Notebook of NAC

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Contents

1 Introduction

1.1 Why Nonlinear Control?

1. All systems are nonlinear—nonlinear control extends range of possible operation and linear systems are not rich enough to describe many commonly observed phenomena.
2. Some hardware nonlinearities do not have linear approximations.
3. System uncertainty should be treated with nonlinear control.
4. Some nonlinear control designs based on physical principles—provide better intuition and understanding.
5. Better cost/performance than assembly of local linear controls.
6. Empowered by modern computational tools.

Definition 1 *An* ***adaptive controller*** *is a fixed-structure controller with adjustable parameters and a mechanism for automatically adjusting those parameters.*

Why adaptive control?

1. Systems to be controlled have parameter uncertainty.
2. Systems experience unpredictable parameter variations.
3. Unknown disturbance characteristics.

Adaptive Control

1. Adaptive control is superior to robust control in dealing with uncertainties in constant or slow-varying parameters.
2. An adaptive controller improves its performance as adaptation goes on.
3. An adaptive controller requires litter or no priori information about the unknown parameters.

Robust Control

1. Robust control has advantages in dealing with disturbance, quickly varying parameters and unmodeled dynamics.
2. A robust controller attempts to keep consistent performance.
3. A robust controller requires reasonable a prior estimates of the parameter bounds.

1.2 Nonlinear System Representation

Consider a simple nonlinear system whose dynamics is described by the following first-order differential equation of the form

$$\dot{x} = f(t, x, u), x(0) = x_0 \tag{1}$$

where $x \in \mathbb{R}^n$ is the state vector, x_0 is the state at time 0, and $u \in \mathbb{R}^m$ is the control input. And

$$x = \begin{bmatrix} x_1 \\ \dots \\ x_n \end{bmatrix}, u = \begin{bmatrix} u_1 \\ \dots \\ u_m \end{bmatrix}, f \in \begin{bmatrix} f_1 \\ \dots \\ f_n \end{bmatrix}$$

$f : [0, \infty) \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$.

One could expect that corresponding to each input u

1. has at least one solution. (Existence of solutions)
2. has exactly one solution. (Uniqueness of solutions)
3. has exactly one solution for all time. (Existence of solution up to $t \to \infty$)

For linear system $\dot{x} = Ax$, the solution is $x = e^{At}x$, all the above statements are true.

Example 1

1. Lack of existence of solution

   $$\dot{x} = -sign(x) = \begin{cases} -1 & x \geq 0 \\ 1 & x < 0 \end{cases}$$

   Condition argument ?

2. Lack of uniqueness of solutions

   $$\dot{x} = 3x^{2/3}, x(0) = 0$$

(a) $x(t) = t^3$

(b) $x(t) = 0$

$$\forall t^* > 0, x(t) = \begin{cases} 0 & t \le t^* \\ (t - t^*)^3 & t > t^* \end{cases}$$

3. Finite escape time

   Consider the system

$$\dot{x} = 1 + x^2, x(0) = 0$$

   It has a solution $x(t) = \tan(t)$. There is no solution defined outside the interval $\left[0, \frac{\pi}{2}\right)$.

Question 1 What requirements on the existence, uniqueness and extension of the solution of (1)?

Answer 1 $f : [0, \infty) \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is piecewise continuous in t and locally `Lipschitz` in x and u.

Definition 2 *Given two metric spaces (X, d_X) and (Y, d_Y), where d_X denotes the metric on the set X and d_Y is the metric on set Y, a function $f : X \to Y$ is called **Lipschitz continuous** if there exists a real constant $K \ge 0$ such that, for all x_1 and x_2 in X,*

$$d_Y(f(x_1), f(x_2)) \le K d_X(x_1, x_2)$$

where K is referred to as a Lipschitz constant for the function f.

1.3 Autonomous Systems and Equilibrium Points

Consider a system of the form (1) with the input $u(t)$ a fixed function of time and state. Then, the system takes the form

$$\dot{x} = f(t, x) \tag{2}$$

Definition 3 *The system (2) is said to be **autonomous** or time-invariant if $f(t, x)$ is not explicitly dependent on time t, i.e., the system equation is*

$$\dot{x} = f(x) \tag{3}$$

Otherwise, the system is non-autonomous or time-varying.

Definition 4 *A Lyapunov function candidate $V(x) : \mathbb{R}^n \to \mathbb{R}$ defined such that $\lim_{||x|| \to \infty} V(x) \to \infty$ is called **radially unbounded**.*

Definition 5 *x^* is said to be an **equilibrium point** iff $f(t, x^*) \equiv 0, \forall t \ge 0$.*

Remark 1 For autonomous systems, find the equilibrium points corresponding to solve $f(x) = 0$. In the linear case $Ax = 0$

1. An unique solution: A is nonsingular.

2. A continuum of solutions: A is singular.

Definition 6 *An equilibrium point x_0 is* ***isolated*** *if there exists some $\delta > 0$, such that there is no other equilibrium points in the ball $\mathcal{B} = \{x : ||x - x_0|| < \delta\}$.*

Example 2
1. Pendulum Equation
Using Newton's second law of motion, the equation of the motion of the pendulum in the tangential direction

$$ml\ddot{\theta} = -mgsin\theta - kl\dot{\theta}$$

To obtain a state model for the pendulum, let $x_1 = \theta, x_2 = \dot{\theta}$. Then the system equations are

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\frac{g}{l}sinx_1 - \frac{k}{m}x_2 \end{aligned}$$

To find the equilibrium points, we set $\dot{x}_1 = 0, \dot{x}_2 = 0$. We have $x_2 = 0, \sin x_1 = 0$. We can obtain that the equilibrium points are $(k\pi, 0), k = 0, \pm 1, \ldots$

2. Mass-Spring System

$$\begin{aligned} m\ddot{y} &= F - F_{sp} - F_f \\ F_f &= c\dot{y} \end{aligned}$$

Here, $F_{sp} = g(y) = k(1 + a^2y^2)y$. Therefore, for an unforced mass-spring system, $F = 0$, we have

$$m\ddot{y} = -c\dot{y} - k(1 + a^2y^2)y$$

Let $x_1 = y, x_2 = \dot{y}$. The state model is

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\frac{c}{m}x_2 - \frac{k}{m}(1 + a^2x_1^2)x_1 \end{aligned}$$

Equilibrium points: $x_2 = 0, x_1 = \left\{0, \pm\frac{i}{a}\right\}$. ($x_1 = \pm\frac{i}{a}$ is meaningless.)

3. Negative-Resistance Oscillator

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4. Common nonlinearities

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2 Stability for Autonomous Systems

2.1 Concepts of Stability

Definition 7 *The equilibrium point $x = 0$ of* (3) *is*

stable*, if for each $\varepsilon > 0$, $\exists\delta = \delta(\varepsilon) > 0$ such that*

$$\|x(0)\| < \delta \Rightarrow \|x(t)\| < \varepsilon, \forall t \geq 0$$

unstable, *if it is not stable;*

asymptotically stable, *if it is stable and δ can be chosen such that*

$$\|x(0)\| < \delta \Rightarrow \lim_{t\to\infty} x(t) = 0$$

exponentially stable, *if there exist two real constants, $\alpha, \lambda > 0$ such that*

$$\|x(0)\| < \delta \Rightarrow \|x(t)\| \leq \alpha\|x(0)\|e^{-\lambda t}, \forall t \geq 0$$

marginally stable, *if it is stable but not asymptotically stable;*

globally asymptotically/exponentially stable, *if it holds for all initial state.*

Remark 2 Exponentially stable means that $\lim_{t\to\infty} x(t) = 0$ at a rate faster than an exponential function. Generally, exponentially stable $\Rightarrow$ asymptotically stable. For linear systems, exponentially stable $\Leftrightarrow$ asymptotically stable.

Example 3

$$\dot{x} = -(1 + \sin^2 x)x$$

The solution is $x(t) = x(0)e^{-\int_0^t (1+\sin^2 \tau)d\tau} \Rightarrow \|x(t)\| \leq \|x(0)\|e^{-t}$.
$\Rightarrow$ exponentially converge to $x = 0$ with a rate $\lambda = 1$.

Example 4

$$\dot{x} = -x^2, x(0) = 1$$

The solution is $x(t) = \frac{1}{t+1} \Rightarrow$ asymptotically stable but not exponentially stable.

2.2 Positive Definite Functions

Definition 8 *A function $V : D \to R$ is said to be positive semidefinite in D containing the or-gin, if it satisfies the following conditions:*

1. *$V(0) = 0$*
2. *$V(x) \geq 0, \forall xinD$-$\{0\}$*

Further, V is said to be positive definite, if condition 2 is replaced by $V(x) > 0, \forall xinD$-$\{0\}$.
And V is said to be negative definite (semidefinite) in D if $-V$ is positive definite (semidefinite).

Example 5 The simplest and perhaps most important class of positive definite functions is a quadratic form $V(x) : \mathbb{R}^n \to \mathbb{R} = x^T Px, P \in \mathbb{R}^{n\times n}, P = P^T$, we have

$$V is \begin{cases} PD & \Leftrightarrow & x^T Px > 0, \forall x \neq 0 & \Leftrightarrow & \lambda_i(P) > 0 \\ PSD & \Leftrightarrow & x^T Px \geq 0, \forall x \neq 0 & \Leftrightarrow & \lambda_i(P) \geq 0 \end{cases}$$

Example 6 Let $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, then

$$V(x) = \begin{cases} x_1^2 & PSD \\ x_1^2 + x_2^2 & PD \\ x_1^2 + x_2^2 + 1 & NA \\ \frac{x_1^2}{1+x_1^2} + x_2^2 & PD \\ \begin{bmatrix} x_1^2 & x_2^2 \end{bmatrix}^T & NA \\ x^2(4 - x^2) & \begin{cases} PD & x \in (-2, 2) \\ PSD & x \in [-2, 2] \end{cases} \end{cases}$$

Remark 3 Positive definite functions can be seen as an abstraction of the total "energy" stored in the system. All the Lyapunov stability theorems focus on the study of the time derivative of a positive definite function along the trajectories of the system.

2.3 Stability Theorems

Theorem 1 *Let $x = 0$ be an equilibrium point for (3) and $D \subset \mathbb{R}^n$ be a domain containing $x = 0$. Let $V : D \to \mathbb{R}$ be a continuously differentiable function such that*

1. $V(0) = 0$ *and* $V(x) > 0 in D\text{-}\{0\}$
2. $\dot{V}(x) \leq 0 in D$

*Then, $x = 0$ is **stable**.*

*Moreover, if $\dot{V}(x) < 0 in D\text{-}\{0\}$ then $x = 0$ is **asymptotically stable**.*

Remark 4 In other word, the origin is stable if there is a continuously differentiable positive definite function $V(x)$ so that $\dot{V}$ is negative semidefinite, and it is asymptotically stable if $\dot{V}(x)$ is negative definite.

Example 7 Consider the pendulum equation without friction

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -a \sin x_1, a > 0$$

Let us study the stability of the equilibrium point at the origin.

A natural Lyapunov function candidate candidate is the energy function

$$V(x) = \frac{1}{2}x_2^2 + k(1 - \cos x_1), k > 0, x_1 \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

The time derivative of $V(x)$ along the trajectories of the system is given by

$$\begin{aligned} \dot{V} &= x_2\dot{x}_2 + k\dot{x}_1 \sin x_1 \\ &= (k - a)x_2 \sin x_1 \end{aligned}$$

If $k = a$, then $\dot{V} = 0$, NSD. The origin is stable.

Example 8 When considering friction

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -a \sin x_1 - bx_2; a, b > 0$$

Then we have $\dot{V} = -bx_2^2$, NSD.

We can only conclude that the origin is stable. But actually, when $b > 0$, the origin is asymptotically stable. The chosen Lyapunov function candidate fails to show this fact.

Replace the term $\frac{1}{2}x_2^2$ by the more general quadratic form $\frac{1}{2}x^T Px, x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, for some positive definite matrix P.

$$\begin{aligned} V(x) &= \frac{1}{2}x^T Px + a(1 - \cos x_1) \\ &= \frac{1}{2}\begin{bmatrix} x_1 & x_2 \end{bmatrix}\begin{bmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + a(1 - \cos x_1) \end{aligned}$$

where $P_{11} > 0, P_{11}P_{22} - P_{12}^2 > 0$.

The time derivative of V is given

$$\begin{aligned} \dot{V}(x) &= \begin{bmatrix} x_1 & x_2 \end{bmatrix}\begin{bmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{bmatrix}\begin{bmatrix} x_2 \\ -a \sin x_1 - bx_2 \end{bmatrix} + ax_2 \sin x_1 \\ &= \begin{bmatrix} x_1 & x_2 \end{bmatrix}\begin{bmatrix} P_{11}x_2 + P_{12}(-a \sin x_1 - bx_2) \\ P_{12}x_2 + P_{22}(-a \sin x_1 - bx_2) \end{bmatrix} + ax_2 \sin x_1 \\ &= P_{11}x_1x_2 + P_{12}x_1(-a \sin x_1 - bx_2) + P_{12}x_2^2 + P_{22}x_2(-a \sin x_1 - bx_2) + ax_2 \sin x_1 \\ &= (P_{11} - bP_{12})x_1x_2 - aP_{12}x_1 \sin x_1 + (P_{12} - bP_{22})x_2^2 - a(P_{22} - 1)x_2 \sin x_1 \end{aligned}$$

Our objective is to choose P_{11}, P_{12}, P_{22} such that $\dot{V}$ is ND.

Let $P_{22} = 1, P_{11} = bP_{12}$, $P_{12} < bP_{22} \Rightarrow P_{12} < b$, by $P_{11}P_{22} - P_{12}^2 > 0 \Rightarrow P_{12} < b$.

Choose $P_{12} = \frac{b}{2}, P_{11} = \frac{b^2}{2}$, we have $\dot{V} = -\frac{ab}{2}x_1 \sin x_1 - \frac{b}{2}x_2^2$. For $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, $\dot{V}$ is ND.

Taking $D = \left\{x \in \mathbb{R}^n : |x_1| < \frac{\pi}{2}\right\}$, we see that V is PD and $\dot{V}$ is ND over D.

Thus, we conclude that the origin is asymptotically stable.

Remark 5 This example emphasizes an important feature of Lyapunov stability theory, namely, the conditions are only sufficient.

Example 9 Let us study the following nonlinear system

$$\dot{x}_1 = x_1(x_1^2 + x_2^2 - 2) - 4x_1x_2^2$$
$$\dot{x}_1 = x_2(x_1^2 + x_2^2 - 2) + 4x_1^2x_2$$

with the equilibrium point at the origin. Given the positive definite function

$$V = \frac{1}{2}(x_1^2 + x_2^2)$$

Its derivative along the system trajectories is

$$\begin{aligned}\dot{V} &= x_1\dot{x}_1 + x_2\dot{x}_2 \\ &= x_1(x_1(x_1^2+x_2^2-2) - 4x_1x_2^2) + x_2(x_2(x_1^2+x_2^2-2) + 4x_1^2x_2) \\ &= (x_1^2+x_2^2)(x_1^2+x_2^2-2)\end{aligned}$$

In $\mathcal{B}_r = \{x : x_1^2 + x_2^2 < 2\}$, $\dot{V}$ is ND.

Therefore, the origin is **locally asymptotically stable**.

Theorem 2 *(Barbashin-Krasovskii)*

Let $x = 0$ be an equilibrium point of (3). Let $V : \mathbb{R}^n \to \mathbb{R}$ be a continuously differentiable function such that

1. $V(0) = 0$ *and* $V(x) > 0, \forall x \neq 0$
2. $\dot{V}(x) < 0, \forall x \neq 0$
3. $\|x\| \to \infty \Rightarrow V(x) \to \infty$

*then $x = 0$ is **globally asymptotically stable**.*

Remark 6 The reason for radial unboundedness condition is to assure that the contour curves $V(x)$ correspond to closed curves. If the curves are not closed, it is possible for state trajectories to drift away from the equilibrium point, even though the state keeps going contours corresponding to smaller and smaller $V(x)$. For example, $V(x) = \frac{x_1^2}{1+x_1^2} + x_2^2$.

2.4 Invariance Principle

2.4.1 LaSalle's Theorem

Asymptotically stability is always more desirable than stability. However, it is often the case that a Lyapunov function candidate fails to identify an asymptotically stable equilibrium by having $\dot{V}(x)$ negative semidefinite. An extension of Lyapunov theorem due to LaSalle studies this problem in detail.

Definition 9 *A set M is said to be an **invariant set** with respect to (3) if*

$$x(0) \in M \Rightarrow x(t) \in M, \forall t \geq 0.$$

Example 10 Some common invariant sets:

1. Equilibrium point.
2. Limit cycle.
3. The whole space $\mathbb{R}^n$.
4. $\Omega_c = \{x \in \mathbb{R}^n : V(x) \leq c\}$ with $\dot{V}(x) \leq 0$.

Theorem 3 ***(LaSalle's theorem)***

Let $\Omega \subset D$ be a compact set that is invariant with respect to (3). Let $V : D \to \mathbb{R}$ be a continuously differentiable function such that $\dot{V}(x) \le 0$ in Ω. Let E be the set of all points in Ω where $\dot{V}(x) = 0$. Let M be the largest invariant set in E. Then every solution starting in Ω approaches M as $t \to \infty$.

