Additional Q&A of NAC

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Question 1. ★★★★★★

Consider the following scalar system

$$\dot{x} = -ax + bu + \varphi\theta$$

Solution:

$$u = \hat{k}_1(t)x + \hat{k}_2(t)u_c - \varphi(x)\frac{\hat{\theta}(t)}{\hat{b}(t)} \triangleq \hat{k}_1(t)x + \hat{k}_2(t)u_c - \varphi(x)\hat{\theta}_1(t)$$

Answer 1.

Consider the following scalar system

$$\dot{x} = -ax + bu + \varphi\theta$$

where $\varphi = \varphi(x)$ is a bounded and continuous known function, and a, b, θ are unknown constant parameters. $\mathrm{sgn}(b)$ is known.

The reference model is given by

$$\dot{x}_{\mathrm{ref}} = -a_{\mathrm{ref}}x_{\mathrm{ref}} + b_{\mathrm{ref}}u_c$$

We propose the following control law

$$u = \hat{k}_1(t)x + \hat{k}_2(t)u_c - \varphi\hat{\theta}_1(t)$$

Then we have

$$\dot{x} = -ax + b\hat{k}_1(t)x + b\hat{k}_2(t)u_c - \varphi(b\hat{\theta}_1(t) - \theta)$$

By $k_1^* = \frac{a - a_{\mathrm{ref}}}{b}$, $k_2^* = \frac{b_{\mathrm{ref}}}{b}$, the error dynamics is

$$\begin{aligned}
\dot{e} &= \dot{x} - \dot{x}_{\mathrm{ref}} \\
&= -ax + b\hat{k}_1(t)x + b\hat{k}_2(t)u_c - \varphi(b\hat{\theta}_1(t) - \theta) + a_{\mathrm{ref}}x_{\mathrm{ref}} - b_{\mathrm{ref}}u_c + a_{\mathrm{ref}}x - a_{\mathrm{ref}}x \\
&= -a_{\mathrm{ref}}e + (a_{\mathrm{ref}} - a + b\hat{k}_1(t))x + (b\hat{k}_2(t) - b_{\mathrm{ref}})u_c - \varphi(b\hat{\theta}_1(t) - \theta) \\
&= -a_{\mathrm{ref}}e + b(\hat{k}_1(t) - k_1^*)x + b(\hat{k}_2(t) - k_2^*)u_c - \varphi b\left(\hat{\theta}_1(t) - \frac{\theta}{b}\right) \\
&= -a_{\mathrm{ref}}e + b\tilde{k}_1x + b\tilde{k}_2u_c - b\varphi\tilde{\theta}
\end{aligned}$$

where $\tilde{k}_1 = \hat{k}_1(t) - k_1^*$, $\tilde{k}_2 = \hat{k}_2(t) - k_2^*$, $\tilde{\theta} = \hat{\theta}_1(t) - \frac{\theta}{b}$.

Consider the following Lyapunov function candidate

$$V = \frac{1}{2}e^2 + \frac{|b|}{2\gamma_1}\tilde{k}_1^2 + \frac{|b|}{2\gamma_2}\tilde{k}_2^2 + \frac{|b|}{2\gamma_3}\tilde{\theta}^2$$

where $\gamma_1, \gamma_2, \gamma_3 > 0$.

Taking the time derivative of V gives

$$\begin{aligned}\dot{V} &= e\dot{e}+\frac{|b|}{\gamma_1}\tilde{k}_1\dot{\hat{k}}_1+\frac{|b|}{\gamma_2}\tilde{k}_2\dot{\hat{k}}_2+\frac{|b|}{\gamma_3}\tilde{\theta}\dot{\hat{\theta}}_1 \\ &= -a_{\text{ref}}e^2+b\tilde{k}_1xe+b\tilde{k}_2u_ce-b\varphi\tilde{\theta}e+\frac{|b|}{\gamma_1}\tilde{k}_1\dot{\hat{k}}_1+\frac{|b|}{\gamma_2}\tilde{k}_2\dot{\hat{k}}_2+\frac{|b|}{\gamma_3}\tilde{\theta}\dot{\hat{\theta}}_1 \\ &= -a_{\text{ref}}e^2+\frac{|b|}{\gamma_1}\tilde{k}_1\big(\dot{\hat{k}}_1+\gamma_1\text{sgn}(b)xe\big)+\frac{|b|}{\gamma_2}\tilde{k}_2\big(\dot{\hat{k}}_2+\gamma_2\text{sgn}(b)u_ce\big)+\frac{|b|}{\gamma_3}\tilde{\theta}\big(\dot{\hat{\theta}}_1-\gamma_3\text{sgn}(b)\varphi e\big)\end{aligned}$$

