon(t) \in \mathbb{R}^n$ is the disturbance. Assume that the pair $(A, B\Lambda)$ is controllable. $\Phi(x) = (\phi_1(x), \ldots, \phi_n(x))^T \in \mathbb{R}^N$ denotes the known regressor vector, whose compononts $\phi_i(x)$ are assumed to be Lipschitz-continuous in x.

Assume that $\|\varepsilon(t)\| \le \varepsilon_f, \forall t > 0$, where $\varepsilon_f > 0$, and A, B, Λ are all known, while Λ is a positive diagonal matrix. Design a σ-modification robust control algorithm that can stabilize the system and regulate x towards the neighborhood of zero.

(*Hint:* For positive definite matrices P and Γ, we have

$$\lambda_{\min}(P)x^Tx \le x^TPx \le \lambda_{\max}(P)x^Tx$$
$$x^TP\varepsilon(t) \le \lambda_{\max}(P)\|x\|\|\varepsilon(t)\|$$
$$\frac{1}{\lambda_{\max}(\Gamma)}\operatorname{tr}(\Theta\Theta^T) \le \operatorname{tr}(\Theta\Gamma^{-1}\Theta^T) \le \frac{1}{\lambda_{\min}(\Gamma)}\operatorname{tr}(\Theta\Theta^T)$$
$$2\operatorname{tr}(\Theta\tilde{\Theta}^T) \le \operatorname{tr}(\Theta\Theta^T) + \operatorname{tr}(\tilde{\Theta}\tilde{\Theta}^T)$$

where $\lambda_{\min}(\cdot)$ and $\lambda_{\max}(\cdot)$ denote, respectively, the minimum and maximum eigenvalues of a positive definite matrix.)

6. (20 points) We denote by $\|x\|$ the absolute value of x if x is a scalar and Euclidean norm of x if x is a vector. For functions of time, the L_p norm is given by

$$\|x\|_p = \left(\int_0^\infty \|x(\tau)\|^p \mathrm{d}\tau\right)^{\frac{1}{p}},$$

for $p \in [1, \infty)$, while

$$\|x\|_\infty = \sup_{t\ge 0} \|x(t)\|.$$

We say that $x \in \mathcal{L}_p$ when $\|x\|_p < \infty$.

   (a) Write down Barbalat's lemma and use it to prove the following corollary.

Corollary 0.1 *If $x \in \mathcal{L}_2 \bigcap \mathcal{L}_\infty$ and $\dot{x} \in \mathcal{L}_\infty$, then $\lim_{t\to\infty} x(t) = 0$.*

(b) Consider the following first-order system

$$\dot{x} = u + d_1(t) + d_2(t)x + d_3(t)x^2,$$

where $d_1(t)$, $d_2(t)$, and $d_3(t)$ are time-varying continuous functions satisfying

$$\max_{t\geq 0} \{\|d_1(t)\|, \|d_2(t)\|, \|d_3(t)\|\} \leq d_{\max},$$

for some unknown positive constant $d_{\max}$. Design a control algorithm to stabilize the system and regulate x towards zero. (Use the corollary in (a) to prove the result.)

7. (20 points) The nonlinear dynamic equations for an m-link robot take the form

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + D\dot{q} + g(q) + d(q,\dot{q},t) = u,$$

where $q \in \mathbb{R}^p$ is the vector of generalized coordinates representing the joint positions, $M(q) \in \mathbb{R}^{p\times p}$ is the symmetric positive-definite inertia matrix, $C(q,\dot{q})\dot{q} \in \mathbb{R}^p$ is the vector of Coriolis and centrifugal torques, $D\dot{q}$ is the vector of viscous damping with D being a constant matrix, $g(q) \in \mathbb{R}^p$ is the gravitational torque, and $u \in \mathbb{R}^p$ is the control torque. The following assumptions hold:

(A1) There exist positive constants $k_{\underline{m}}$ and $k_{\overline{m}}$ such that $0 < k_{\underline{m}} I_p \leq M(q) \leq k_{\overline{m}} I_p$. For $x, y, z \in \mathbb{R}^p$, $0 \leq \|C(x,y)z\| \leq K_C\|y\|\|z\|$.

(A2) $\dot{M}(q) - 2\dot{C}(q,\dot{q})$ is skew symmetric and D is positive semidefinite.

(A3) $g(q) = 0$ has an isolated root at $q = 0$.

(A4) There are parameter uncertainties in $M(q), C(q,\dot{q})$, D and $g(q)$. For $x, y, z \in \mathbb{R}^p$, $M(q)x + C(q,\dot{q})y + Dz + g(q) = Y(q,\dot{q},x,y,z)\Theta$, with Θ being an unknown constant vector.

(a) Assume that $d(q,\dot{q},t) = 0$. With $u = g(q) - K_p(q - q_d) - K_v\dot{q}$, where K_p and K_v are positive diagonal matrices, and q_d is a constant desired position. Show that the point $(q = q_d, \dot{q} = 0)$ is asymptotically stable.

(b) Design an adaptive controller (using sliding mode variables) such that $q(t)$ asymptotically tracks a reference trajectory $q_d(t)$, where $q_d(t)$, $\dot{q}_d(t)$, and $\ddot{q}_d(t)$ are continuous and bounded.