Example 11 The pendulum model. Consider $V(x) = \frac{1}{2}x_2^2 + a(1 - \cos x_1)$.

1. Without friction: $\begin{cases} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -a \sin x_1; a > 0 \end{cases}$, $\dot{V} = 0$, $\Omega = E = M$.
2. With friction: $\begin{cases} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -a \sin x_1 - bx_2; a, b > 0 \end{cases}$, $\dot{V} = -bx_2^2$, $E = \{x \in \mathcal{D} : x_2 = 0\}$, $M = \{(0, 0)\}$.

Remark 7 Besides, often yielding conclusion on asymptotic stability when $\dot{V}$ is only semidefinite, the invariant set theorems also allow us to extend the concept of Lyapunov function so as to describe convergence to dynamic behaviors more general that equilibrium, e.g. convergence to a limit cycle.

Corollary 1

Let $x = 0$ be an equilibrium point for $\dot{x} = f(x)$.

Let $V : \mathcal{D} \to \mathbb{R}$ be a continuously differentiable positive definite function, such that $\dot{V} \le 0$ in $\mathcal{D}$.

Let $\mathcal{S} = \{x \in \mathcal{D} : \dot{V}(x) = 0\}$ and suppose that no solution can stay identically in $\mathcal{S}$ other than the trivial solution $x(t) \equiv 0$. Then the origin is ***asymptotically stable****. Furthermore, if $\mathcal{D} \in \mathbb{R}^n$ and $V(x)$ is radially unbounded, the origin is* ***globally asymptotically stable****.*

Remark 8 When $\dot{V}(x)$ is negative definite, $\mathcal{S} = \{0\}$, the above corollary conclusion coincides with the Lyapunov theorem.

Example 12

For $x_1 \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, $\begin{cases} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -a \sin x_1 - bx_2; a, b > 0 \end{cases}$, $V(x) = \frac{1}{2}x_2^2 + a(1 - \cos x_1)$, we have $\dot{V} = -bx_2^2$.

Then $\mathcal{S} = \{x \in \mathcal{D} : \dot{V}(x) = 0\} = \{x \in \mathcal{D} : x_2 = 0\}$.

By $\dot{V}(x) \equiv 0 \Rightarrow x_2 \equiv 0 \Rightarrow \dot{x}_2 \equiv 0 \Rightarrow x_1 \equiv 0$.

Example 13 Consider the first-order system

$$\dot{y} = ay + u$$

together with the adaptive control law

$$u = -ky, \dot{k} = \gamma y^2, \gamma > 0.$$

Taking $x_1 = y, x_2 = k$, we can obtain

$$\begin{cases} \dot{x}_1 = ax_1 - x_1 x_2 \\ \dot{x}_2 = \gamma x_1^2 \end{cases}$$

The line $x_1 = 0$ is an equilibrium set. We want t show that the trajectories approach this equilibrium set as $t \to \infty$, which means that the adaptive controller regulates y to zero.

Consider the following Lyapunov function candidate

$$V(x) = \frac{1}{2}x_1^2 + \frac{1}{2\gamma}(x_2 - k_0)^2$$

where $k_0 = 0$ is a constant.

The time derivative of V along the trajectories is given by

$$\begin{aligned}\dot{V} &= x_1\dot{x}_1 + \frac{1}{\gamma}(x_2 - k_0)\dot{x}_2 \\ &= x_1(ax_1 - x_1x_2) + (x_2 - k_0)x_1^2 \\ &= (a - k_0)x_1^2\end{aligned}$$

Design $k_0 > a$, then $\dot{V} \leq 0$.

Define the set $\Omega_c = \{x \in \mathbb{R}^n : V(x) \leq c\}$, which is an invariant set. The set E is given by $E = \{x \in \Omega_c : x_1 = 0\}$. Therefore, in this example, $M = E$. From LaSalle's Theorem, we can conclude that $\lim_{t\to\infty} x_1(t) = 0$. Moreover, since $V(x)$ is radially unbounded, the conclusion is global.

2.4.2 Limit Cycle

Definition 10 *An isolated periodic/closed orbit is called a **limit cycle**.*

Stable: all trajectories in the vicinity of the limit cycle converge to it as $t \to \infty$.

Unstable: all trajectories in the vicinity of the limit cycle diverge from it as $t \to \infty$.

Semi-Stable: some trajectories in the vicinity of the limit cycle converge to it while others diverge from it as $t \to \infty$.

Remark 9 Consider the system defined by

$$\begin{aligned}\dot{x}_1 &= x_2 + x_1(\beta^2 - x_1^2 - x_2^2) \\ \dot{x}_2 &= -x_1 + x_2(\beta^2 - x_1^2 - x_2^2)\end{aligned}$$

The origin $x = (0, 0)$ is an equilibrium point.

For $x_1^2 + x_2^2 = \beta^2$, the system is reduced to $\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -x_1 \end{cases}$, its solution is $x_1(0) = x_{10}, x_1(t) = x_{10}\cos t, x_2(0) = \dot{x}_1(0) = 0, x_2(t) = -x_{10}\sin t$.

Thus, the set of points defined by the circle $x_1^2 + x_2^2 = \beta^2$ constitutes an invariant set (limit cycle).

Assume $r = \sqrt{x_1^2 + x_2^2}, \theta = \arctan\frac{x_2}{x_1}$, we have

$$\begin{aligned}\dot{r} &= r(\beta^2 - r^2) \\ \dot{\theta} &= -1\end{aligned}$$

By $\left\{ \begin{array}{l} r < \beta, \dot{r} > 0 \Rightarrow r \to \beta \\ r > \beta, \dot{r} < 0 \Rightarrow r \to \beta \end{array} \right. \Rightarrow$ stable limit cycle.

We now investigate the stability of this limit cycle using LaSalle's theorem.

To end this, consider the following Lyapunov function

$$V(x) = \frac{1}{4}(x_1^2 + x_2^2 - \beta^2)^2$$

which represents a measure of the "distance" to the limit cycle.

Clearly, $V(x) \geq 0$.

$$\begin{aligned} \dot{V}(x) &= \frac{1}{2}(x_1^2 + x_2^2 - \beta^2)(2x_1\dot{x}_1 + 2x_2\dot{x}_2) \\ &= -(x_1^2 + x_2^2 - \beta^2)^2(x_1^2 + x_2^2) \\ &\leq 0 \end{aligned}$$

Step 1: Given any real number $c > \frac{1}{4}\beta^4$, define $\Omega_c = \{x \in \mathbb{R}^2 : V(x) \leq c\}$. Since $\dot{V} \leq 0$, Ω_c is an invariant set.

Step 2: Find $E = \{x \in \Omega_c : \dot{V}(x) = 0\}$. Clearly, $E = \{(0,0)\} \cup \{x : x_1^2 + x_2^2 = \beta^2\}$.

Step3: Find M, the largest invariant set in E. Clearly, $M = E$.

Thus, choosing c such that $0 < c < \frac{1}{4}\beta^4$, $\Omega_c = \{x \in \mathbb{R}^2 : V(x) \leq c\}$ includes the limit cycle but not the origin. The application of LaSalle's theorem with $c \in \left(0, \frac{1}{4}\beta^4\right)$ shows that every motion starting in Ω_c converges to the limit cycle, and therefore the limit cycle is stable.

Further, assume $c = \frac{1}{4}\beta^4 - \varepsilon \downarrow$. The origin is unstable.

2.5 Linear System and Linearization

$$\dot{x} = Ax, A \in \mathbb{R}^{n \times n}$$

Remark 10 Some reasons for investigating the stability of LTI systems via Lyapunov method.

1. The Lyapunov analysis permits studying linear and nonlinear systems under the same framework, where LTI is a special case.
2. We will introduce a very useful class of Lyapunov functions that appears frequently in the literature.
3. We will study the stability of nonlinear systems via linearization of the state equation and try to get some insight into the limitations associated with this process.

Consider a quadratic Lyapunov function candidate

$$V(x) = x^T P x$$

where $P \in \mathbb{R}^{n \times n}$ is PD.

$$\dot{V}(x) = x^T(PA + A^T P)x \triangleq -x^T Q x$$

where Q is a symmetric matrix defined by

$$PA + A^T P = -Q \tag{4}$$

If Q is PD, the origin is asymptotically stable.

Definition 11 *Matrix M is Hurwitz iff $Re[\lambda_i(M)] < 0$.*

Theorem 4 *Matrix A is Hurwitz, if and only if for any given positive definite symmetric matrix Q, there exists a unique positive definite symmetric matrix P satisfies the Lyapunov equation (4).*

Theorem 5 *For $\dot{x} = f(x)$, the Jacobian matrix is*

$$A = \left.\frac{\partial f}{\partial x}\right|_{x=0} = \left.\begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}\right|_{x=0}$$

If $\forall i, Re[\lambda_i(M)] < 0$, the origin of $\dot{x} = f(x)$ is asymptotically stable.

If $\exists i, Re[\lambda_i(M)] > 0$, the origin of $\dot{x} = f(x)$ is unstable.

If $\exists i, Re[\lambda_i(M)] < 0$,and $\exists j, Re[\lambda_j(M)] = 0$,no conclusion.

Example 14 Consider the pendulum equation

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -a \sin x_1 - b x_2; a, b > 0 \end{aligned}$$

Check the stability for two equilibrium points $(0, 0)$ and $(\pi, 0)$.

$$A_1 = \left.\frac{\partial f}{\partial x}\right|_{x=(0,0)} = \left.\begin{bmatrix} 0 & 1 \\ -acosx_1 & -b \end{bmatrix}\right|_{x=(0,0)} = \begin{bmatrix} 0 & 1 \\ -a & -b \end{bmatrix}$$

The characteristic equation is $\lambda^2 + b\lambda + a = 0 \Rightarrow$ stable.

$$A_2 = \left.\frac{\partial f}{\partial x}\right|_{x=(\pi,0)} = \left.\begin{bmatrix} 0 & 1 \\ -acosx_1 & -b \end{bmatrix}\right|_{x=(\pi,0)} = \begin{bmatrix} 0 & 1 \\ a & -b \end{bmatrix}$$

The characteristic equation is $\lambda^2 + b\lambda - a = 0 \Rightarrow$ unstable.

Note 1 For $ax^2 + bx + c = 0$, its solution is $x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, then we have

$$x_1 + x_2 = \frac{-b}{a}, \quad x_1 \times x_2 = \frac{c}{a}$$

For $a = 1$, if $c > 0$, then $\text{Re}(x_{1,2})$ have the same sign.

Further, if $b > 0$, then $x_{1,2}$ both have negative real parts.

Otherwise, if $a = 1, c < 0$, then $x_{1,2}$ are real numbers with different sign.

3 Stability for Non-autonomous Systems

3.1 Comparison Functions

Definition 12 *A continuous function $\alpha : [0, a) \to [0, \infty)$ is said to belong to class $\mathcal{K}$ if it is strictly increasing and $\alpha(0) = 0$. It is said to belong to class $\mathcal{K}_\infty$ if $a = \infty$ and $\alpha(r) \to \infty$ as $r \to \infty$.*

Definition 13 *A continuous function $\beta : [0, a) \times [0, \infty) \to [0, \infty)$ is said to belong to class $\mathcal{KL}$ if, for each fixed s, the mapping $\beta(r, s)$ belongs to class $\mathcal{K}$ with respect to r and, for each fixed r, the mapping $\beta(r, s)$ is decreasing with respect to s and $\beta(r, s) \to 0$ as $s \to \infty$.*

Example 15

1. $\alpha(r) = \arctan(r), \alpha'(r) = \frac{1}{r^2+1} \geq 0, \alpha(0) = 0, \lim_{r\to\infty} \alpha(r) = \frac{\pi}{2}$. So $\alpha(r)$ belongs to class $\mathcal{K}$, but not belong to class $\mathcal{K}_\infty$.
2. $\alpha(r) = r^2, \alpha'(r) = 2r \geq 0, \alpha(0) = 0, \lim_{r\to\infty} \alpha(r) = \infty$. So $\alpha(r)$ belongs to class $\mathcal{K}_\infty$.
3. $\alpha(r) = \min\{r, r^2\}$ belongs to class $\mathcal{K}_\infty$.
4. $\beta(r, s) = r^2 e^{-s}$ belongs to class $\mathcal{KL}$.

Remark 11 Class $\mathcal{K}$ and $\mathcal{KL}$ functions enter into Lyapunov analysis through the next lemmas.

Lemma 1 *$V : D \to \mathbb{R}$ is positive definite iff there exist $\mathcal{K}$ functions such that*

$$\alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|), \forall x \in B_r \subset D.$$

Moreover, if $D = \mathbb{R}^n$ and $V(.)$ is radially unbounded, α_1 and α_2 should be chosen in the class $\mathcal{K}_\infty$.

Example 16 $V(x) = x^T P x$, P is positive definite, we have

$$\lambda_{\min}(P)\|x\|^2 \leq V(x) \leq \lambda_{\max}(P)\|x\|^2.$$

Lemma 2 *The equilibrium point $x = 0$ of $\dot{x} = f(x)$ is **stable** iff there exists a class $\mathcal{K}$ function $\alpha(.)$ and a constant δ such that*

$$\|x(0)\| \leq \delta \quad \Rightarrow \quad \|x(t)\| \leq \alpha(\|x(0)\|), \forall t \geq 0.$$

Lemma 3 *The equilibrium point $x = 0$ of $\dot{x} = f(x)$ is **asymptotically stable** iff there exists a class $\mathcal{KL}$ function $\beta(., .)$ and a constant δ such that*

$$\|x(0)\| \leq \delta \quad \Rightarrow \quad \|x(t)\| \leq \beta(\|x(0)\|, t), \forall t \geq 0.$$

3.2 Stability of Non-autonomous systems

Consider the non-autonomous system

$$\dot{x} = f(x,t) \tag{5}$$

where $f : [0,\infty) \times D \to \mathbb{R}^n$ is piecewise continuous in t and locally Lipschitz in x on $[0,\infty) \times D$ and $D \subset \mathbb{R}^n$ is a domain containing the origin. The origin is an equilibrium point for (5), i.e., $f(0,t) \equiv 0, \forall t \geq 0$.

Definition 14 *The equilibrium point $x = 0$ of (5) is*

1. stable, *if for each $\varepsilon > 0$, there is $\delta = \delta(\varepsilon, t_0)$ such that*

$$\|x(t_0)\| < \delta \quad \Rightarrow \quad \|x(t)\| < \varepsilon, \forall t \geq t_0 \geq 0. \tag{6}$$

2. unstable, *if it is not stable.*

3. uniformly stable, *if for each $\varepsilon > 0$, there is $\delta = \delta(\varepsilon) > 0$, independent of t_0, such that (6) is satisfied.*

4. asymptotically stable, *if it is stable and there is a positive constant $c = c(t_0)$ such that*

$$\forall \|x(t_0)\| < c, \lim_{t\to\infty} x(t) \to 0. \tag{7}$$

5. uniformly asymptotically stable, *if it is uniformly stable and there is positive constant c, independent of t_0, such that (7) is satisfied.*

6. globally uniformly asymptotically stable, *if it is uniformly asymptotically stable and c can be chosen as ∞.*

Remark 12 The new element here is that, while the solution of an autonomous system depends only on $(t - t_0)$, the solution of a non-autonomous system may depend on both t and t_0.

The next lemma gives equivalent, more transparent, definitions of uniform stability and uniform asymptotic stability by using class $\mathcal{K}$ and class $\mathcal{KL}$ functions.

Lemma 4 *The equilibrium point $x = 0$ of (5) is*

1. uniformly stable, *iff there exist a class $\mathcal{K}$ function α and a positive constant c, independent of t_0, such that*

$$\|x(t)\| \leq \alpha(\|x(t_0)\|), \forall t \geq t_0 \geq 0, \forall \|x(t_0)\| < c.$$

2. uniformly asymptotically stable, *iff there exist a class $\mathcal{KL}$ function β and a positive constant c, independent of t_0, such that*

$$\|x(t)\| \leq \beta(\|x(t_0)\|, t - t_0), \forall t \geq t_0 \geq 0, \forall \|x(t_0)\| < c. \tag{8}$$

3. globally uniformly asymptotically stable, *iff inequality (8) is satisfied for any initial state $x(t_0)$.*

Example 17 Consider the following system

$$\dot{x} = (6t \sin t - 2t)x$$

Its solution is

$$\begin{aligned} x(t) &= x(t_0) \exp\left\{ \int_{t_0}^{t} (6\tau \sin\tau - 2\tau) d\tau \right\} \\ &= x(t_0) \exp\{6 \sin t - 6t \cos t - t^2 - 6 \sin t_0 + 6t_0 \cos t_0 + t_0^2\} \end{aligned}$$

Note $6 \sin t - 6t \cos t - t^2 \leq 6 + 6t - t^2 = -(t-3)^2 + 15 \leq 15$.

Choose $c(t_0) = \exp(15 - 6 \sin t_0 + 6t_0 \cos t_0 + t_0^2)$.

Clearly, we have $\|x(t)\| \leq \|x(t_0)\| \cdot c(t_0), \forall t \geq t_0 \geq 0$.