If we choose

$$\dot{\hat{k}}_1=-\gamma_1\text{sgn}(b)xe,\, \dot{\hat{k}}_2=-\gamma_2\text{sgn}(b)u_ce,\, \dot{\hat{\theta}}_1=\gamma_3\text{sgn}(b)\varphi e$$

which leads to

$$\dot{V}=-a_{\text{ref}}e^2\leq 0$$

Thus $V(t)\leq V(0)$ which implies that $e,\tilde{k}_1,\tilde{k}_2,\tilde{\theta}\in\mathbb{L}_\infty$. Since x_{ref} is bouded, thus $x=e+x_{\text{ref}}\in\mathbb{L}_\infty$, $\dot{e}=-a_{\text{ref}}e+b\tilde{k}_1x+b\tilde{k}_2u_c-\varphi\tilde{\theta}\in\mathbb{L}_\infty$. Then we have $\ddot{V}=-2a_{\text{ref}}e\dot{e}\in\mathbb{L}_\infty$.

We can conclude from Barbalat's lemma that $\lim_{t\to\infty}\dot{V}(t)=\lim_{t\to\infty}e(t)=0$.

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Question 2. ⋆⋆⋆

1. Λ with all negative elements?

   Solution: choose $\text{tr}\{\tilde{K}_1^T(-\Lambda)\tilde{K}_1\}$

2. Λ with some negative elements and some positive elements?

   Solution: choose

   $$V=e^TPe+\text{tr}\{\tilde{K}_1^T|\Lambda|\tilde{K}_1\}+\text{tr}\{\tilde{K}_2^T|\Lambda|\tilde{K}_2\}$$

   and

   $$\dot{\hat{K}}_1=-\text{sgn}(\Lambda)B^TPex^T,\, \dot{\hat{K}}_2=-\text{sgn}(\Lambda)B^TPeu_c^T$$

   where $|\Lambda|\triangleq\Lambda\text{sgn}(\Lambda)$, $\text{sgn}(\Lambda)\triangleq\text{diag}\{\text{sgn}(\lambda_i)\}$.

Answer 2.

We consider a linear system described by

$$\dot{x}=Ax+B\Lambda u$$

where $A\in\mathbb{R}^{n\times n},\Lambda\in\mathbb{R}^{m\times m}$ are unknown constant matrix. In addition, assume that Λ is diagonal, and $(A,B\Lambda)$ is controllable. The uncertainty in Λ is introduced to model the control failure.

Control objective: design u such that all signals in the closed-loop system are bounded and x tracks the state x_{ref} of the following reference model.

$$\dot{x}_{\text{ref}}=A_{\text{ref}}x_{\text{ref}}+B_{\text{ref}}u_c$$

where $A_{\text{ref}} \in \mathbb{R}^{n\times n}$ is Hurwitz, $B_{\text{ref}} \in \mathbb{R}^{n\times m}$, $u_c \in \mathbb{R}^m$ is the bounded command vector.

If matrices A, Λ were known, we can apply the control law

$$u = K_1^* x + K_2^* u_c$$

and we can obtain

$$\dot{x} = (A + B\Lambda K_1^*)x + B\Lambda K_2^* u_c$$

Then the matching condition is

$$\begin{cases} A + B\Lambda K_1^* = A_{\text{ref}} \\ B\Lambda K_2^* = B_{\text{ref}} \end{cases}$$