For any $\varepsilon > 0$, the choice $\delta = \frac{\varepsilon}{c(t_0)}$ shows that

$$\|x(t_0)\| < \delta = \frac{\varepsilon}{c(t_0)} \quad \Rightarrow \quad \|x(t)\| \leq \|x(t_0)\| \cdot c(t_0) < \frac{\varepsilon}{c(t_0)} \cdot c(t_0) = \varepsilon$$

which implies that the origin is stable.

We study the sensitivity of the system solutions due to the changes in t_0.

Let $t_0 = 2k\pi$. We can examine $x(t)$ at $t = t_0 + \pi = (2k+1)\pi$.

$$\begin{aligned} x(t_0 + \pi) &= x(t_0) \exp\{6 \sin[(2k+1)\pi] - 6(2k+1)\pi \cos[(2k+1)\pi] - [(2k+1)\pi]^2 \\ &\quad - 6 \sin(2k\pi) - 6(2k\pi) \cos(2k\pi) - (2k\pi)^2\} \\ &= x(t_0) \exp\{6(2k+1)\pi - [(2k+1)\pi]^2 - 12k\pi - (2k\pi)^2\} \\ &= x(t_0) e^{(4k+1)(6-\pi)\pi} \end{aligned}$$

$$\frac{x(t_0 + \pi)}{x(t_0)} = e^{(4k+1)(6-\pi)\pi} \to \infty, ask \to \infty$$

Therefore, given any $\varepsilon > 0$, there is no $\delta(\varepsilon)$ independent of t_0 that could satisfy the uniform stability definition.

3.3 Time-Dependent Positive Definite Functions

Definition 15 *Let $V(x,t) = D \times [0, \infty) \to \mathbb{R}$ is a continuously differential function.*

*$V(x,t)$ is said to be **positive semidefinite**, if*

1. $V(0,t) \equiv 0, \forall t \geq 0$
2. $V(x,t) \geq 0, \forall t \geq 0, x \in D$

*$V(x,t)$ is said to be **positive definite**, if*

1. $V(0,t) \equiv 0, \forall t \geq 0$
2. *there exists a positive definite function $\omega_1(x)$ such that*

$$\omega_1(x) \leq V(x,t), \forall t \geq 0, x \in D$$

*$V(x,t)$ is said to be **decrescent** in D if there exists a positive definite function $\omega_2(x)$ such that*

$$V(x,t) \le \omega_2(x)$$

*$V(x,t)$ is **radially unbounded** if $V(x,t) \to \infty as \|x\| \to \infty$.*

Example 18 $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

1. $V(x,t) = (1+t^2)(x_1^2 + x_2^2)$: $\omega_1(x) = x_1^2 + x_2^2$, PD, not decrescent.
2. $V(x,t) = (1+\sin^2 t)(x_1^2 + x_2^2)$: $\omega_1(x) = x_1^2 + x_2^2, \omega_2 = 2(x_1^2 + x_2^2)$, PD and decrescent.
3. $V(x,t) = \frac{1+t^2}{2+t^2}(x_1^2 + x_2^2)$: $\omega_1(x) = \frac{1}{2}(x_1^2 + x_2^2), \omega_2 = x_1^2 + x_2^2$, PD and decrescent.

Theorem 6 *Let $x = 0$ be an equilibrium point for (5) and $D \subset \mathbb{R}^n$ be a domain containing the origin. Let $V : D \times [0, \infty) \to \mathbb{R}$ be continuously differentiable function such that*

1. *$V(x,t)$ is PD; $(\omega_1(x) \le V(x,t), \omega_1 is PD)$*
2. *$\dot{V}(x,t) = \frac{\partial V}{\partial x} f(x,t) + \frac{\partial V}{\partial t} \le 0$,*

*then $x = 0$ is **stable**.*

In addition, if $V(x,t)$ is also `decrescent`, i.e.,

$$\omega_1(x) \le V(x,t) \le \omega_2(x) \qquad (\omega_1, \omega_2 are PD)$$

*then $x = 0$ is **uniformly stable**.*

If the second condition is strengthened to

$$\dot{V}(x,t) = \frac{\partial V}{\partial x} f(x,t) + \frac{\partial V}{\partial t} \le -\omega_3(x) \qquad (\omega_3 is PD)$$

*then $x = 0$ is **uniformly asymptotically stable**.*

*Finally, if $D \subset \mathbb{R}^n$ and $\omega_1(x)$ is `radially unbounded`, then $x = 0$ is **globally uniformly asymptotically stable**.*

*If $k_1\|x\|^a \le V(x,t) \le k_2\|x\|^a, \dot{V} \le -k_3\|x\|^a; k_1, k_2, k_3, a > 0$, then $x = 0$ is **exponentially stable**.*

Lemma 5 *$V(t,x) \le \omega_2(x)$ is necessary to **asymptotically stable**.*

Example 19 Consider the following system

$$\dot{x}_1 = -x_1 - e^{-2t} x_2$$
$$\dot{x}_2 = x_1 - x_2$$

We consider the following Lyapunov function candidate

$$V(x,t) = x_1^2 + (1 + e^{-2t}) x_2^2$$

It can be easily seen that

$$\|x\|^2 = \omega_1(x) = x_1^2 + x_2^2 \le V(x,t) \le x_1^2 + 2x_2^2 = \omega_2(x) \le 2\|x\|^2$$

Hence, $V(x,t)$ is positive definite, decrescent, and radially unbounded.

The derivative of V along the trajectories of the system is given by

$$\begin{aligned}\dot{V}(x,t) &= 2x_1\dot{x}_1 + (1+e^{-2t})2x_2\dot{x}_2 - 2e^{-2t}x_2^2 \\ &= -(x_1^2+x_2^2) - (x_1-x_2)^2 - 4e^{-2t}x_2^2 \\ &\le -(x_1^2+x_2^2) \\ &= -\omega_3(x) = -\|x\|^2\end{aligned}$$

Hence, the origin is globally uniformly exponentially stable.

3.4 Linear Time-Varying Systems

Consider the system

$$\dot{x} = A(t)x \tag{9}$$

The solution of (9) is given by

$$x(t) = \Phi(t,t_0)x(t_0)$$

where $\Phi(t,t_0)$ is the state transition matrix.

Theorem 7 *$x = 0$ of (9) is* ***uniformly exponentially stable*** *iff the state transition matrix satisfies the following inequality*

$$\|\Phi(t,t_0)\| \le ke^{-\lambda(t-t_0)}, \forall t \ge t_0 \ge 0$$

for some positive constants k and λ.

Remark 13 For linear time-varying systems, its stability cannot be characterized by the location of the eigenvalues of the matrix $A(t)$.

Example 20 Consider a second-order linear system with

$$A(t) = \begin{bmatrix} -1+1.5\cos^2 t & 1-1.5\sin t\cdot\cos t \\ -1-1.5\sin t\cdot\cos t & -1+1.5\sin^2 t\end{bmatrix}$$

$|\lambda I - A| = \lambda^2 + 0.5\lambda + 0.5 = 0$ has two eigenvalues with negative real parts.

Actually, $\Phi(t,0) = \begin{bmatrix} e^{0.5t}\cos t & e^{-t}\sin t \\ e^{0.5t}\sin t & e^{-t}\cos t\end{bmatrix}$.

Corollary 2 *The system* (9) *is* ***uniformly asymptotically stable*** *if $\exists\lambda > 0$ such that*

$$\lambda_i(A(t) + A^T(t)) \le -\lambda, \forall t \ge 0, i = 1, \dots, n$$

Proof *Let $V = x^T x$, then*

$$\begin{aligned}\dot{V} &= x^T(A(t) + A^T(t))x \\ &\leq -\lambda x^T x \\ &= -\lambda V\end{aligned}$$

□

Corollary 3 *Assume that there exists $P(t) \in \mathbb{R}^{n\times n}$ which is continuously differentiable and symmetric, and there exist $0 < c_1 < c_2 < \infty$ such that*

$$c_1 I \leq P(t) \leq c_2 I, \forall t \geq 0$$

Further assume for some $Q(t) \in \mathbb{R}^{n\times n}$, continuous and symmetric such that

$$Q(t) \geq c_3 I > 0$$

$$\dot{P}(t) + P(t)A(t) + A^T(t)P(t) = -Q(t)$$

Then (9) *is globally uniformly asymptotically stable.*
Proof *Let $V(x,t) = x^T P(t)x \Rightarrow \omega_1(x) = c_1 x^T x \leq V(x,t) \leq c_2 x^T x = \omega_2(x)$. Its derivative is*

$$\begin{aligned}\dot{V} &= x^T P(t)A(t)x + x^T A^T(t)P(t)x + x^T \dot{P}(t)x \\ &= -x^T Q(t)x \\ &\leq -c_3 x^T x = \omega_3(x)\end{aligned}$$

□

Corollary 4 *Assume that at any time $t \geq 0$, all the eigenvalues of $A(t)$ have negative real parts. In addition, if $A(t)$ is bounded and $\int_0^\infty A^T(t)A(t)dt < \infty$, then $x = 0$ is globally asymptotically stable.*
Proof *Hint: linearization.* □

3.5 Barbalat's Lemma

How about the asymptotical stability when $\dot{V}(x,t)$ is only negative semidefinite? For autonomous system, LaSalle's invariance theorem can be used. But it is not valid for non-autonomous system.

Barbalat's Lemma is purely mathematical result concerning the asymptotical properties of functions and their derivatives.

Example 21 Given a differentiable f at time t, f converges $\not\Leftrightarrow \dot{f} \to 0$.

1. $f = e^{-t}\sin(e^{2t}) \to 0, \dot{f} = -e^{-t}\sin(e^{2t}) + e^t\cos(e^{2t}) \cdot 2e^{2t} \to \infty$
2. $f = \ln t \to \infty, \dot{f} = \frac{1}{t} \to 0$.

Definition 16 *A function $f : \mathbb{R} \to \mathbb{R}$ is said to be* ***uniformly continuous*** *if $\forall \varepsilon > 0, \exists \delta = \delta(\varepsilon) > 0, \forall |t_2 - t_1| \leq \delta \Rightarrow |f(t_2) - f(t_1)| \leq \varepsilon$.*

Lemma 6 *A continuous function is* `uniformly continuous` *on a closed set.*

Lemma 7 *A sufficient condition for a differentiable function to be* `uniformly continuous` *is that its derivative is bounded.*

Lemma 8 *(**Barbalat's lemma**) If the differentiable function $f(t)$ has a* `finite limit` *as $t \to \infty$, and $\dot{f}(t)$ is uniformly continuous, then $\lim_{t\to\infty} \dot{f}(t) = 0$.*

Lemma 9 *If f is lower bounded and $\dot{f} \leq 0$, it converges to a* `finite limit`*.*

Note 2 $\lim_{t\to\infty} \dot{f}(t) = 0$ if

1. f has a finite limit;
2. $\dot{f}$ is uniformly continuous. ($\ddot{f}$ bounded)

Lemma 10 *(**Lyapunov-Like lemma**) For a scalar function $V(x,t)$, $\lim_{t\to\infty} \dot{V}(x,t) = 0$ if the following conditions are satisfied*

1. *$V(x,t)$ has a finite limit :* $\begin{cases} V(x,t) \text{ is lower bounded} \\ \dot{V}(x,t) \text{ is negative semidefinite } (\dot{V} \leq 0) \end{cases}$
2. *$\dot{V}(x,t)$ is uniformly continuous ($\Leftarrow \ddot{V} is bounded$)*

Example 22 In adaptive control, we will often encounter the following non-autonomous system

$$\dot{e} = -e + \theta\varphi(t)$$
$$\dot{\theta} = -e\varphi(t)$$

where $\varphi(t)$ is a bounded continuous function, e and θ are two states of the closed-loop system, representing the tracking error and parameter error.

Consider the quadratic scalar function $V = \frac{1}{2}e^2 + \frac{1}{2}\theta^2 \geq 0$.

Clearly, V is lower bounded.

Its derivative is

$$\dot{V} = -e^2 \leq 0$$

Check $\ddot{V} = -2e\dot{e} = -2e(-e + \theta\varphi(t))$.

Note that $\dot{V} \leq 0$ which implies that $V(t) \leq V(0)$. Therefore, e, θ are bounded.

With $\varphi(t)$ being bounded, we get that $\ddot{V}$ is also bounded.

Then from Barbalat's lemma, we can conclude that $\lim_{t\to\infty} \dot{V}(t) = 0$, i.e., $\lim_{t\to\infty} e(t) = 0$.

Theorem 8 *(**LaSalle-Yoshizawa**) Let $x = 0$ be an equilibrium point of (5) and suppose that f is locally Lipschitz in x and uniformly in t. Let V be a continuously differentiable function such that $\forall t \geq 0, x \in R^n$,*

$$\gamma_1(\|x\|) \leq V(x,t) \leq \gamma_2(\|x\|)$$

$$\dot{V} = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(x,t) \leq -\omega(x) \leq 0$$

where γ_1 and γ_2 are class $\mathcal{K}_\infty$ functions and ω is a continuous function.

Then all solutions of (5) satisfy $\lim_{t\to\infty} \omega(x(t)) = 0$.

*In addition, if $\omega(x)$ is positive definite, $x = 0$ is **globally uniformly asymptotically stable**.*

Proof *Note $\omega(x) \geq 0 \Rightarrow \int_0^t \omega(x(\tau))d\tau \uparrow$.*

Since $\dot{V} \leq -\omega(x), V(t) - V(0) \leq -\int_0^t \omega(x(\tau))d\tau \Rightarrow \int_0^t \omega(x(\tau))d\tau \leq V(0) - V(t) \leq V(0)$

Thus $\int_0^t \omega(x(\tau))d\tau$ has a ***`finite limit`****.*

Since $\dot{V} \leq 0, \gamma_1(\|x\|) \leq V(x,t) \leq \gamma_2(\|x\|) \Rightarrow V$ is bounded and x is bounded, i.e., $\|x\| \leq B_r$.

Note that a continuous function is uniformly continuous on a closed set.

Thus $\omega(x)$ is uniformly continuous in x.

By f is locally Lipschitz, $\forall t_2 > t_1 \geq 0$, , one has

$$\begin{aligned} |x(t_2) - x(t_1)| &= \left| \int_{t_1}^{t_2} f(x(\tau), \tau) d\tau \right| \\ &\leq L \left| \int_{t_1}^{t_2} \|x(\tau)\| d\tau \right| \\ &\leq Lr|t_2 - t_1| \end{aligned}$$

Thus $x(t)$ is uniformly continuous in t.

Thus $\omega(x)$ is uniformly continuous in t.

So $\lim_{t\to\infty} \omega(x(t)) = 0$. □

3.6 Boundedness and Ultimate Boundedness

The concept of stability in the sense of Lyapunov are formulated with respect to an equilibrium point. Often, systems are designed to operate in the presence of disturbance and other uncertainties. In this subsection, we show that Lyapunov analysis can be used to show the boundedness of the solution, even when there is no equilibrium point at the origin.

Consider the scalar non-autos

$$\dot{x} = -x + \delta \sin t, x(t_0) = a, a > \delta > 0$$

which has no equilibrium point and whose solution is given by

$$x(t) = e^{-(t-t_0)} x(t_0) + \delta \int_{t_0}^{t} e^{-(t-\tau)} \sin \tau d\tau$$

The solution satisfies

$$\begin{aligned} \|x(t)\| &\leq e^{-(t-t_0)} a + \delta \int_{t_0}^{t} e^{-(t-\tau)} d\tau \\ &= (a - \delta) e^{-(t-t_0)} + \delta \\ &\leq a \end{aligned}$$

which shows that the solution is uniformly bounded. Further, for any number $b \in (\delta, a)$, it can be easily seen that

$$\|x(t)\| \leq b, \forall t \geq t_0 + \ln\left(\frac{a-\delta}{b-\delta}\right)$$

The bound b, which again is independent of t_0, gives a better estimate of the solution after a transient period has passed. In this case, the solution is said to be **uniformly ultimately bounded** (UUB) and b is called the ultimate bound.

This can be also done via Lyapunov analysis without using the explicit solution of the state equation. Starting with $V(x) = \frac{1}{2}x^2$, its derivative along the trajectories of the system is

$$\begin{aligned}\dot{V} &= -x^2 + \delta x \sin t \\ &\leq -x^2 + \delta|x| \\ &= -|x|(|x| - \delta)\end{aligned}$$

$\dot{V}$ is not negative definite near the origin. However, $\dot{V}$ is negative definite outside the set $\{|x| < \delta\}$.

With $c > \frac{1}{2}\delta^2, (|x| > \delta)$, solutions starting in the set $\{V(x) \leq c\}$ will remain therein for all future time since $\dot{V}$ is negative on the boundary $V = c$. Hence, the solutions are uniformly bounded.

Pick up any number ε such that $\frac{\delta^2}{2} < \varepsilon < c$. Then $\dot{V} < 0$ in the set $\{\varepsilon \leq V(x) \leq c\}$, which shows that, in the set, V will decrease monotonically until the solution enters the set $\{V(x) \leq \varepsilon\}$. From that time on, the solution cannot leave the set $\{V(x) \leq \varepsilon\}$ because $\dot{V} < 0$ on the boundary $V = \varepsilon$. Then we can conclude that the solution is uniformly bounded with the ultimate bound $|x| \leq \sqrt{2\varepsilon}$.