Let us assume that K_1^*, K_2^* exist. We propose the control law

$$u = \hat{K}_1(t)x + \hat{K}_2(t)u_c$$

Then we obtain

$$\dot{x} = Ax + B\Lambda\hat{K}_1(t)x + B\Lambda\hat{K}_2(t)u_c$$

Define the tracking error $e \triangleq x - x_{\text{ref}}$. Its dynamic is

$$\begin{aligned}
\dot{e} &= \dot{x} - \dot{x}_{\text{ref}} \\
&= Ax + B\Lambda\hat{K}_1(t)x + B\Lambda\hat{K}_2(t)u_c - A_{\text{ref}}x_{\text{ref}} - B_{\text{ref}}u_c + A_{\text{ref}}x - A_{\text{ref}}x \\
&= A_{\text{ref}}e + (A - A_{\text{ref}})x - B_{\text{ref}}u_c + B\Lambda\hat{K}_1(t)x + B\Lambda\hat{K}_2(t)u_c \\
&= A_{\text{ref}}e + B\Lambda K_1^* x - B\Lambda K_2^* u_c + B\Lambda\hat{K}_1(t)x + B\Lambda\hat{K}_2(t)u_c \\
&= A_{\text{ref}}e + B\Lambda[\hat{K}_1(t) - K_1^*]x + B\Lambda[\hat{K}_2(t) - K_2^*]u_c \\
&= A_{\text{ref}}e + B\Lambda\tilde{K}_1 x + B\Lambda\tilde{K}_1 u_c
\end{aligned}$$

where $\tilde{K}_1 = \hat{K}_1 - K_1, \tilde{K}_2 = \hat{K}_2 - K_2$.

Since A_{ref} is Hurwitz, we can get from Lyapunov theorem that for any positive definte $Q \in \mathbb{R}^{n\times n}$, there exists a unique positive definite $P \in \mathbb{R}^{n\times n}$ such that

$$A_{\text{ref}}^T P + P A_{\text{ref}} = -Q < 0$$

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Case 1: Λ is diagonal with positive diagonal elements.

Consider the following Lyapunov function candidate

$$V = e^T P e + \text{tr}\{\tilde{K}_1^T \Lambda \tilde{K}_1\} + \text{tr}\{\tilde{K}_2^T \Lambda \tilde{K}_2\}$$

Its derivative is

$$\begin{aligned}
\dot{V} &= 2e^T P\dot{e} + 2\text{tr}\left\{\tilde{K}_1^T \Lambda \dot{\tilde{K}}_1\right\} + 2\text{tr}\left\{\tilde{K}_2^T \Lambda \dot{\tilde{K}}_2\right\} \\
&= 2e^T P A_{\text{ref}} e + 2e^T P B\Lambda\tilde{K}_1 x + 2e^T P B\Lambda\tilde{K}_1 u_c + 2\text{tr}\left\{\tilde{K}_1^T \Lambda \dot{\tilde{K}}_1\right\} + 2\text{tr}\left\{\tilde{K}_2^T \Lambda \dot{\tilde{K}}_2\right\}
\end{aligned}$$

Since $e^T P B\Lambda\tilde{K}_1 x = \text{tr}\{e^T P B\Lambda\tilde{K}_1 x\} = \text{tr}\{x e^T P B\Lambda\tilde{K}_1\} = \text{tr}\{\tilde{K}_1 \Lambda B^T P e x^T\}$, we have

$$\dot{V} = 2e^T P A_{\text{ref}} e + 2\text{tr}\{\tilde{K}_1 \Lambda B^T P e x^T\} + 2\text{tr}\{\tilde{K}_2 \Lambda B^T P e u_c^T\} + 2\text{tr}\left\{\tilde{K}_1^T \Lambda \dot{\tilde{K}}_1\right\} + 2\text{tr}\left\{\tilde{K}_2^T \Lambda \dot{\tilde{K}}_2\right\}$$

we can choose

$$\dot{\tilde{K}}_1 = -B^T P e x^T, \dot{\tilde{K}}_2 = -B^T P e u_c^T$$

Then we have

$$\dot{V} = -e^T Q e \le 0$$

Thus, $V(t) \le V(0)$, which implies that $e, \tilde{K}_1, \tilde{K}_2 \in \mathbb{L}_\infty$. Since u_c is bounded and A_{ref} is Hurwitz, $x_{\text{ref}} \in \mathbb{L}_\infty$, $x = e + x_{\text{ref}} \in \mathbb{L}_\infty$, $\dot{e} = A_{\text{ref}} e + B\Lambda\tilde{K}_1 x + B\Lambda\tilde{K}_1 u_c \in \mathbb{L}_\infty$. Then $\ddot{V} = -2e^T Q\dot{e} \in \mathbb{L}_\infty$.