Definition 17 *The solution of* (5) *is*

uniformly bounded *if there exits a positive constant* c*, independent of* $t_0 \geq 0$*, and for every* $a \in (0, c)$*, there is* $\beta = \beta(a) > 0$*, independent of* t_0*, such that*

$$\|x(t_0)\| \leq a \Rightarrow \|x(t)\| \leq \beta, \forall t \geq t_0 \geq 0$$

uniformly ultimately bounded *with ultimate bound* b *if there exist positive constants* b, c*, independent of* $t_0 \geq 0$*, and for every* $a \in (0, c)$*, there is* $T = T(a, b) > 0$*, independent of* t_0*, such that*

$$\|x(t_0)\| \leq a \Rightarrow \|x(t)\| \leq b, \forall t \geq t_0 + T$$

globally uniformly bound *if* a *is chosen arbitrarily large.*

3.7 Input-to-State Stability

Consider the system

$$\dot{x} = f(t, x, u) \tag{10}$$

where $f : (0, \infty) \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is piecewise continuous in t and locally Lipschitz in x and u. The input is a bounded continuous function of t. Suppose that the unforced system

$$\dot{x} = f(t, x, 0) \tag{11}$$

has a globally uniformly asymptotically stable equilibrium point at $x = 0$.

Question 2 What can we say about the behavior of (10) in the presence of a bounded input?

Example 23 For the linear time-invariant system

$$\dot{x} = Ax + Bu$$

with a Hurwitz matrix A. Its solution is

$$x(t) = e^{A(t-t_0)}x(t_0) + \int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)d\tau$$

We can use $\|e^{A(t-t_0)}\| \leq ke^{-\lambda(t-t_0)}$ with $k, \lambda \geq 0$, to estimate the solution

$$\begin{aligned}\|x(t)\| &\leq ke^{-\lambda(t-t_0)}\|x(t_0)\| + k\int_{t_0}^{t} e^{-\lambda(t-\tau)}\|B\|\|u(\tau)\|d\tau \\ &\leq ke^{-\lambda(t-t_0)}\|x(t_0)\| + k\|B\| \sup_{t_0\leq\tau\leq t}\|u(\tau)\| \int_{t_0}^{t} e^{-\lambda(t-\tau)}d\tau \\ &\leq ke^{-\lambda(t-t_0)}\|x(t_0)\| + \frac{k\|B\|}{\lambda} \sup_{t_0\leq\tau\leq t}\|u(\tau)\|\end{aligned}$$

The zero-input response decays to zero exponentially fast, while the state is bounded for every bounded input.

Definition 18 *The system* (10) *is said to be* ***input-to-state stable*** *(ISS) if there exist a class* $\mathcal{KL}$ *function* β *and a class* $\mathcal{K}$ *function* γ *such that for any initial state* $x(t_0)$ *and any bounded input* $u(t)$*, the solution* $x(t)$ *exists for all* $t \geq t_0$ *and satisfies*

$$\|x(t)\| \leq \beta(\|x(t_0)\|, t-t_0) + \gamma(\sup_{t_0\leq\tau\leq t}\|u(\tau)\|)$$

Note 3

1. For any bounded $u(t)$, $x(t)$ will be bounded.
2. as t increases, $x(t)$ will be `ultimately bounded` by a class $\mathcal{K}$ function of $\sup_{t_0\leq\tau\leq t}\|u(\tau)\|$.
3. If $u(t) \equiv 0$, then $x = 0$ is asymptotically stable.
4. If $u(t) \to 0$ as $t \to \infty$, so does $x(t)$.

Lemma 11 *If the unforced system* $\dot{x} = f(t, x, 0)$ *has a globally exponentially stable equilibrium point at* $x = 0$*, then the system* $\dot{x} = f(t, x, u)$ *is ISS.*

Example 24 An interesting application. The cascade system

$$\dot{x}_1 = f_1(t, x_1, x_2) \quad (12)$$

$$\dot{x}_2 = f_2(t, x_2) \quad (13)$$

Suppose both (12) and (13) has globally asymptotically stable equilibrium points at their respective origins.

Lemma 12 *Under the stated assumptions, if the system* (12) *with* x_2 *as input, is ISS and the origin of* (13) *is globally uniformly asymptotically stable, then the origin of the cascade system* (12) *and* (13) *is globally uniformly asymptotically stable.*

4 Adaptive control

4.1 Methods & Types

1. Characterize the desired behavior of the closed-loop systems.
2. Determine a suitable control law containing adjustable parameters.
3. Find a mechanism (an adaptation law) for adjusting those parameters.
4. Analyze the convergence properties and implement the control law.

Gain Scheduling

1. Controller parameters change in a predetermined fashion with the operating conditions.
2. AE: the parameters can be changed quickly in response to changes in the dynamics.
3. DE: it is an open-loop adaptation scheme, with no real "learning".

Self-Tuning

1. Controller parameters change in a predetermined fashion with the operating conditions.
2. Performs simultaneous parameter identification and control.
3. Uses Certainty Equivalence Principle

Model Reference

1. Plant: containing unknown parameters and having a known structure.
2. RM: specifying the desired output of the control system.
3. Feedback control law: containing adjustable parameters.
4. Adaptation mechanism: updating the adjustable parameters.

Types

1. Indirected: estimate plant parameters $\Rightarrow$ design controller parameters. The process model and possibly the disturbance characteristics are first determined. The controller parameters are designed on the basis of this information.
2. Directed: directly design controller parameter. The controller parameters are changed directly without the characteristics of the process and its disturbance first being determined.

Example 25 Consider the following first-order system

$$\dot{x} = ax + u$$

where a is an unknown parameter, x is the state and u is the control input.

Control objective: design u such that all signals in the closed-loop system are bounded, and x tracks the state x_{ref} of the following reference model

$$\dot{x}_{ref} = -a_{ref}x_{ref} + b_{ref}u_c$$

where $a_{ref} > 0$ and b_{ref} are known, $u_c(t)$ is the input command which is bounded and piecewise continuous.

We propose the control law

$$u^* = k_1^* x + k_2^* u_c$$

$$e = x - x_{ref}$$

$$\dot{e} = \dot{x} - \dot{x}_{ref} = ax + k_1^* x + k_2^* u_c + a_{ref} x_{ref} - b_{ref} u_c$$

If $k_1^* = -a - a_{ref}, k_2^* = b_{ref}$, we have $\dot{e} = -a_{ref} e$.

$$u = \begin{cases} \hat{k}_1 x + b_{ref} u_c & \textit{directed} \\ (-\hat{a}(t) - a_{ref})x + b_{ref} u_c & \textit{indirected} \end{cases}$$

4.2 Direct MRAC Design for Scalar Systems

Consider the following first-order system

$$\dot{x} = -ax + bu \tag{14}$$

where a, b are unknown parameters but the sign of b is known.

Control objective: design u such that all signals in the closed-loop system are bounded, and x tracks the state x_{ref} of the following reference model

$$\dot{x}_{ref} = -a_{ref} x_{ref} + b_{ref} u_c \tag{15}$$

where $a_{ref} > 0$ and b_{ref} are the desired known constants, $u_c(t)$ is the input command which is bounded and piecewise continuous.

We propose the control law

$$u = k_1^* x + k_2^* u_c \tag{16}$$

where k_1^*, k_2^* are the desired known constants. The closed-loop system of (14) and (16) is

$$\begin{aligned} \dot{x} &= -ax + b(k_1^* x + k_2^* u_c) \\ &= -(a - bk_1^*)x + bk_2^* u_c \end{aligned}$$

Then the perfect model condition is

$$k_1^* = \frac{a - a_{ref}}{b}, k_2^* = \frac{b_{ref}}{b} \tag{17}$$

When a, b are unknown, (17) cannot be implemented. Therefore, instead of (16), we propose the following control law with adjustable parameters.

$$u = \hat{k}_1(t)x + \hat{k}_2(t)u_c \tag{18}$$

where $\hat{k}_1(t), \hat{k}_2(t)$ are the estimates of k_1^*, k_2^*, respectively.

We substitute (18) into (14)

$$\dot{x} = -(a - b\hat{k}_1(t))x + b\hat{k}_2(t)u_c \tag{19}$$

Define the tracking error

$$e \triangleq x - x_{ref} \tag{20}$$

We can get from (17) that

$$\begin{aligned}
\dot{e} &= \dot{x} - \dot{x}_{ref} \\
&= -(a - b\hat{k}_1(t))x + b\hat{k}_2(t)u_c + a_{ref}x_{ref} - b_{ref}u_c \\
&= -(a - a_{ref} - b\hat{k}_1(t))x + (b\hat{k}_2(t) - b_{ref})u_c - a_{ref}e \\
&= -a_{ref}e - b(k_1^* - \hat{k}_1(t))x + b(\hat{k}_2(t) - k_2^*)u_c \\
&= -a_{ref}e + b\tilde{k}_1 x + b\tilde{k}_2 u_c
\end{aligned} \tag{21}$$

where $\tilde{k}_1 = \hat{k}_1(t) - k_1^*$ and $\tilde{k}_2 = \hat{k}_2(t) - k_2^*$.

Consider a Lyapunov function candidate

$$V = \frac{1}{2}e^2 + \frac{1}{2}\tilde{k}_1^2 + \frac{1}{2}\tilde{k}_2^2 \tag{22}$$

Taking the time derivative of V, along the trajectories of (21), gives

$$\begin{aligned}
\dot{V} &= e\dot{e} + \tilde{k}_1\dot{\hat{k}}_1 + \tilde{k}_2\dot{\hat{k}}_2 \\
&= -a_{ref}e^2 + b\tilde{k}_1 xe + b\tilde{k}_2 u_c e + \tilde{k}_1\dot{\hat{k}}_1 + \tilde{k}_2\dot{\hat{k}}_2 \\
&= -a_{ref}e^2 + \tilde{k}_1(\dot{\hat{k}}_1 + bxe) + \tilde{k}_2(\dot{\hat{k}}_2 + bu_c e)
\end{aligned} \tag{23}$$

If we choose

$$\dot{\hat{k}}_1 = -bxe, \dot{\hat{k}}_1 = -bu_c e \tag{24}$$

We can get that $\dot{V} \leq 0$. However, (24) cannot be implemented since b is unknown.

We consider the following Lyapunov function candidate

$$V = \frac{1}{2}e^2 + \frac{|b|}{2\gamma_1}\tilde{k}_1^2 + \frac{|b|}{2\gamma_2}\tilde{k}_2^2 \tag{25}$$

where $\gamma_1, \gamma_2 > 0$.

Taking the time derivative of V, along the trajectories of (21), gives

$$\dot{V} = -a_{ref}e^2 + \frac{|b|}{\gamma_1}\tilde{k}_1(\dot{\hat{k}}_1 + \gamma_1 sgn(b)xe) + \frac{|b|}{\gamma_2}\tilde{k}_2(\dot{\hat{k}}_2 + \gamma_2 sgn(b)u_c e)$$

If we choose

$$\dot{\hat{k}}_1 = -\gamma_1 sgn(b)xe, \dot{\hat{k}}_2 = -\gamma_2 sgn(b)u_c e \tag{26}$$

which leads to

$$\dot{V} = -a_{ref}e^2 \leq 0 \tag{27}$$

Thus, $V(t) \leq V(0)$. From (25), $e, \tilde{k}_1, \tilde{k}_2$ are bounded and x is bounded.

From (21), $\ddot{V} = -2a_{ref}e\dot{e}$ is also bounded.

We can conclude from Barbalat's lemma that $\lim_{t\to\infty}\dot{V}(t) = 0, i.e. \lim_{t\to\infty} e(t) = 0$.

Design steps:

Open-loop plant	$\dot{x} = -ax + bu$; $\{a, b\}$ are unknown, $sgn(b)$ is known
Reference model	$\dot{x}_{ref} = -a_{ref}x_{ref} + b_{ref}u_c$
Tracking error	$e \triangleq x - x_{ref}$
Control input	$u = \hat{k}_1(t)x + \hat{k}_2(t)u_c$
Direct MRAC laws	$\dot{\hat{k}}_1 = -\gamma_1 sgn(b)xe, \dot{\hat{k}}_1 = -\gamma_2 sgn(b)u_c e; \gamma_1, \gamma_2 > 0$

Table 1: Direct MRAC design summary for a scalar system

1. Find a controller structure (based on known parameters)
2. Derive the error dynamics
3. Find a Lyapunov function
4. Design parameter updating laws such that $\dot{V} \leq 0$.

4.3 Direct MRAC Design With A Nonlinear Term

We consider the following scalar system

$$\dot{x} = -ax + b(u + f(x)) \tag{28}$$

where $f(x) = \varphi(x)\theta$ with φ being a bounded and continuous known function, and a, b, θ are unknown constant parameters.

The reference model is given by

$$\dot{x}_{ref} = -a_{ref}x_{ref} + b_{ref}u_c \tag{29}$$

We propose the control law

$$u = \hat{k}_1(t)x + \hat{k}_2(t)u_c - \varphi(x)\hat{\theta}(t) \tag{30}$$

Then (28) can be written as

$$\dot{x} = -ax + b\hat{k}_1(t)x + b\hat{k}_2(t)u_c - b\varphi(x)(\hat{\theta}(t) - \theta)$$

By $k_1^* = \frac{a-a_{ref}}{b}, k_2^* = \frac{b_{ref}}{b}$, the error dynamics is

$$\begin{aligned}
\dot{e} &= -ax + b\hat{k}_1(t)x + b\hat{k}_2(t)u_c - b\varphi(x)(\hat{\theta}(t) - \theta) + a_{ref}x_{ref} - b_{ref}u_c \\
&= (a_{ref} - a)x + b\hat{k}_1(t)x + (b\hat{k}_2(t) - b_{ref})u_c - b\varphi(x)(\hat{\theta}(t) - \theta) - a_{ref}e \\
&= b(\hat{k}_1(t) - k_1^*)x + b(\hat{k}_2(t) - k_2^*)u_c - b\varphi(x)(\hat{\theta}(t) - \theta) - a_{ref}e \\
&= -a_{ref}e + b\tilde{k}_1 x + b\tilde{k}_2 u_c - b\varphi\tilde{\theta}
\end{aligned} \tag{31}$$

where $\tilde{k}_1 = \hat{k}_1(t) - k_1^*$ and $\tilde{k}_2 = \hat{k}_2(t) - k_2^*$ and $\tilde{\theta} = \hat{\theta}(t) - \theta$.

Consider the following Lyapunov function candidate

$$V = \frac{1}{2}e^2 + \frac{|b|}{2\gamma_1}\tilde{k}_1^2 + \frac{|b|}{2\gamma_2}\tilde{k}_2^2 + \frac{|b|}{2\gamma_3}\tilde{\theta}^2 \tag{32}$$

where $\gamma_1, \gamma_2, \gamma_3 > 0$.

Taking the time derivative of V, along the trajectories of (31), gives

$$\begin{aligned}\dot{V} &= e\dot{e} + \frac{|b|}{\gamma_1}\widetilde{k}_1\dot{\hat{k}}_1 + \frac{|b|}{\gamma_2}\tilde{k}_2\dot{\hat{k}}_2 + \frac{|b|}{\gamma_3}\tilde{\theta}\dot{\hat{\theta}} \\ &= -a_{ref}e^2 + \frac{|b|}{\gamma_1}\widetilde{k}_1(\dot{\hat{k}}_1 + \gamma_1 sgn(b)xe) + \frac{|b|}{\gamma_2}\tilde{k}_2(\dot{\hat{k}}_2 + \gamma_2 sgn(b)u_c e) + \frac{|b|}{\gamma_3}\tilde{\theta}(\dot{\hat{\theta}} - \gamma_3 sgn(b)\varphi e)\end{aligned}$$

If we choose

$$\dot{\hat{k}}_1 = -\gamma_1 sgn(b)xe, \dot{\hat{k}}_2 = -\gamma_2 sgn(b)u_c e, \dot{\hat{\theta}} = \gamma_3 sgn(b)\varphi e \tag{33}$$

which leads to

$$\dot{V} = -a_{ref}e^2 \leq 0 \tag{34}$$

Thus, $V(t) \leq V(0)$. From (32), $e, \tilde{k}_1, \tilde{k}_2, \dot{\hat{\theta}}$ are bounded and x is bounded.

$$\ddot{V} = -2a_{ref}e\dot{e}$$

From (31), $\ddot{V}$ is bounded.

We can conclude from Barbalat's lemma that $\lim_{t\to\infty}\dot{V}(t) = 0, i.e.\ \lim_{t\to\infty} e(t) = 0$.

Question 3

Consider the following scalar system

$$\dot{x} = -ax + bu + \varphi\theta$$

Answer 2

We propose the following control law

$$u = \hat{k}_1(t)x + \hat{k}_2(t)u_c - \varphi(x)\frac{\hat{\theta}(t)}{\hat{b}(t)} \triangleq \hat{k}_1(t)x + \hat{k}_2(t)u_c - \varphi(x)\hat{\theta}_1(t)$$

4.4 Indirect MRAC Design for Scalar Systems

Consider the following first-order scalar system

$$\dot{x} = -ax + bu \tag{35}$$

where a, b are unknown parameters.

With the following reference model

$$\dot{x}_{ref} = -a_{ref}x_{ref} + b_{ref}u_c \tag{36}$$

The modeling condition is

$$k_1^* = \frac{a - a_{ref}}{b}, k_2^* = \frac{b_{ref}}{b} \tag{37}$$

We propose the control law

$$u = \hat{k}_1(t)x + \hat{k}_2(t)u_c \tag{38}$$

In direct adaptive control, $\hat{k}_1(t), \hat{k}_2(t)$ are generated directly by adaptation laws. In indirect adaptive control, we follow a different approach. We calculate $\hat{k}_1(t), \hat{k}_2(t)$ by busing the relationship (35) and the estimates and of the unknot parameters a, b as

$$\hat{k}_1(t) = \frac{\hat{a} - a_{ref}}{\hat{b}}, \hat{k}_2(t) = \frac{b_{ref}}{\hat{b}}$$

which implies

$$u = \frac{1}{\hat{b}}[(\hat{a} - a_{ref})x + b_{ref}u_c] \tag{39}$$

Using (39), (35) can be written as

$$\begin{aligned} \dot{x} &= -(a - \hat{a} + \hat{a})x + (b - \hat{b} + \hat{b})u \\ &= \tilde{a}x - \tilde{b}u - \hat{a}x + \hat{b}u \\ &= \tilde{a}x - \tilde{b}u - a_{ref}x + b_{ref}u \end{aligned} \tag{40}$$

where $\tilde{a} \triangleq \hat{a} - a$ and $\tilde{b} \triangleq \hat{b} - b$.

Then the error dynamics can be written as

$$\dot{e} = -a_{ref}e + \tilde{a}x - \tilde{b}u \tag{41}$$

Consider a Lyapunov function candidate

$$V = \frac{1}{2}e^2 + \frac{1}{2\gamma_1}\tilde{a}^2 + \frac{1}{2\gamma_2}\tilde{b}^2 \tag{42}$$

Taking the time derivative of V, gives

$$\dot{V} = -a_{ref}e^2 + \frac{1}{\gamma_1}\tilde{a}(\dot{\hat{a}} + \gamma_1 ex) + \frac{1}{\gamma_2}\tilde{b}(\dot{\hat{b}} - \gamma eu)$$

If we choose

$$\dot{\hat{a}} = -\gamma_1 ex, \dot{\hat{b}} = \gamma_2 eu \tag{43}$$

We can get that $\dot{V} = -a_{ref}e^2 \leq 0$.

And consequently, $e, \tilde{a}, \tilde{b} \in L_\infty$. Since $x_{ref} \in L_\infty$ and a, b are constants, we have $x, \hat{a}, \hat{b} \in L_\infty$.

$$\ddot{V} = -2a_{ref}e\dot{e}$$

In order to claim the boundedness of u, we need to modify the adaptation law and prevent $\tilde{b}(t)$ from going through zero. Such a modification can be achieved using the following a prior knowledge.

Assumption: The $sgn(b)$ and a lower bound $b_{\min} > 0$ for $|b|$ are known.

Let us the following modification of the second equation in (43)

$$\dot{\hat{b}} = \begin{cases} \gamma_2 eu, & if|\hat{b}| > b_{\min} or if|\hat{b}| = b_{\min} and uesgn(b) \geq 0 \\ 0, & otherwise(if|\hat{b}| = b_{\min} and uesgn(b) < 0) \end{cases} \tag{44}$$

The main motivation here is to stop adaptation of $\hat{b}$ if the parameter reaches its lower absolute limit value $b_{\min}$ with a nonzero time derivative.

We need to argue that the modification (44) does indeed preserve $\hat{b}$ from crossing its allowable bound and at the same time, it preserves the closed-loop system ability. For this to be true, it is sufficient to show that

$$\tilde{b}(\dot{\hat{b}} - \gamma eu) \leq 0$$

We only need to check $if|\hat{b}| = b_{\min} anduesgn(b) < 0$, then $\dot{\hat{b}} = 0, -\tilde{b}eu = -(\hat{b} - b)sgn(b) \cdot uesgn(b)$

$$-\tilde{b}eu \begin{cases} \leq 0, sgn(b) > 0, \hat{b} = b_{\min} \leq b \Rightarrow \hat{b} - b \leq 0 \\ \leq 0, sgn(b) < 0, \hat{b} = -b_{\min} \geq b \Rightarrow \hat{b} - b \geq 0 \end{cases}$$

Overall, we can get under the modification (44)

$$\tilde{b}(\dot{\hat{b}} - \gamma eu) \leq 0$$

which implies

$$\dot{V} \leq -a_{ref} e^2 \leq 0 \tag{45}$$

We can get $x, \hat{a}, \hat{b}, u, \dot{e} \in L_\infty$. Integrating both sides of (45)

$$V(t) - V(0) \leq a_{ref} \int_0^t e^2(\tau) d\tau \leq 0$$

$$\Rightarrow \int_0^t e^2(\tau) d\tau \leq \frac{1}{a_{ref}} (V(0) - V(t)) \leq \frac{1}{a_{ref}} V(0)$$

$$(e(t) \in \mathbb{L}_2)$$

Let $f(t) \triangleq \int_0^t e^2(\tau) d\tau$, $f(t)$ has a finite limit.

$$\dot{f} = e^2; \ddot{f} = 2e\dot{e} \in L_\infty$$

From Barbalat's Lemma,

$$\lim_{t \to \infty} \dot{f}(t) = 0, i.e. \lim_{t \to \infty} e(t) = 0.$$

Lemma 13 *If* $e \in \mathbb{L}_2 \cap \mathbb{L}_\infty$, $\dot{e} \in \mathbb{L}_\infty$, *we have* $e \to 0$ *as* $t \to \infty$.

Lemma 14 *If* $e \in \mathbb{L}_1 \cap \mathbb{L}_\infty$, $\dot{e} \in \mathbb{L}_\infty$, *we have* $e \to 0$ *as* $t \to \infty$.

4.5 Direct MRAC Design for General Linear System

We consider a linear system described by

$$\dot{x} = Ax + B\Lambda u \tag{46}$$

where $x \in \mathbb{R}^n$ is the system state, $u \in \mathbb{R}^m$ is the control input, and $B \in \mathbb{R}^{n \times m}$ is the known control matrix, while $A \in \mathbb{R}^{n \times n}, \Lambda \in \mathbb{R}^{m \times m}$ are unknown constant matrix. In addition, assume that Λ is diagonal with positive diagonal elements, and $(A, B\Lambda)$ is controllable. The uncertainty in Λ is introduced to model the control failure.

Open-loop plant	$\dot{x} = -ax + bu$
Reference model	$\dot{x}_{ref} = -a_{ref}x + b_{ref}u_c$
Tracking error	$e = x - x_{ref}$
Control input	$u = \frac{1}{\hat{b}}[(\hat{a} - a_{ref})x + b_{ref}u_c]$
Indirect MRAC laws	$\dot{\hat{a}} = -\gamma_1 ex \ \dot{\hat{b}} = \begin{cases} \gamma_2 eu, & if\|\hat{b}\| > b_{\min} or if \|\hat{b}\| = b_{\min} and ue sgn(b) \geq 0 \\ 0, & otherwise (if \|\hat{b}\| = b_{\min} and ue sgn(b) < 0) \end{cases}$

Table 2: Indirect MRAC design summary for a scalar system

Control objective: design u, such that all signals in the closed-loop system are bounded and x tracks the state x_{ref} of the following reference model.

$$\dot{x}_{ref} = A_{ref}x_{ref} + B_{ref}u_c \tag{47}$$

where $A_{ref} \in \mathbb{R}^{n\times n}$ is Hurwitz, $B_{ref} \in \mathbb{R}^{n\times m}$, $u_c \in \mathbb{R}^m$ is the bounded command vector.

If matrices $A \in \mathbb{R}^{n\times n}$ and $\Lambda \in \mathbb{R}^{m\times m}$ were known, we can apply the control law

$$u = K_1^* x + K_2^* u_c$$

where $K_1^* \in \mathbb{R}^{m\times n}, K_2^* \in \mathbb{R}^{m\times m}$ and we can obtain

$$\dot{x} = (A + B\Lambda K_1^*)x + B\Lambda K_2^* u_c$$

Then the matching condition is

$$\begin{cases} A + B\Lambda K_1^* = A_{ref} \\ B\Lambda K_2^* = B_{ref} \end{cases} \tag{48}$$

We should note that in general, there is no guarantee that the idea gains K_1^*, K_2^* exist such that (48) is satisfied. However, in practice, if the structure of A, B is known. A_{ref}, B_{ref} may be designed so that (48) has a solution.

Let us assume that K_1^*, K_2^* in (48) exist, i.e., there is sufficient structure flexibility to meet the control objective. We propose the control law

$$u = \hat{K}_1(t)x + \hat{K}_1(t)u_c$$

By adding and subtracting the desired term, we obtain

$$\dot{x} = A_{ref}x + B_{ref}u_c + B\Lambda\tilde{K}_1 x + B\Lambda\tilde{K}_2 u_c$$

where $\tilde{K}_1 \triangleq \hat{K}_1 - K_1^*, \tilde{K}_2 \triangleq \hat{K}_2 - K_2^*$.

Define the tracking error $e \triangleq x - x_{ref}$. Its dynamic is

$$\dot{e} = A_{ref}e + B\Lambda\tilde{K}_1 x + B\Lambda\tilde{K}_2 u_c \tag{49}$$

Since A_{ref} is Hurwitz, we can get from Lyapunov theorem that for any positive definite $Q \in \mathbb{R}^{n\times n}$, there exists a unique positive definite $P \in \mathbb{R}^{n\times n}$ such that

$$A_{ref}^T P + PA_{ref} = -Q < 0$$

Remark 14 Property of trace: For $A \in \mathbb{R}^{m\times n}, B \in \mathbb{R}^{n\times m}$, we have $tr(AB) = tr(BA)$.

we then consider the following Lyapunov function candidate

$$V = e^T P e + tr\{\tilde{K}_1^T \Lambda \tilde{K}_1\} + tr\{\tilde{K}_2^T \Lambda \tilde{K}_2\}$$

Since

$$\begin{aligned} e^T PB\Lambda\tilde{K}_1 x &= tr\{e^T PB\Lambda\tilde{K}_1 x\} \\ &= tr\{x e^T PB\Lambda\tilde{K}_1\} \\ &= tr\{\tilde{K}_1 \Lambda B^T P e x^T\} \end{aligned}$$

Its derivative is

$$\begin{aligned} \dot{V} &= 2e^T P\dot{e} + 2tr\{\tilde{K}_1^T \Lambda \dot{\hat{K}}_1\} + 2tr\{\tilde{K}_2^T \Lambda \dot{\hat{K}}_2\} \\ &= 2e^T P(A_{ref} e + B\Lambda\tilde{K}_1 x + B\Lambda\tilde{K}_2 u_c) + 2tr\{\tilde{K}_1^T \Lambda \dot{\hat{K}}_1\} + 2tr\{\tilde{K}_2^T \Lambda \dot{\hat{K}}_2\} \\ &= 2e^T PA_{ref} e + 2e^T PB\Lambda\tilde{K}_1 x + 2e^T PB\Lambda\tilde{K}_2 u_c + 2tr\{\tilde{K}_1^T \Lambda \dot{\hat{K}}_1\} + 2tr\{\tilde{K}_2^T \Lambda \dot{\hat{K}}_2\} \\ &= 2e^T PA_{ref} e + 2tr\{\tilde{K}_1 \Lambda B^T P e x^T\} + 2tr\{\tilde{K}_2 \Lambda B^T P e u_c^T\} + 2tr\{\tilde{K}_1^T \Lambda \dot{\hat{K}}_1\} + 2tr\{\tilde{K}_2^T \Lambda \dot{\hat{K}}_2\} \end{aligned}$$

we can choose

$$\dot{\hat{K}}_1 = -B^T P e x^T, \dot{\hat{K}}_2 = -B^T P e u_c^T$$

Then we have

$$\dot{V} = -e^T Q e \le 0$$

Thus, $V(t) \le V(0)$, which implies that $e, \tilde{K}_1, \tilde{K}_2 \in \mathbb{L}_\infty$. Since u_c is bounded and A_{ref} is Hurwitz, $x_{ref} \in \mathbb{L}_\infty$. Since $x = e + x_{ref}$, $x \in \mathbb{L}_\infty$. From (49), $\dot{e} \in \mathbb{L}_\infty$. Then

$$\ddot{V} = -2e^T Q\dot{e} \in \mathbb{L}_\infty$$

Using Barbalat's Lemma, we can get $\lim_{t\to\infty} \dot{V} = 0$, i.e., $\lim_{t\to\infty} e(t) = 0$.

Question 4 ⋆⋆⋆

1. Λ with all negative elements?

   Solution: choose $tr\{\tilde{K}_1^T(-\Lambda)\tilde{K}_1\}$

2. Λ with some negative elements and some positive elements?

   Solution: choose

   $$V = e^T P e + tr\{\tilde{K}_1^T |\Lambda| \tilde{K}_1\} + tr\{\tilde{K}_2^T |\Lambda| \tilde{K}_2\}$$

   and

   $$\dot{\hat{K}}_1 = -sgn(\Lambda) B^T P e x^T, \dot{\hat{K}}_2 = -sgn(\Lambda) B^T P e u_c^T$$

   where $|\Lambda| \triangleq \Lambda sgn(\Lambda)$, $sgn(\Lambda) \triangleq diag\{sgn(\lambda_i)\}$.

Question 5

$$\dot{x} = Ax + B\Lambda(u + \Phi(t)\cdot\Theta)$$

where $\Theta \in \mathbb{R}^{p\times 1}, \Phi(t) \in \mathbb{R}^{m\times p}$ with Φ being bounded and Θ unknown.

Solution:

$$u = \hat{K}_1(t)x + \hat{K}_2(t)u_c - \Phi(t)\cdot\hat{\Theta}$$

4.6 Adaptive Control Design Without RM

Consider a first-order scalar system

$$\dot{x} = ax + bu \tag{50}$$

where both a and b are unknown constants and $b \neq 0$.

The objective is to design u such that $\lim_{t\to\infty} x(t) = 0$.

If a and b are known, the following control input

$$u = k^* x$$

with $a + bk^* = a_0 < 0$ can stabilize the system with $\lim_{t\to\infty} x(t) = 0$. Since k^* is unknown, we design the following control input with a time-varying gain

$$u = \widehat{k}(t)x$$

The closed-loop system is

$$\begin{aligned}
\dot{x} &= ax + b\widehat{k}(t)x + a_0 x - (a + bk^*)x \\
&= a_0 x + b(\widehat{k}(t) - k^*)x \\
&= a_0 x + b\tilde{k}(t)x
\end{aligned}$$

Consider a Lyapunov function candidate

$$V = \frac{1}{2}x^2 + \frac{|b|}{2\gamma}\tilde{k}^2$$

$$\begin{aligned}
\dot{V} &= x\dot{x} + \frac{|b|}{\gamma}\tilde{k}\dot{\hat{k}} \\
&= a_0 x^2 + b\tilde{k}(t)x^2 + \frac{|b|}{\gamma}\tilde{k}\dot{\hat{k}} \\
&= a_0 x^2 + \frac{|b|}{\gamma}\tilde{k}(\dot{\hat{k}} + \gamma sgn(b)x^2)
\end{aligned}$$

Choose $\dot{\hat{k}} = \gamma sgn(b)x^2$.

When the sign of b is unknown, the following controller structure was suggested by Nussbaum.

$$\begin{aligned}
u &= \mathcal{N}(k)x \\
\mathcal{N}(k) &= k^2 \cos k \\
\dot{k} &= x^2 \\
\lim_{s\to\infty} \sup \frac{1}{s}\int_0^s \mathcal{N}(\tau)d\tau &= +\infty \\
\lim_{s\to\infty} \inf \frac{1}{s}\int_0^s \mathcal{N}(\tau)d\tau &= -\infty
\end{aligned}$$

$\mathcal{N}(k)$ changes its sign, an infinite number of time as $k \to \infty$.

Note that the closed-loop system is

$$\begin{aligned}\dot{x} &= (a + b\mathcal{N}(k))x \\ &= (a + bk^2 \cos k)x\end{aligned}$$

We derive the following express

$$\begin{aligned}\frac{d(x^2)}{dk} &= \frac{d(x^2)}{dt}\frac{dt}{dk} \\ &= 2x\dot{x} \cdot \frac{1}{\dot{k}} \\ &= 2x(a + bk^2 \cos k)x\frac{1}{x^2} \\ &= 2(a + bk^2 \cos k) \qquad (51)\end{aligned}$$

Integrating both sides of (51) from $k(t_0)$ to $k(t)$

$$x^2(k(t)) - x^2(k(t_0)) = 2\int_{k(t_0)}^{k(t)} (a + b\tau^2 \cos \tau)d\tau$$

$\Rightarrow$

$$x^2(k(t)) = x^2(k(t_0)) + 2\varphi(k(t)) - 2\varphi(k(t_0))$$

with

$$\varphi(k(t)) = ak(t) + b(k^2(t) \sin k(t) + 2k(t) \cos(k(t)) - 2 \sin k(t)) \qquad (52)$$

Note that $k(t)$ is monotone nondecreasing. $k(t)$ must either approach a finite or grow without bound. Assume that $k(t)$ grows without bound. Then the sign of $\varphi(k(t))$ will be dominated by the term $bk^2(t) \sin k(t)$, which can assume a large negative value independent of the sign of b. As a result, the right side of (52) is negative. Clearly, $k(t)$ must be bounded. Then we can get $x \in \mathbb{L}_\infty, \dot{x} \in \mathbb{L}_\infty$. Note that $\dot{k} = x^2$. We have $\int_0^t x^2(\tau)d\tau = k(t) - k(0)$. Therefore, $x \in \mathbb{L}_2$. By Barbalat's Lemma, $\lim_{t\to\infty} x(t) = 0$.