Using Barbalat's lemma, we can get $\lim_{t\to\infty}\dot{V}(t) = \lim_{t\to\infty} e(t) = 0$.

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Case 2: Λ is diagonal with negative diagonal elements.

Denote $\bar{\Lambda} = -\Lambda$.

Consider the following Lyapunov function candidate

$$V = e^T P e + \text{tr}\{\tilde{K}_1^T \Lambda \tilde{K}_1\} - \text{tr}\{\tilde{K}_2^T \bar{\Lambda} \tilde{K}_2\}$$

Its derivative is

$$\begin{aligned} \dot{V} &= 2e^T P\dot{e} + 2\text{tr}\left\{\tilde{K}_1^T \bar{\Lambda} \dot{\tilde{K}}_1\right\} + 2\text{tr}\left\{\tilde{K}_2^T \bar{\Lambda} \dot{\tilde{K}}_2\right\} \\ &= 2e^T P A_{\text{ref}} e + 2e^T P B\Lambda\tilde{K}_1 x + 2e^T P B\Lambda\tilde{K}_1 u_c + 2\text{tr}\left\{\tilde{K}_1^T \bar{\Lambda} \dot{\tilde{K}}_1\right\} + 2\text{tr}\left\{\tilde{K}_2^T \bar{\Lambda} \dot{\tilde{K}}_2\right\} \end{aligned}$$

Since $e^T P B\Lambda\tilde{K}_1 x = \text{tr}\{e^T P B\Lambda\tilde{K}_1 x\} = \text{tr}\{x e^T P B\Lambda\tilde{K}_1\} = \text{tr}\{\tilde{K}_1 \Lambda B^T P e x^T\}$, we have

$$\dot{V} = 2e^T P A_{\text{ref}} e + 2\text{tr}\{\tilde{K}_1 \Lambda B^T P e x^T\} + 2\text{tr}\{\tilde{K}_2 \Lambda B^T P e u_c^T\} + 2\text{tr}\left\{\tilde{K}_1^T \bar{\Lambda} \dot{\tilde{K}}_1\right\} + 2\text{tr}\left\{\tilde{K}_2^T \bar{\Lambda} \dot{\tilde{K}}_2\right\}$$

we can choose

$$\dot{\tilde{K}}_1 = +B^T P e x^T, \dot{\tilde{K}}_2 = +B^T P e u_c^T$$

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Case 3: Λ is diagonal with some negative elements and some positive elements

Denote $|\Lambda| \triangleq \Lambda\text{sgn}(\Lambda)$, $\text{sgn}(\Lambda) \triangleq \text{diag}\{\text{sgn}(\lambda_i)\}$.

Consider the following Lyapunov function candidate

$$V = e^T P e + \text{tr}\{\tilde{K}_1^T \Lambda \tilde{K}_1\} - \text{tr}\{\tilde{K}_2^T |\Lambda| \tilde{K}_2\}$$

Its derivative is

$$\begin{aligned} \dot{V} &= 2e^T P\dot{e} + 2\text{tr}\left\{\tilde{K}_1^T |\Lambda| \dot{\tilde{K}}_1\right\} + 2\text{tr}\left\{\tilde{K}_2^T |\Lambda| \dot{\tilde{K}}_2\right\} \\ &= 2e^T P A_{\text{ref}} e + 2e^T P B\Lambda\tilde{K}_1 x + 2e^T P B\Lambda\tilde{K}_1 u_c + 2\text{tr}\left\{\tilde{K}_1^T |\Lambda| \dot{\tilde{K}}_1\right\} + 2\text{tr}\left\{\tilde{K}_2^T |\Lambda| \dot{\tilde{K}}_2\right\} \end{aligned}$$