5 Nonlinear Control System Design

5.1 Feedback Control

In a typical control problem, our interest is usually in the analysis and design of feedback control systems. Consider the following system

$$\dot{x} = f(t, x, u)$$

$$y = h(t, x, u)$$

1. State feedback stabilization

   Design a feedback control law

   $$u = \gamma(t, x)$$

such that $x = 0$ is uniformly asymptotical stable equilibrium point of the closed-loop system

$$\dot{x} = f(t, x, \gamma(t, x))$$

Static state feedback: $u = \gamma(t, x)$.

Dynamic state feedback: $u = \gamma(t, x, z), \dot{z} = g(t, x, z)$. (such as adaptive control)

2. Output feedback stabilization

   Static output feedback: $u = \gamma(t, y)$.

   Dynamic output feedback: $u = \gamma(t, y, z), \dot{z} = g(t, y, z)$.

3. For linear system

$$\begin{aligned}\dot{x} &= Ax + Bu \\ y &= Cx + Du\end{aligned}$$

   State feedback: $u = -Kx$, $\dot{x} = (A - BK)x$, $A - BK$ being Hurwitz.

   Output feedback: $u = -K\hat{x}, \hat{y} = C\hat{x} + Du, \dot{\hat{x}} = A\hat{x} + Bu + H(y - \hat{y})$.

   Define $e = x - \hat{x}$. Then the error dynamic is $\dot{e} = (A - HC)e$, $A - HC$ being Hurwitz.

   $\dot{x} = (A - BK)x + BKe$, $A - BK$ being Hurwitz. (ISS)

4. Tracking

   Consider a nonlinear dynamics

$$\begin{aligned}\dot{x} &= f(t, x, u) \\ y &= h(t, x, u) \triangleq h(x)\end{aligned}$$

   and a desired output trajectory y_d, find u such that the tracking error $y - y_d$ goes to zero.

5.2 Feedback Linearization

Central idea Can we transform the nonlinear dynamics into a linear one? So that linear control technique can be used. This differs entirely from approximate linearization through a Jacobian matrix, in that feedback linearization is achieved by exact state transformation and feedback, rather by linear approximates of the dynamics.

Problem statement Given $\dot{x} = f(x, u), x \in \mathbb{R}^n, u \in \mathbb{R}^m$, find function g, h and new variables $z \in \mathbb{R}^n, v \in \mathbb{R}^m$, such that $u = g(x, v), z = h(x), \dot{z} = Az + Bv, A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times m}$.

5.2.1 Controllability Canonical Form

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ &\cdots \\ \dot{x}_{n-1} &= x_n \\ \dot{x}_n &= f(x) + b(x)u\end{aligned}$$

Assume that $b \neq 0$, use the control input

$$u = \frac{1}{b(x)}(v - f(x))$$

We can cancel the nonlinearities and obtain

$$\dot{x}_n = v$$

Thus the control law

$$v = -k_0 x_1 - k_1 x_2 - \cdots - k_{n-1} x_n$$

with k_i chosen such that the polynomial

$$p^n + k_{n-1} p^{n-1} + \cdots + k_0 = 0$$

has all the roots in the LHP, which leads to $x(t) \to 0$.

5.2.2 An Input-State Linearization

The technique of input-state linearization solves the problem in two steps.

1. Find a state transformation $z = z(x)$ and an input transformation $u = u(x, v)$ so that the nonlinear system is transformed into a linear one, in the familiar form $\dot{z} = Az + bv$.
2. Use standard linear techniques to design u.

Example 26 Consider the system

$$\begin{aligned}\dot{x}_1 &= -2x_1 + ax_2 + \sin x_1 \\ \dot{x}_2 &= -x_2 \cos x_1 + u \cos(2x_1)\end{aligned}$$

A specific difficulty is the nonlinearity in the first equation, which cannot directly canceled by u.

Pick $z_1 = x_1, z_2 = ax_2 + \sin x_1 \Rightarrow ax_2 = z_2 - \sin z_1$

$$\begin{aligned}\dot{z}_1 &= -2z_1 + z_2 \\ \dot{z}_2 &= a\dot{x}_2 + \cos x_1 \cdot \dot{x}_1 \\ &= a(-x_2 \cos x_1 + u \cos(2x_1)) + \cos x_1(-2x_1 + ax_2 + \sin x_1) \\ &= -(z_2 - \sin z_1) \cos z_1 + ua \cos(2z_1) + (-2z_1 + z_2) \cos z_1 \\ &= (\sin z_1 - 2z_1) \cos z_1 + ua \cos(2z_1)\end{aligned}$$

Design $u = \frac{1}{a \cos(2z_1)}(v - (\sin z_1 - 2z_1) \cos z_1)$, we obtain

$$\begin{cases} \dot{z}_1 = -2z_1 + z_2 \\ \dot{z}_2 = v \end{cases}$$

For stabilization, use $v = -k_1 z_1 - k_2 z_2$, we can place the poles anywhere with proper choice of feedback control gains k_1 and k_2.

Remark 15

1. The result is not global since u is not well defined if $\cos(2z_1) = 0$.
2. In order to implement the control law, the new state components (z_1, z_2) must be available. That is, the full state (x_1, x_2) must be measured to compute u.

5.2.3 Input-Output Linearization

We now add an output y to the original system

$$\begin{cases} \dot{x} = f(x,u) \\ y = h(x) \end{cases}$$

Design u such that $y \to 0$. We want to find a direct and simple relation between y and u.

Consider the third-order system

$$\begin{aligned} \dot{x}_1 &= \sin x_2 + (x_2+1)x_3 \\ \dot{x}_2 &= x_1^5 + x_3 \\ \dot{x}_3 &= x_1^2 + u \\ y &= x_1 \end{aligned}$$

Differentiate the output

$$\dot{y} = \dot{x}_1 = \sin x_2 + (x_2+1)x_3$$

Since $\dot{y}$ is still not directly related to u, let us differentiate again, we obtain

$$\begin{aligned} \ddot{y} &= \dot{x}_2 \cos x_2 + \dot{x}_2 x_3 + (x_2+1)\dot{x}_3 \\ &= (x_1^5 + x_3)\cos x_2 + x_3(x_1^5 + x_3) + (x_2+1)(x_1^2+u) \\ &= (x_1^5 + x_3)(\cos x_2 + x_3) + (x_2+1)x_1^2 + u(x_2+1) \\ &\triangleq f_1(x) + u(x_2+1) \end{aligned}$$

Design $u = \frac{1}{1+x_2}(v - f_1(x))$, we have $\ddot{y} = v$.

Design $v = -k_1 y - k_2 \dot{y}$.

Remark 16

1. Full state (x_1, x_2, x_3) is still necessary to compute u.
2. Singular at $x_2 = -1$
3. Pick $v = -k_1 y - k_2 \dot{y}$, $k_1, k_2 > 0$ to stabilize y, but will x_2, x_3 be stable?

Remark 17 Limitation:

1. It cannot be used fo all nonlinear systems.
2. The full state must be measured.
3. Existence of singular points.

Remark 18 The basic philosophy of feedback linearization is to cancel the nonlinear terms of the system. However, we should examine the philosophy itself: Is it a good idea to cancel nonlinear term?

Example 27 $\dot{x} = ax - x^3 + u$

5.3 Sliding Mode Control

5.3.1 Motivating Example

Consider a simple second-order system

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = f(x) + g(x)u$$

where $f(x)$ and $g(x)$ are nonlinear functions, $g(x) \geq g_0 > 0$ for all x.

Objective: Design u to stabilize the origin, i.e., $x_1, x_2 \rightarrow \infty$, as $t \rightarrow \infty$.

Idea:

Define a sliding surface

$$s = x_2 + \lambda x_1 = 0 \quad with \lambda > 0$$

On the surface

$$s = 0 \Rightarrow \begin{cases} \dot{x}_1 = -\lambda x_1 \Rightarrow x_1(t) = x_1(0)e^{-\lambda t} \\ x_2 = -\lambda x_1 = -\lambda x_1(0)e^{-\lambda t} \end{cases}$$

$x_1(t)$ and $x_2(t)$ tend to zero as $t \rightarrow \infty$ and the rate of convergence can be controlled by the choice of λ. And the motion on $s = 0$ is independent of f and g.

1. Reaching phase: design u to force the state (x_1, x_2) to be on $s = 0$ in finite time;
2. Sliding phase: (x_1, x_2) slides to the origin.

The dynamics of s is $\dot{s} = \dot{x}_2 + \lambda \dot{x}_1 = f(x) + g(x)u + \lambda x_2$.
Design a Lyapunov function candidate $V = \frac{1}{2}s^2$.
Its derivative is $\dot{V} = s \cdot \dot{s} = s(f(x) + g(x)u + \lambda x_2)$.
Design $u = \frac{1}{g(x)}(-f(x) - \lambda x_2 - ksgn(s))$, $k > 0, sgn(s) = \begin{cases} 1 & s > 0 \\ 0 & s = 0 \\ -1 & s < 0 \end{cases}$.
We obtain $\dot{V} = -ks \cdot sgn(s) = -k|s| = -k\sqrt{2V} \Rightarrow \frac{\dot{V}}{\sqrt{V}} = -\sqrt{2}k \Rightarrow$

$$\int_0^t \frac{\dot{V}}{\sqrt{V}} dt = 2\left.\sqrt{V}\right|_0^t = 2\sqrt{V(t)} - 2\sqrt{V(0)} = -\sqrt{2}kt$$

$$\Rightarrow$$

$$V(t) \equiv 0, s(t) \equiv 0, \forall t \geq \frac{\sqrt{2V(0)}}{k}$$

Remark 19 By $s = x_2 + \lambda x_1 = \dot{x}_1 + \lambda x_1$, we have

$$\dot{x}_1 = -\lambda x_1 + s$$

where x_1 can be viewed as the state and s can be viewed as input.
Then by ISS, we have $s \rightarrow 0 \Rightarrow x_1 \rightarrow 0, x_2 \rightarrow 0$.

5.3.2 Tracking Problem

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = f(x) + g(x)u \end{cases}$$

Objective: state $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ to track $\begin{bmatrix} x_d \\ \dot{x}_d \end{bmatrix}$.

Define the tracking error $\tilde{x}_1 = x_1(t) - x_d(t), \tilde{x}_2 = x_2(t) - \dot{x}_d(t)$ and the sliding surface

$$s = \tilde{x}_2 + \lambda\tilde{x}_1 = 0.$$

$\Leftrightarrow \tilde{x}_2 = \dot{\tilde{x}}_1 = -\lambda\tilde{x}_1$ on the surface.

The dynamics of s is given by

$$\begin{aligned} \dot{s} &= \dot{\tilde{x}}_2 + \lambda\dot{\tilde{x}}_1 \\ &= \dot{x}_2 - \ddot{x}_d(t) + \lambda\tilde{x}_2 \\ &= f(x) + g(x)u - \ddot{x}_d(t) + \lambda\tilde{x}_2 \end{aligned}$$

Design a Lyapunov function candidate $V = \frac{1}{2}s^2$.

Its derivative is $\dot{V} = s \cdot \dot{s} = s(f(x) + g(x)u - \ddot{x}_d(t) + \lambda\tilde{x}_2)$.

Design $u = \frac{1}{g(x)}(-f(x) + \ddot{x}_d(t) - \lambda\tilde{x}_2 - ksgn(s))$, $k > 0$, yields $\dot{V} \leq -k|s|$.

5.3.3 Tracking With Uncertainties

Function $f(x)$ is not exactly known but can be characterized by an estimate $\hat{f}(x)$ and an error bound $F(x)$:

$$|\hat{f}(x) - f(x)| \leq F(x)$$

Function $g(x)$ is uncertain with $0 < g_{\min}(x) \leq g(x) \leq g_{\max}(x)$.

Define $\hat{g}(x) = \sqrt{g_{\min}(x)g_{\max}(x)}$, we have

$$\frac{1}{\beta} \triangleq \frac{\sqrt{g_{\min}(x)g_{\max}(x)}}{g_{\max}(x)} \leq \frac{\hat{g}(x)}{g(x)} \leq \frac{\sqrt{g_{\min}(x)g_{\max}(x)}}{g_{\min}(x)} = \sqrt{\frac{g_{\max}(x)}{g_{\min}(x)}} \triangleq \beta \geq 1$$

Design $u = \frac{1}{\hat{g}(x)}(-\hat{f}(x) + \ddot{x}_d(t) - \lambda\tilde{x}_2 - ksgn(s))$, $k > 0$, yields

$$\begin{aligned} \dot{V} &= s\left\{f(x) - \ddot{x}_d(t) + \lambda\tilde{x}_2 + \frac{g(x)}{\hat{g}(x)}[-\hat{f}(x) + \ddot{x}_d(t) - \lambda\tilde{x}_2 - ksgn(s)]\right\} \\ &= s\left\{[f(x) - \hat{f}(x)] + \left(\frac{g(x)}{\hat{g}(x)} - 1\right)(-\hat{f}(x) + \ddot{x}_d(t) - \lambda\tilde{x}_2) - k\frac{g(x)}{\hat{g}(x)}sgn(s)\right\} \\ &\leq |s|\left\{F(x) + \left|\frac{g(x)}{\hat{g}(x)} - 1\right| \cdot |-\hat{f}(x) + \ddot{x}_d(t) - \lambda\tilde{x}_2| - k\frac{g(x)}{\hat{g}(x)}\right\} \\ &= -|s|\left\{k\frac{g(x)}{\hat{g}(x)} - F(x) - \left|\frac{g(x)}{\hat{g}(x)} - 1\right| \cdot |-\hat{f}(x) + \ddot{x}_d(t) - \lambda\tilde{x}_2|\right\} \end{aligned}$$

Design k such that

$$k\frac{g(x)}{\hat{g}(x)} \geq F(x) + \left|\frac{g(x)}{\hat{g}(x)} - 1\right| \cdot |-\hat{f}(x) + \ddot{x}_d(t) - \lambda\tilde{x}_2| + \eta, \eta > 0$$

$$\Leftrightarrow$$

$$k \geq \frac{\hat{g}(x)}{g(x)}[F(x) + \eta] + \left|1 - \frac{\hat{g}(x)}{g(x)}\right| \cdot |-\hat{f}(x) + \ddot{x}_d(t) - \lambda\tilde{x}_2|$$

Note that $\frac{1}{\beta} \leq \frac{\hat{g}(x)}{g(x)} \leq \beta, \beta > 1$.

Choose $k = \beta[F(x) + \eta] + (\beta - 1) \cdot |-\hat{f}(x) + \ddot{x}_d(t) - \lambda\tilde{x}_2|$, then we have $\dot{V} \leq -\eta|s|$.

Example 28

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = u + d(t)$$

where $d(t)$ is a external disturbance with $|d(t)| \leq d_{\max}$.

Design a sliding surface: $s = x_2 + \lambda x_1, \lambda > 0$.

The dynamics of s is $\dot{s} = \dot{x}_2 + \lambda\dot{x}_1 = u + d(t) + \lambda x_2$.

Design a Lyapunov function candidate $V = \frac{1}{2}s^2$.

Its derivative is $\dot{V} = s \cdot \dot{s} = s(u + d(t) + \lambda x_2)$.

Design $u = -\lambda x_2 - ksgn(s)$, $k > 0$, yields

$$\begin{aligned}\dot{V} &= s(d(t) - ksgn(s)) \\ &= -k|s| + sd(t) \\ &\leq -(k - d_{\max})|s|\end{aligned}$$

Choose $k = d_{\max} + \eta, \eta > 0$, then we have $\dot{V} \leq -\eta|s|$.

Example 29 If $d_{\max}$ is unknown, we can design

$$u = -\lambda x_2 - \hat{k}sgn(s)$$

Denote $\tilde{k} = \hat{k} - d_{\max} - \eta$.

Consider the following Lyapunov function candidate

$$V = \frac{1}{2}s^2 + \frac{1}{2\gamma}\tilde{k}^2, \gamma > 0,$$

Its derivative is

$$\begin{aligned}\dot{V} &= s\dot{s} + \frac{1}{\gamma}\tilde{k}\dot{\hat{k}} \\ &\leq -(\hat{k} - d_{\max})|s| + \frac{1}{\gamma}\tilde{k}\dot{\hat{k}} \\ &= -(\hat{k} - d_{\max} - \eta)|s| - \eta|s| + \frac{1}{\gamma}\tilde{k}\dot{\hat{k}} \\ &= \frac{1}{\gamma}\tilde{k}(\dot{\hat{k}} - \gamma|s|) - \eta|s|\end{aligned}$$

Design $\dot{\hat{k}} = \gamma|s|$, we obtain $\dot{V} \leq -\eta|s|$. Integrating both sides yields

$$V(t) - V(0) \leq -\eta \int_0^t |s| d\tau$$

$\Rightarrow s \in \mathbb{L}_1$.