Since $e^T P B\Lambda\tilde{K}_1 x = \text{tr}\{e^T P B\Lambda\tilde{K}_1 x\} = \text{tr}\{x e^T P B\Lambda\tilde{K}_1\} = \text{tr}\{\tilde{K}_1 \Lambda B^T P e x^T\}$, we have

$$\dot{V} = 2e^T P A_{\text{ref}} e + 2\text{tr}\{\tilde{K}_1 \Lambda B^T P e x^T\} + 2\text{tr}\{\tilde{K}_2 \Lambda B^T P e u_c^T\} + 2\text{tr}\left\{\tilde{K}_1^T |\Lambda| \dot{\tilde{K}}_1\right\} + 2\text{tr}\left\{\tilde{K}_2^T |\Lambda| \dot{\tilde{K}}_2\right\}$$

we can choose

$$\dot{\tilde{K}}_1 = -\text{sgn}(\Lambda) B^T P e x^T, \dot{\tilde{K}}_2 = -\text{sgn}(\Lambda) B^T P e u_c^T$$

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Question 3.

$$\dot{x} = Ax + B\Lambda(u + \Phi(t) \cdot \Theta)$$

where $\Theta \in \mathbb{R}^{p\times 1}, \Phi(t) \in \mathbb{R}^{m\times p}$ with Φ being bounded and Θ unknown.

Solution:

$$u = \hat{K}_1(t)x + \hat{K}_2(t)u_c - \Phi(t) \cdot \hat{\Theta}$$

Answer 3. For the following system

$$\dot{x} = Ax + B\Lambda(u + \Theta\Phi(t))$$

where $\Theta, \Phi(t)$ with Φ being bounded and Θ unknown.

Control objective: design u, such that all signals in the closed-loop system are bounded and x tracks the state x_{ref} of the following reference model.

$$\dot{x}_{\text{ref}} = A_{\text{ref}}x_{\text{ref}} + B_{\text{ref}}u_c \tag{1}$$

where $A_{\text{ref}} \in \mathbb{R}^{n\times n}$ is Hurwitz, $B_{\text{ref}} \in \mathbb{R}^{n\times m}$, $u_c \in \mathbb{R}^m$ is the bounded command vector.

If matrices $A \in \mathbb{R}^{n\times n}$, $\Lambda \in \mathbb{R}^{m\times m}$ and $\Theta \in \mathbb{R}^{p\times 1}$ were known, we can apply the control law

$$u = K_1^* x + K_2^* u_c - \Theta\Phi$$

where $K_1^* \in \mathbb{R}^{m\times n}, K_2^* \in \mathbb{R}^{m\times m}$ and we can obtain

$$\dot{x} = (A + B\Lambda K_1^*)x + B\Lambda K_2^* u_c$$

Then the matching condition is

$$\begin{cases} A + B\Lambda K_1^* = A_{\text{ref}} \\ B\Lambda K_2^* = B_{\text{ref}} \end{cases} \tag{2}$$

Let us assume that K_1^*, K_2^* in (2) exist, i.e., there is sufficient structure flexibility to meet the control objective. We propose the control law

$$u = \hat{K}_1(t)x + \hat{K}_1(t)u_c - \hat{\Theta}\Phi$$

By adding and subtracting the desired term, we obtain

$$\dot{x} = A_{\text{ref}}x + B_{\text{ref}}u_c + B\Lambda\tilde{K}_1 x + B\Lambda\tilde{K}_2 u_c - B\Lambda\hat{\Theta}\Phi$$

where $\tilde{K}_1 \triangleq \hat{K}_1 - K_1^*, \tilde{K}_2 \triangleq \hat{K}_2 - K_2^*, \tilde{\Theta} = \hat{\Theta} - \Theta$.

Define the tracking error $e \triangleq x - x_{\text{ref}}$. Its dynamic is

$$\dot{e} = A_{\text{ref}} e + B\Lambda\tilde{K}_1 x + B\Lambda\tilde{K}_2 u_c - B\Lambda\tilde{\Theta}\Phi \tag{3}$$

we then consider the following Lyapunov function candidate

$$V = e^T P e + \text{tr}\{\tilde{K}_1^T \Lambda \tilde{K}_1\} + \text{tr}\{\tilde{K}_2^T \Lambda \tilde{K}_2\} + \tilde{\Theta}^T \Lambda \tilde{\Theta}$$

Since

$$\begin{aligned} e^T PB\Lambda\tilde{K}_1 x &= \text{tr}\{e^T PB\Lambda\tilde{K}_1 x\} \\ &= \text{tr}\{x e^T PB\Lambda\tilde{K}_1\} \\ &= \text{tr}\{\tilde{K}_1 \Lambda B^T P e x^T\} \end{aligned}$$

Its derivative

$$\begin{aligned} \dot{V} &= 2e^T P\dot{e} + 2\text{tr}\left\{\tilde{K}_1^T \Lambda \dot{\hat{K}}_1\right\} + 2\text{tr}\left\{\tilde{K}_2^T \Lambda \dot{\hat{K}}_2\right\} + 2\tilde{\Theta}^T \Lambda \dot{\hat{\Theta}} \\ &= 2e^T P(A_{\text{ref}} e + B\Lambda\tilde{K}_1 x + B\Lambda\tilde{K}_2 u_c - B\Lambda\tilde{\Theta}\Phi) \\ &\quad + 2\text{tr}\left\{\tilde{K}_1^T \Lambda \dot{\hat{K}}_1\right\} + 2\text{tr}\left\{\tilde{K}_2^T \Lambda \dot{\hat{K}}_2\right\} + 2\tilde{\Theta}^T \Lambda \dot{\hat{\Theta}} \\ &= 2e^T P A_{\text{ref}} e + 2e^T PB\Lambda\tilde{K}_1 x + 2e^T PB\Lambda\tilde{K}_2 u_c - 2e^T PB\Lambda\tilde{\Theta}\Phi \\ &\quad + 2\text{tr}\left\{\tilde{K}_1^T \Lambda \dot{\hat{K}}_1\right\} + 2\text{tr}\left\{\tilde{K}_2^T \Lambda \dot{\hat{K}}_2\right\} + 2\tilde{\Theta}^T \Lambda \dot{\hat{\Theta}} \\ &\leq 2e^T P A_{\text{ref}} e + 2\text{tr}\{\tilde{K}_1 \Lambda B^T P e x^T\} + 2\text{tr}\{\tilde{K}_2 \Lambda B^T P e u_c^T\} - 2\text{tr}\{\tilde{\Theta}^T \Lambda B^T P e \Phi^T\} \\ &\quad + 2\text{tr}\left\{\tilde{K}_1^T \Lambda \dot{\hat{K}}_1\right\} + 2\text{tr}\left\{\tilde{K}_2^T \Lambda \dot{\hat{K}}_2\right\} + 2\text{tr}\left\{\tilde{\Theta}^T \Lambda \dot{\hat{\Theta}}\right\} \end{aligned}$$

we can choose

$$\dot{\hat{K}}_1 = -B^T P e x^T, \dot{\hat{K}}_2 = -B^T P e u_c^T, \dot{\hat{\Theta}} = B^T P e \Phi^T$$

Then we have

$$\dot{V} = -e^T Q e \leq 0$$

Thus, $V(t) \leq 0$, which implies that $e, \tilde{K}_1, \tilde{K}_2, \tilde{\Theta} \in \mathbb{L}_\infty$. Since u_c is bounded and A_{ref} is Hurwitz, $x_{\text{ref}} \in \mathbb{L}_\infty$. Then $x = e + x_{\text{ref}} \in \mathbb{L}_\infty$. Since Φ is bounded, $\dot{e} = A_{\text{ref}} e + B\Lambda\tilde{K}_1 x + B\Lambda\tilde{K}_2 u_c - B\Lambda\tilde{\Theta}\Phi \in \mathbb{L}_\infty$. Thus $\ddot{V} = -2e^T Q\dot{e} \in \mathbb{L}_\infty$. By Barbalats's lemma, $\lim_{t\to\infty}\dot{V}(t) = \lim_{t\to\infty}\dot{e}(t) = 0$.

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Question 4. ⋆⋆⋆⋆⋆

$$\|d(t)\| \leq d_{\max}$$

$$\|d(t)\| \leq d_{\max}(1 + \|x_1\| + \|x_2\|^2)$$

$d_{\max}$ is unknown.