By $s \in \mathbb{L}_1 \cap \mathbb{L}_\infty, \dot{s} \in \mathbb{L}_\infty$, we have $\lim_{t\to\infty} s = 0$.

Question 6 ⋆⋆⋆⋆⋆

$$\|d(t)\| \leq d_{\max}$$

$$\|d(t)\| \leq d_{\max}(1 + \|x_1\| + \|x_2\|^2)$$

$d_{\max}$ is unknown.

5.4 Backstepping

5.4.1 Basic Backstepping

Consider the system

$$\dot{z} = f(z) + g(z)\xi$$
$$\dot{\xi} = u$$

where $[z^T, \xi^T]^T \in \mathbb{R}^{n+1}$ is the state and $u \in \mathbb{R}$ is the control input. The functions of f, g are smooth in a domain D containing $z = 0$ and $f(0) = 0$.

We want to design a state feedback control law to stabilize the origin. We view this as a cascade connection of two components.

Suppose that we can asymptotically stabilize the first system

$$\dot{z} = f(z) + g(z)\xi$$

by a state feedback control law $\xi = \Phi(z)$, with $\Phi(0) = 0$. This implies that the origin of

$$\dot{z} = f(z) + g(z)\Phi(z)$$

is asymptotically stable.

Suppose that we know a Lyapunov function $V(z)$ that satisfies

$$\dot{V} = \frac{\partial V}{\partial z}[f(z) + g(z)\Phi(z)] \leq -\omega(z)$$

where $\omega(z)$ is positive definite.

Add and subtract $g(z)\Phi(z)$ to the original system

$$\dot{z} = f(z) + g(z)\Phi(z) + g(z)(\xi - \Phi(z))$$
$$\dot{\xi} = u$$

We now introduce the change of variable

$$y = \xi - \Phi(z)$$

We can get the following system

$$\begin{aligned}\dot{z} &= f(z) + g(z)\Phi(z) + g(z)y \\ \dot{y} &= u - \dot{\Phi}(z)\end{aligned}$$

This change of variable is often called **backstepping** since it "back-steps" the control $-\Phi(z)$ though the integrator.

Since f, g and Φ are known

$$\dot{\Phi}(z) = \frac{\partial \Phi}{\partial z}\dot{z} = \frac{\partial \Phi}{\partial z}(f(z) + g(z)\xi)$$

Letting $v = u - \dot{\Phi}(z)$ reduces our system to

$$\begin{aligned}\dot{z} &= f(z) + g(z)\Phi(z) + g(z)y \\ \dot{y} &= v\end{aligned}$$

which has the same "form" as the system we started with the exception that we know the first component is asymptotically stable at the origin when y is zero.

Consider a Lyapunov function candidate

$$V_c(z, y) = V(z) + \frac{1}{2}y^2$$

Its derivative is given by

$$\begin{aligned}\dot{V}_c(z, y) &= V(z) + y \cdot \dot{y} \\ &= \frac{\partial V}{\partial z}[f(z) + g(z)\Phi(z) + g(z)y] + yv \\ &\le -\omega(z) + \frac{\partial V}{\partial z}g(z)y + yv\end{aligned}$$

Design $v = -\frac{\partial V}{\partial z}g(z) - ky, k > 0$. We obtain $\dot{V}(z, y) \le -\omega(z) - ky^2$, (ND)

which implies that the origin $(z = 0, y = 0)$ is asymptotically stable. Since $\Phi(0) = 0$, we conclude that the origin $(z = 0, \xi = 0)$ is asymptotically stable. And the state feedback control law is

$$\begin{aligned}u &= v + \dot{\Phi} \\ &= -\frac{\partial V}{\partial z}g(z) - ky + \frac{\partial \Phi}{\partial z}(f(z) + g(z)\xi) \\ &= -\frac{\partial V}{\partial z}g(z) - k(\xi - \Phi(z)) + \frac{\partial \Phi}{\partial z}(f(z) + g(z)\xi)\end{aligned}$$

Example 30 Consider the system

$$\begin{aligned}\dot{x}_1 &= x_1^2 - x_1^3 + x_2 \\ \dot{x}_2 &= u\end{aligned}$$

we start with the system

$$\dot{x}_1 = x_1^2 - x_1^3 + x_2$$

with x_2 viewed as the input and proceed to design a feedback control $x_2 = \Phi(x_1)$ to stabilize the origin $x_1 = 0$.

Design $x_2 = \Phi(x_1) = -x_1^2$, we have $\dot{x}_1 = -x_1^3$. Consider $V(x_1) = \frac{1}{2}x_1^2$, whose derivative is

$$\dot{V}(x_1) = x_1\dot{x}_1 = -x_1^4 \quad ND$$

To backstep, we use the change of variable

$$y = x_2 - \Phi(x_1) = x_2 + x_1^2$$

Then the original system can be transformed into the form

$$\begin{aligned}\dot{x}_1 &= -x_1^3 + y\\ \dot{y} &= v\end{aligned}$$

where $v = u + 2x_1(x_1^2 - x_1^3 + x_2)$.

We then consider the following combined Lyapunov function candidate

$$V_c = \frac{1}{2}x_1^2 + \frac{1}{2}y^2$$

Its derivative is

$$\begin{aligned}\dot{V}_c &= x_1\dot{x}_1 + y\dot{y}\\ &= -x_1^4 + x_1 y + yv\end{aligned}$$

Design $v = -x_1 - ky, k > 0$. Then $\dot{V}_c = -x_1^4 - ky^2$ is ND. $\Rightarrow x_1, y \to 0$.

The control input is designed as follows

$$u = v - 2x_1(x_1^2 - x_1^3 + x_2) = -x_1 - ky - 2x_1(x_1^2 - x_1^3 + x_2)$$

Remark 20

$$\begin{aligned}\dot{x}_1 &= x_1^2 - x_1^3 + x_2\\ x_2 &= -x_1^2 + x_1x_2 + \cdots +\\ \dot{x}_3 &= u + x_3x_1 + \cdots +\end{aligned}$$

5.4.2 Adaptive Backstepping

Consider the following system

$$\begin{aligned}\dot{x}_1 &= x_2 + \varphi(x_1)\theta\\ \dot{x}_2 &= u\end{aligned}$$

where θ is an unknown constant, and $\varphi(x_1)$ is a bounded function with $\varphi(0) = 0$.

Assume that θ is known. If x_2 were the control input, it should be designed as

$$x_2 = -k_1 x_1 - \varphi(x_1)\theta \triangleq \alpha_1(x_1), k_1 > 0$$

Since θ is unknown, we can still use the idea of backstepping.

Step 1:

Define $z_1 = x_1$. Under the condition that θ is unknown, if x_2 were the control input, an adaptive controller would be given

$$\begin{aligned} \alpha_1(x_1, v_1) &= -k_1 z_1 - \varphi(z_1) v_1 \\ \dot{v}_1 &= \gamma_1 z_1 \varphi(z_1) \end{aligned}$$

In the above equation, we have replaced the parameter estimate $\hat{\theta}$ with v_1, which denotes the estimate generated in the first design step. There will be another estimate generated in the next step.

Define the change of variable

$$z_2 \triangleq x_2 - \alpha_1(x_1, v_1)$$

we then have the following dynamics

$$\begin{aligned} \dot{z}_1 &= \dot{x}_1 \\ &= x_2 + \varphi(z_1)\theta \\ &= x_2 - \alpha_1(x_1, v_1) + \alpha_1(x_1, v_1) + \varphi(z_1)\theta \\ &= z_2 - k_1 z_1 - \varphi(z_1)(v_1 - \theta) \end{aligned}$$

Consider the following Lyapunov function candidate

$$V_1(z_1, v_1) = \frac{1}{2} z_1^2 + \frac{1}{2\gamma_1}(v_1 - \theta)^2$$

Its derivative is

$$\begin{aligned} \dot{V}_1 &= z_1 \dot{z}_1 + \frac{1}{\gamma_1}(v_1 - \theta)\dot{v}_1 \\ &= z_1(z_2 - k_1 z_1 - \varphi(z_1)(v_1 - \theta)) + \frac{1}{\gamma_1}(v_1 - \theta)\gamma_1 z_1 \varphi(z_1) \\ &= -k_1 z_1^2 + z_1 z_2 \end{aligned}$$

Step 2:

The dynamics of z_2 is now expressed as

$$\begin{aligned} \dot{z}_2 &= \dot{x}_2 - \dot{\alpha}_1(z_1, v_1) \\ &= u - \frac{\partial \alpha_1}{\partial z_1}\dot{z}_1 - \frac{\partial \alpha_1}{\partial v_1}\dot{v}_1 \\ &= u - \frac{\partial \alpha_1}{\partial z_1}(x_2 + \varphi(z_1)\theta) - \frac{\partial \alpha_1}{\partial v_1}\gamma_1 z_1 \varphi(z_1) \\ &= u - \frac{\partial \alpha_1}{\partial z_1} x_2 - \frac{\partial \alpha_1}{\partial v_1}\gamma_1 z_1 \varphi(z_1) - \theta \frac{\partial \alpha_1}{\partial z_1}\varphi(z_1) \end{aligned}$$

At this point, we need to design u and a Lyapunov function candidate such that the closed-loop system is asymptotically stable. One first attempt is the combined Lyapunov function candidate

$$V_c(z_1, z_2, v_1) = V_1(z_1, v_1) + \frac{1}{2}z_2^2$$

whose derivative is

$$\begin{aligned}\dot{V}_c &= \dot{V}_1 + z_2\dot{z}_2 \\ &= -k_1z_1^2 + z_1z_2 + z_2\left(u - \frac{\partial\alpha_1}{\partial z_1}x_2 - \frac{\partial\alpha_1}{\partial v_1}\gamma_1z_1\varphi(z_1) - \theta\frac{\partial\alpha_1}{\partial z_1}\varphi(z_1)\right)\end{aligned}$$

To deal with the term containing θ, we will try to employ the existing estimate v_1.

$$u = -z_1 - k_2 + z_2\frac{\partial\alpha_1}{\partial z_1}x_2 + \frac{\partial\alpha_1}{\partial v_1}\gamma_1z_1\varphi(z_1) + v_1\frac{\partial\alpha_1}{\partial z_1}\varphi(z_1), k_2 > 0$$

we then can get

$$\dot{V}_c = -k_1z_1^2 - k_2z_2^2 + z_2(v_1 - \theta)\frac{\partial\alpha_1}{\partial z_1}\varphi(z_1)$$

There is no design freedom left to cancel the $(v_1 - \theta)$ term. To overcome this difficulty, we replace v_1 with a new estimate v_2.

$$u = -z_1 - k_2 + z_2\frac{\partial\alpha_1}{\partial z_1}x_2 + \frac{\partial\alpha_1}{\partial v_1}\gamma_1z_1\varphi(z_1) + v_2\frac{\partial\alpha_1}{\partial z_1}\varphi(z_1), k_2 > 0$$

The presence of the new estimate v_2 suggests the following Lyapunov function candidate

$$V_c = V_1(z_1, v_1) + \frac{1}{2}z_2^2 + \frac{1}{2\gamma_2}(v_2 - \theta)^2$$

Its derivative is

$$\dot{V}_c = -k_1z_1^2 - k_2z_2^2 + z_2(v_2 - \theta)\frac{\partial\alpha_1}{\partial z_1}\varphi(z_1) + \frac{1}{\gamma_2}(v_2 - \theta)\dot{v}_2$$

Choose $\dot{v}_2 = -\gamma_2z_2\frac{\partial\alpha_1}{\partial z_1}\varphi(z_1)$ which yields $\dot{V}_c = -k_1z_1^2 - k_2z_2^2$, NSD.

Integrating both sides yields

$$k_1\int_0^t z_1^2d\tau + k_2\int_0^t z_2^2d\tau = V_c(0) - V_c(t) \leq V_c(0)$$

Note that the closed-loop adaptive system is

$$\begin{aligned}\dot{z}_1 &= -k_1z_1 + z_2 - (v_1 - \theta)\varphi(z_1) \\ \dot{z}_2 &= -z_1 - k_2z_2 + (v_2 - \theta)\frac{\partial\alpha_1}{\partial z_1}\varphi(z_1) \\ \dot{v}_1 &= \gamma_1z_1\varphi(z_1) \\ \dot{v}_2 &= -\gamma_2z_2\frac{\partial\alpha_1}{\partial z_1}\varphi(z_1)\end{aligned}$$

The matrix form is

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} -k_1 & 1 \\ -1 & -k_2 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} -\varphi & 0 \\ 0 & \frac{\partial \alpha_1}{\partial z_1}\varphi \end{bmatrix} \begin{bmatrix} v_1 - \theta \\ v_2 - \theta \end{bmatrix}$$
$$\begin{bmatrix} \dot{v}_1 \\ \dot{v}_2 \end{bmatrix} = - \begin{bmatrix} \gamma_1 & 0 \\ 0 & \gamma_2 \end{bmatrix} \begin{bmatrix} -\varphi & 0 \\ 0 & \frac{\partial \alpha_1}{\partial z_1}\varphi \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

We can get $z_1, z_2 \in \mathbb{L}_2 \cap \mathbb{L}_\infty, \dot{z}_1, \dot{z}_2 \in \mathbb{L}_\infty$.

From Barbalat's Lemma, $\lim_{t\to\infty} z_1 = \lim_{t\to\infty} z_2 = 0$.

Note that $z_1 = x_1, x_2 = z_2 - \alpha_1(x_1, v_1)$ and $\varphi(0) = 0$.

We can conclude that $\lim_{t\to\infty} x_1 = \lim_{t\to\infty} x_2 = 0$.

Question 7

$$\begin{cases} \dot{x}_1 = x_2 + \theta_1\varphi_1(x_1) \\ \dot{x}_2 = u + \theta_2\varphi_2(x_2) \end{cases}$$

Design u (cancel known extra term or use adaptive for unknown extra term)

Reducing the overparametrization

Step 1:

Define $z_1 = x_1$ and view x_2 as the virtual control input

$$x_2 = -k_1x_1 - \varphi(x_1)\hat{\theta} \triangleq \alpha_1(x_1, \hat{\theta})$$

Define $z_2 = x_2 - \alpha_1(x_1, \hat{\theta})$. And we have

$$\begin{aligned} \dot{z}_1 &= x_2 + \varphi\theta \\ &= x_2 - \alpha_1(x_1, \hat{\theta}) + \alpha_1(x_1, \hat{\theta}) + \varphi\theta \\ &= z_2 - k_1z_1 - \varphi\tilde{\theta} \end{aligned}$$

where $\tilde{\theta} \triangleq \hat{\theta} - \theta$.

Step 2:

The dynamic of z_2 is now expressed as

$$\begin{aligned} \dot{z}_2 &= \dot{x}_2 - \dot{\alpha}_1(x_1, \hat{\theta}) \\ &= u - \frac{\partial \alpha_1}{\partial z_1}\dot{z}_1 - \frac{\partial \alpha_1}{\partial \hat{\theta}}\dot{\hat{\theta}} \\ &= u - \frac{\partial \alpha_1}{\partial z_1}(z_2 - k_1z_1 - \varphi\tilde{\theta}) + \varphi(z_1)\dot{\hat{\theta}} \end{aligned}$$

We consider the combined Lyapunov function candidate

$$V_c = \frac{1}{2}z_1^2 + \frac{1}{2}z_2^2 + \frac{1}{2\gamma}\tilde{\theta}^2$$

Its derivative is given as

$$\begin{aligned}
\dot{V}_c &= z_1\dot{z}_1 + z_2\dot{z}_2 + \frac{1}{\gamma}\tilde{\theta}\dot{\hat{\theta}} \\
&= z_1(z_2 - k_1z_1 - \varphi\tilde{\theta}) + z_2\left(u - \frac{\partial\alpha_1}{\partial z_1}(z_2 - k_1z_1 - \varphi\tilde{\theta}) + \varphi\dot{\hat{\theta}}\right) + \frac{1}{\gamma}\tilde{\theta}\dot{\hat{\theta}} \\
&= -k_1z_1^2 + \frac{\tilde{\theta}}{\gamma}\left(\dot{\hat{\theta}} - \gamma\varphi z_1 + \gamma z_2\varphi\frac{\partial\alpha_1}{\partial z_1}\right) + z_2\left(u - \frac{\partial\alpha_1}{\partial z_1}z_2 + \frac{\partial\alpha_1}{\partial z_1}k_1z_1 + z_1 + \varphi\dot{\hat{\theta}}\right)
\end{aligned}$$

Then we choose

$$\dot{\hat{\theta}} = \gamma\varphi z_1 - \gamma z_2\varphi\frac{\partial\alpha_1}{\partial z_1}$$

$$u = \frac{\partial\alpha_1}{\partial z_1}z_2 - \frac{\partial\alpha_1}{\partial z_1}k_1z_1 - z_1 - \varphi\dot{\hat{\theta}} - k_2z_2, k_2 > 0$$

Then we obtain

$$\dot{V}_c = -k_1z_1^2 - k_2z_2^2$$

Note that the closed-loop adaptive system is

$$\begin{aligned}
\dot{z}_1 &= -k_1z_1 + z_2 - \tilde{\theta}\varphi(z_1) \\
\dot{z}_2 &= -z_1 - k_2z_2 + \tilde{\theta}\frac{\partial\alpha_1}{\partial z_1}\varphi(z_1) \\
\dot{\hat{\theta}} &= \gamma\varphi z_1 - \gamma z_2\varphi\frac{\partial\alpha_1}{\partial z_1}
\end{aligned}$$

The matrix form is

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} -k_1 & 1 \\ -1 & -k_2 \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} -\varphi \\ \frac{\partial\alpha_1}{\partial z_1}\varphi \end{bmatrix}\tilde{\theta}$$

$$\dot{\hat{\theta}} = -\gamma\begin{bmatrix} -\varphi & \frac{\partial\alpha_1}{\partial z_1}\varphi \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

6 Application of Robotic Manipulators

Lemma 15 *If $A = -A^T$, then $x^TAx = 0$.*

6.1 Euler-Lagrange Equation

A dynamical system with p degrees of freedom can be described by the EL equations as

$$M(q)\ddot{q} + C(q, \dot{q})\dot{q} + g(q) = \tau$$

where $q \in \mathbb{R}^p$ is the vector of generalized coordinates, $M(q) \in \mathbb{R}^{p\times p}$ is the symmetric positive definite inertia matrix, $C(q, \dot{q}) \in \mathbb{R}^p$ is the vector of Coriolis and centrifugal forces, $g(q)$ is the vector of gravitational force, and $\tau \in \mathbb{R}^p$ is the vector of control force.