Answer 4.

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= u + d(t) \end{aligned}$$

where $d(t)$ is a external disturbance with $||d(t)|| \leq d_{\max}(1+\|x_1\|+\|x_2\|^2)$.

Denote $\Phi(x)=1+\|x_1\|+\|x_2\|^2 \geq 1$.

Design a sliding surface: $s=x_2+\lambda x_1, \lambda>0$.

The dynamics of s is $\dot{s}=\dot{x}_2+\lambda\dot{x}_1=u+d(t)+\lambda x_2$.

If $d_{\max}$ is unknown, we can design

$$u=-\lambda x_2-\eta \text{sgn}(s)-\text{sgn}(s)\hat{d}(t)\Phi(x)$$

Then $\dot{s}=d(t)-\text{sgn}(s)\hat{d}(t)\|\Phi(x)\|-\eta\text{sgn}(s)$.

Denote $\tilde{d}=\hat{d}(t)-d_{\max}$.

Consider the following Lyapunov function candidate

$$V=\frac{1}{2}s^2+\frac{1}{2\gamma}\tilde{d}^2, \gamma>0,$$

Its derivative is

$$\begin{aligned}
\dot{V} &= s\dot{s}+\frac{1}{\gamma}\tilde{k}\dot{\hat{k}} \\
&= s[d(t)-\text{sgn}(s)\hat{d}\Phi(x)-\eta\text{sgn}(s)]+\frac{1}{\gamma}\tilde{d}\dot{\hat{d}} \\
&\leq -\eta|s|+d_{\max}\Phi(x)|s|-\hat{d}\Phi(x)|s|+\frac{1}{\gamma}\tilde{d}\dot{\hat{d}} \\
&= -\eta|s|-(\hat{d}-d_{\max})\Phi(x)|s|+\frac{1}{\gamma}\tilde{d}\dot{\hat{d}} \\
&= -\eta|s|-\tilde{d}\Phi(x)|s|+\frac{1}{\gamma}\tilde{d}\dot{\hat{d}} \\
&= -\eta|s|+\frac{1}{\gamma}\tilde{d}\big(\dot{\hat{d}}-\gamma\Phi(x)|s|\big)
\end{aligned}$$

Design $\dot{\hat{d}}=\gamma\Phi(x)|s|$, we obtain $\dot{V}\leq-\eta|s|$. Integrating both sides yields

$$V(t)-V(0)\leq-\eta\int_0^t|s|d\tau$$

$\Rightarrow s\in\mathbb{L}_1$.

By $s\in\mathbb{L}_1\cap\mathbb{L}_\infty, \dot{s}\in\mathbb{L}_\infty$, we have $\lim_{t\to\infty}s(t)=0$.

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Question 5. If the system is $M(q)\ddot{q}+C(q,\dot{q})\dot{q}+g(q)=u$, then choose

$$u=g(q)-K_p\tilde{q}-K_d\dot{\tilde{q}}$$

If the system is $M(q)\ddot{q}+C(q,\dot{q})\dot{q}+g(q)+D\dot{q}=u$, D is PSD (It is useful, don't cancel).

Control Objective: $q\to q_d$, $\dot{q}_d=0$.

Answer 5.

Define the position error: $\tilde{q} = q - q_d$, $\dot{\tilde{q}} = \dot{q}$, $\ddot{\tilde{q}} = \ddot{q}$.

The error dynamics is

$$M\ddot{\tilde{q}} + C\dot{\tilde{q}} + g + D\dot{\tilde{q}} = u$$

Design the following control input

$$u = -K_p\tilde{q} - K_d\dot{\tilde{q}} + g = -K_p(q - q_d) - K_d\dot{q} + g$$

where K_p and K_d are positive definite matrices.

Then the closed-loop system is

$$M\ddot{\tilde{q}} + C\dot{\tilde{q}} + D\dot{\tilde{q}} = -K_p\tilde{q} - K_d\dot{\tilde{q}}$$

Consider the following Lyapunov function candidate

$$V = \frac{1}{2}\dot{\tilde{q}}^T M\dot{\tilde{q}} + \frac{1}{2}\tilde{q}^T K_p\tilde{q}$$

The derivative of V is

$$\begin{aligned}
\dot{V} &= \dot{\tilde{q}}^T M\ddot{\tilde{q}} + \frac{1}{2}\dot{\tilde{q}}^T\dot{M}\dot{\tilde{q}} + \dot{\tilde{q}}^T K_d\tilde{q} \\
&= \dot{\tilde{q}}^T(-K_p\tilde{q} - K_d\dot{\tilde{q}} - C\dot{\tilde{q}} - D\dot{\tilde{q}}) + \frac{1}{2}\dot{\tilde{q}}^T\dot{M}\dot{\tilde{q}} + \tilde{q}^T K_p\dot{\tilde{q}} \\
&= -\dot{\tilde{q}}^T K_d\dot{\tilde{q}} + \frac{1}{2}\dot{\tilde{q}}^T(\dot{M} - 2C)\dot{\tilde{q}} - \dot{\tilde{q}}^T D\dot{\tilde{q}} \\
&= -\dot{\tilde{q}}^T K_d\dot{\tilde{q}} - \dot{\tilde{q}}^T D\dot{\tilde{q}}
\end{aligned}$$

NSD => stable.

Note that the closed-loop system is autonomous, we have

$$E \triangleq \{(\tilde{q}, \dot{\tilde{q}}) | \dot{V} = 0\} = \{(\tilde{q}, \dot{\tilde{q}}) | \dot{\tilde{q}} \equiv 0\}$$

Since $\dot{\tilde{q}}(t) \equiv 0 \Rightarrow \ddot{\tilde{q}}(t) \equiv 0 \Rightarrow \tilde{q}(t) \equiv 0$, the only solution that can stay identically to E is the origin. Thus, from LaSalle's Theorem, the origin is asymptotically stable, i.e., $\lim_{t\to\infty} q(t) = q_d$, $\lim_{t\to\infty}\dot{q}(t) = 0$.

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Question 6. ⋆⋆⋆⋆⋆⋆

If there exist external disturbance in the system, for example

$$M(q)\ddot{q} + C(q, \dot{q})\dot{q} = u + d(t)$$

where $\|d(t)\| \le d_{\max}$. We may use $u = -k\,\mathrm{sgn}(s) + M\ddot{q}_r + C\dot{q}_r$.

Answer 6.

Define a sliding surface

$$s = \dot{\tilde{q}} + \lambda\tilde{q} = \dot{q} - (\dot{q}_d - \lambda\tilde{q}) = \dot{q} - \dot{q}_r, \lambda > 0$$

where $\dot{q}_r$ is an auxiliary variable.

Then we have

$$M(\dot{s} + \ddot{q}_r) + C(s + \dot{q}_r) = u + d(t)$$

or

$$M\dot{s} + Cs = u - M\ddot{q}_r - C\dot{q}_r + d(t) \tag{4}$$

Define the following control input

$$u = -k\text{sgn}(s) + M\ddot{q}_r + C\dot{q}_r \tag{5}$$

where K is positive definite. Then the closed-loop system is

$$M\dot{s} + Cs = -k\text{sgn}(s) + d(t)$$

Consider the following Lyapunov function candidate

$$V = \frac{1}{2}s^T Ms$$

Its derivative is

$$\begin{aligned}
\dot{V} &= s^T M\dot{s} + \frac{1}{2}s^T\dot{M}s \\
&= -s^T(Cs + k\text{sgn}(s) + d(t)) + \frac{1}{2}s^T\dot{M}s \\
&= -s^T k\text{sgn}(s) - s^T d(t) + \frac{1}{2}s^T(\dot{M} - 2C)s \\
&= -s^T k\text{sgn}(s) - s^T d(t) \\
&\leq -||s||k - ||s||d_{\max}
\end{aligned}$$

Choose $k = d_{\max} + \eta$, $\eta > 0$, then $\dot{V} \leq -\eta||s||$.

Therefore, the origin $s = 0$ is globally uniformly expoenentially stable, i.e, $\lim_{t\to\infty} s(t) = 0$.

From the input-to-state stability, $\lim_{t\to\infty}\tilde{q}(t) = \lim_{t\to\infty}\dot{\tilde{q}}(t) = 0$.

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