Properties

1. $M(q)$ is positive definite and $k_{\underline{m}} x^T x \leq x^T M x \leq k_{\bar{m}} x^T x, \|C(x,y)z\| \leq k_c \|y\| \|z\|$.
2. $\dot{M}(q) - 2C(q,\dot{q})$ is skew symmetric.
3. $M(q)y + C(q,\dot{q})x + g(q) = Y(q,\dot{q},y,x)\Theta$, where $Y(q,\dot{q},y,x)$ is the regressor and Θ is an unknown but constant vector.

6.2 Position Control

Consider the following system for a robotic manipulator

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} = u \tag{53}$$

Control objective: $q \to q_d, \dot{q}_d = 0$.

Define the position error: $\tilde{q} = q - q_d, \dot{\tilde{q}} = \dot{q}, \ddot{\tilde{q}} = \ddot{q}$.
The error dynamics is

$$M(q)\ddot{\tilde{q}} + C(q,\dot{q})\dot{\tilde{q}} = u$$

$$\Updownarrow$$

$$M(\tilde{q} + q_d)\ddot{\tilde{q}} + C(\tilde{q} + q_d, \dot{\tilde{q}})\dot{\tilde{q}} = u$$

Note 4 Define $x_1 = \tilde{q}, x_2 = \dot{\tilde{q}}$, it has

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= [M(x_1 + q_d)]^{-1}[u - C(x_1 + q_d, x_2)x_2]
\end{aligned}$$

Design the following control input

$$u = -K_p\tilde{q} - K_d\dot{\tilde{q}} = -K_p(q - q_d) - K_d\dot{q}$$

where K_p and K_d are positive definite matrices.
Then the closed-loop system is

$$M(\tilde{q} + q_d)\ddot{\tilde{q}} + C(\tilde{q} + q_d, \dot{\tilde{q}})\dot{\tilde{q}} = -K_p\tilde{q} - K_d\dot{\tilde{q}}$$

Consider the following Lyapunov function candidate

$$V_1 = \frac{1}{2}\dot{\tilde{q}}^T M \dot{\tilde{q}}$$

The derivative of V_1 is

$$\begin{aligned}
\dot{V}_1 &= \dot{\tilde{q}}^T M \ddot{\tilde{q}} + \frac{1}{2}\dot{\tilde{q}}^T \dot{M} \dot{\tilde{q}} \\
&= \dot{\tilde{q}}^T(-K_p\tilde{q} - K_d\dot{\tilde{q}} - C\dot{\tilde{q}}) + \frac{1}{2}\dot{\tilde{q}}^T \dot{M} \dot{\tilde{q}} \\
&= -\dot{\tilde{q}}^T K_p \tilde{q} - \dot{\tilde{q}}^T K_d \dot{\tilde{q}} + \frac{1}{2}\dot{\tilde{q}}^T(\dot{M} - 2C)\dot{\tilde{q}}
\end{aligned}$$

Since $(\dot{M} - 2C)$ is skew symmetric, we have

$$\dot{V}_1 = -\dot{\tilde{q}}^T K_p \tilde{q} - \dot{\tilde{q}}^T K_d \dot{\tilde{q}}$$

We consider a combined Lyapunov function candidate

$$V = V_1 + \frac{1}{2}\tilde{q}^T K_p \tilde{q}$$

Then we have

$$\begin{aligned}\dot{V} &= \dot{V}_1 + \dot{\tilde{q}}^T K_p \tilde{q} \\ &= -\dot{\tilde{q}}^T K_d \dot{\tilde{q}} - \dot{\tilde{q}}^T K_p \tilde{q} + \tilde{q}^T K_p \dot{\tilde{q}} \\ &= -\dot{\tilde{q}}^T K_d \dot{\tilde{q}}\end{aligned}$$

NSD $\Rightarrow$ stable.

Note that the closed-loop system is autonomous, we have

$$E \triangleq \{x | \dot{V} = 0\} = \{x | x_2 \equiv 0\}$$

Let $x(t)$ be a solution that belongs identically to E.

$$x_2(t) \equiv 0 \Rightarrow \dot{x}_2(t) \equiv 0 \Rightarrow x_1(t) \equiv 0$$

Therefore, the only solution that can stay identically to E is the origin. Thus, from LaSalle's Theorem, the origin is asymptotically stable, i.e., $\lim_{t\to\infty} q(t) = q_d, \lim_{t\to\infty} \dot{q}(t) = 0$.

Question 8 If the system is $M(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q) = u$, then choose

$$u = g(q) - K_p \tilde{q} - K_d \dot{\tilde{q}}$$

If the system is $M(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q) + D\dot{q} = u$, D is PSD (It is useful, don't cancel).

6.3 Tracking Control

Objective: $q(t) \to q_d(t), \dot{q}(t) \to \dot{q}_d(t)$. $q_d(t), \dot{q}_d(t), \ddot{q}_d(t)$ are bounded.

Define the tracking error: $\tilde{q}(t) = q(t) - q_d(t), \dot{\tilde{q}}(t) = \dot{q}(t) - \dot{q}_d(t)$.

The error dynamics:

$$M\ddot{\tilde{q}} + C\dot{\tilde{q}} = u - M\ddot{q}_d - C\dot{q}_d$$

Design the following controller

$$u = M\ddot{q}_d - C\dot{q}_d - K_p \tilde{q} - K_d \dot{\tilde{q}}$$

Then the closed-loop system is

$$M(\tilde{q} + q_d(t))\ddot{\tilde{q}} + C(\tilde{q} + q_d(t), \dot{\tilde{q}} + \dot{q}_d(t))\dot{\tilde{q}} = -K_p \tilde{q} - K_d \dot{\tilde{q}} \tag{54}$$

Actually, (54) is non-autonomous due to the presence of $q_d(t)$ and $\dot{q}_d(t)$.

Consider the following Lyapunov function candidate

$$V = \frac{1}{2}\dot{\tilde{q}}^T M \dot{\tilde{q}} + \frac{1}{2}\tilde{q}^T K_p \tilde{q}$$

Its derivative along (54) is also $\dot{V} = -\dot{\tilde{q}}^T K_d \dot{\tilde{q}} \leq 0$.

Then we have $V(t) \leq V(0)$, which implies that $\dot{\tilde{q}}, \tilde{q} \in \mathbb{L}_\infty$.

Note that $\ddot{V} = -2\dot{\tilde{q}}^T k_d \ddot{\tilde{q}} \in \mathbb{L}_\infty$.

From Barbalat's Lemma, $\lim_{t\to\infty} \dot{V}(t) = 0$, i.e., $\lim_{t\to\infty} \dot{\tilde{q}} = 0$.

Unfortunately, from the study sketched above, it is not possible to derive any immediate conclusion about the asymptotic behavior of the position error $\tilde{q}$.

6.4 Sliding Mode Control for Tracking Problem

Define a sliding surface

$$s = \dot{\tilde{q}} + \lambda\tilde{q} = \dot{q} - (\dot{q}_d - \lambda\tilde{q}) = \dot{q} - \dot{q}_r, \lambda > 0$$

where $\dot{q}_r$ is an auxiliary variable.

Then we have

$$M(\dot{s} + \ddot{q}_r) + C(s + \dot{q}_r) = u$$

or

$$M\dot{s} + Cs = u - M\ddot{q}_r - C\dot{q}_r \tag{55}$$

Define the following control input

$$u = -Ks + M\ddot{q}_r + C\dot{q}_r \tag{56}$$

where K is positive definite. Then the closed-loop system is

$$M\dot{s} + Cs = -Ks$$

Consider the following Lyapunov function candidate

$$V = \frac{1}{2}s^T M s$$

Its derivative is

$$\begin{aligned}\dot{V} &= s^T M\dot{s} + \frac{1}{2}s^T \dot{M} s \\ &= -s^T(Cs + Ks) + \frac{1}{2}s^T\dot{M}s \\ &= -s^T Ks + \frac{1}{2}s^T(\dot{M} - 2C)s \\ &= -s^T Ks\end{aligned}$$

Therefore, the origin $s = 0$ is globally uniformly exponentially stable, i,e, $\lim_{t\to\infty} s(t) = 0$.

Note that $s = \dot{\tilde{q}} + \lambda\tilde{q}$ is ISS with respect to the input s and the state $\tilde{q}$. It follows the fact $\lim_{t\to\infty} s(t) = 0$ that $\lim_{t\to\infty} \tilde{q}(t) = \lim_{t\to\infty} \dot{\tilde{q}}(t) = 0$.

Question 9 ⋆⋆⋆⋆⋆⋆

If there exist external disturbance in the system, for example

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} = u + d(t)$$

where $\|d(t)\| \leq d_{\max}$. We may use $u = -ksgn(s) + M\ddot{q}_r + C\dot{q}_r$.

6.5 Adaptive Sliding Mode Control for Tracking Problem

When there exist parametric uncertainties, (56) cannot be implemented.

Note that $M(q)\ddot{q}_r + C(q,\dot{q})\dot{q}_r = Y(q,\dot{q},\ddot{q}_r,\dot{q}_r)\Theta$.

Since Θ is unknown, we propose the following controller

$$u = -Ks + Y(q,\dot{q},\ddot{q}_r,\dot{q}_r)\hat{\Theta} \tag{57}$$

Using (57) for (55), we have the following closed-loop system

$$M\dot{s} + Cs = -Ks + Y\tilde{\Theta} \tag{58}$$

where $\tilde{\Theta} = \hat{\Theta} - \Theta$.

Consider the following Lyapunov function candidate

$$V = \frac{1}{2}s^T M s + \frac{1}{2}\tilde{\Theta}^T \Gamma^{-1}\tilde{\Theta}$$

where Γ is PD.

Its derivative along (58) is

$$\begin{aligned}\dot{V} &= -s^T K s + s^T Y\tilde{\Theta} + \tilde{\Theta}^T\Gamma^{-1}\dot{\hat{\Theta}} \\ &= -s^T K s + \tilde{\Theta}^T\Gamma^{-1}\Gamma Y^T s + \tilde{\Theta}^T\Gamma^{-1}\dot{\hat{\Theta}} \\ &= -s^T K s + \tilde{\Theta}^T\Gamma^{-1}(\Gamma Y^T s + \dot{\hat{\Theta}})\end{aligned}$$

Then design $\dot{\hat{\Theta}} = -\Gamma Y^T s$.

We have

$$\dot{V} = -s^T K s \leq 0 \tag{59}$$

which implies that $V(t) \leq V(0)$, $s, \tilde{\Theta} \in \mathbb{L}_\infty$.

Integrating both sides of (59),

$$V(t) - V(0) = -\int_0^t s^T K s d$$

$$\Rightarrow$$

$$\int_0^t s^T K s d = V(0) - V(t) \leq V(0) \in \mathbb{L}_\infty$$

$$\Rightarrow$$

$$s \in \mathbb{L}_2$$

From $\dot{\tilde{q}} = -\lambda\tilde{q} + s$ and the ISS stability, $\tilde{q}, \dot{\tilde{q}} \in \mathbb{L}_\infty$.

Since $q_d, \dot{q}_d$ and $\ddot{q}_d$ are all bounded, $q, \dot{q} \in \mathbb{L}_\infty$.

We can get from the properties of EL equation that $\dot{s} \in L_\infty$.

So far, we have $s \in \mathbb{L}_2 \cap \mathbb{L}_\infty$ and $\dot{s} \in \mathbb{L}_\infty$.

From Barbalat's Lemma, $\lim_{t\to\infty} s(t) = 0$.

From the ISS stability, $\lim_{t\to\infty} \tilde{q}(t) = \lim_{t\to\infty} \dot{\tilde{q}}(t) = 0$.

6.6 Backstepping Sliding Mode Control for Tracking Problem

Consider the following dynamics of a robotic manipulator

$$M\ddot{q} + C\dot{q} = K(\theta - q)$$
$$J\ddot{\theta} = -K(\theta - q) + u$$

where $q \in \mathbb{R}^p$ and $\theta \in \mathbb{R}^p$ represent respectively, the vector of link positions and motor angles, K is the positive diagonal matrix representing the point stiffness, and J is the positive diagonal matrix representing the actuator inertia.

Define $x_1 = \theta, x_2 = \dot{\theta}$. The dynamics can be described as

$$\begin{aligned} M\ddot{q} + C\dot{q} + Kq &= Kx_1 \\ \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -J^{-1}K(x_1 - q) + J^{-1}u \end{aligned} \tag{60}$$

Objective: $q(t) \to q_d(t), \dot{q}(t) \to \dot{q}_d(t)$. $q_d(t), \dot{q}_d(t), \ddot{q}_d(t)$ are bounded.

Define

$$\tilde{q}(t) = q(t) - q_d(t), \dot{\tilde{q}}(t) = \dot{q}(t) - \dot{q}_d(t)$$
$$\dot{q}_r = \dot{q}_d - \lambda\tilde{q}, s = \dot{q} - \dot{q}_r = \dot{\tilde{q}} + \lambda\tilde{q}$$

then (60) can be written as

$$M\dot{s} + Cs = Kx_1 - Kq - M\ddot{q}_r - C\dot{q}_r$$

Step 1:

Design a feedback control $x_1 = \Phi_1$ to stabilize the origin $s = 0$.

$$\Phi_1 = K^{-1}[M\ddot{q}_r + C\dot{q}_r - K_1 s] + q$$

where K_1 is positive definite.

We have

$$M\dot{s} + Cs = -K_1 s$$

Take $V_1 = \frac{1}{2}s^T M s \Rightarrow$

$$\dot{V}_1 = -s^T K_1 s.$$

Step 2:

To backstep, define $y_1 = x_1 - \Phi_1$, we have

$$M\dot{s} + Cs = -K_1 s + K y_1 \tag{61}$$

Note that the dynamic of y_1 is

$$\dot{y}_1 = \dot{x}_1 - \dot{\Phi}_1 = x_2 - \dot{\Phi}_1$$

Consider a combined Lyapunov function candidate

$$V_2 = V_1 + \frac{1}{2}y_1^T K y_1$$

Its derivative along (61) is

$$
\begin{aligned}
\dot{V}_2 &= -s^T K_1 s + s^T K y_1 + y_1^T K \dot{y}_1 \\
&= -s^T K_1 s + y_1^T K(\dot{y}_1 + s) \\
&= -s^T K_1 s + y_1^T K(x_2 - \dot{\Phi}_1 + s)
\end{aligned}
$$

Design a feedback control $x_2 = \Phi_2$ to stabilize the origin $(s, y_1) = (0, 0)$

$$\Phi_2 = -y_1 + \dot{\Phi}_1 - s$$

Then we have

$$\dot{V}_2 = -s^T K_1 s - y_1^T K y_1$$

Step 3:

To backstep, define $y_2 = x_2 - \Phi_2$, we have

$$\dot{y}_2 = \dot{x}_2 - \dot{\Phi}_2 = -J^{-1}K(x_1 - q) + J^{-1}u - \dot{\Phi}_2$$

Consider a combined Lyapunov function candidate

$$V_3 = V_2 + \frac{1}{2} y_2^T K y_2$$

The derivative of V_3 is

$$
\begin{aligned}
\dot{V}_2 &= -s^T K_1 s - y_1^T K y_1 + y_1^T K y_2 + y_2^T K \dot{y}_2 \\
&= -s^T K_1 s - y_1^T K y_1 + y_1^T K y_2 + y_2^T K(-J^{-1}K(x_1 - q) + J^{-1}u - \dot{\Phi}_2 + y_1)
\end{aligned}
$$

Design the control input as

$$u = J(\dot{\Phi}_2 - y_1 - y_2) + K(x_1 - q)$$

Then we have

$$\dot{V}_3 = -s^T K_1 s - y_1^T K y_1 - y_2^T K y_2, ND$$

Then from Lyapunov Theorem, $\lim_{t\to\infty} s(t) = \lim_{t\to\infty} y_1(t) = \lim_{t\to\infty} y_2(t) = 0$.