# Lecture 5
# Propagation of states and covariances

- Discrete-time systems
- Sampled-data systems
- Continuous-time systems

# What is this chapter about

- mathematical description of a dynamic system
- derive the equations that govern the propagation of the state mean and covariance
- is fundamental to the state estimation algorithm (the Kalman filter)

# Contents

- Discrete-time systems
- Sampled-data systems
- Continuous-time systems

# Linear discrete-time system

Suppose we have the following linear discrete-time system:

$$x_k = F_{k-1}x_{k-1} + G_{k-1}u_{k-1} + w_{k-1} \tag{1}$$

in which $u_k$ is a known input and $w_k$ is the process noise drawn from a zero-mean multivariate normal distribution with covariance $Q_k$. Besides, the initial state, and the noise vector at each step $\{x_0, w_1, \ldots, w_k\}$ are all assumed to be mutually independent.

# Mean and covariance of $x_k$

- Mean: take the expected value of both sides of Equation (1) we obtain

$$\bar{x}_k = E(x_k) = F_{k-1}\bar{x}_{k-1} + G_{k-1}u_{k-1}$$

- Covariance ($P_k = E[x_k - \bar{x}_k][x_k - \bar{x}_k]^T$):

$$\begin{aligned}
(x_k - \bar{x}_k)(\cdots)^T &= (F_{k-1}x_{k-1} + G_{k-1}u_{k-1} + w_{k-1} - \bar{x}_k)(\cdots)^T \\
&= [F_{k-1}(x_{k-1} - \bar{x}_{k-1}) + w_{k-1}][\cdots]^T \\
&= F_{k-1}(x_{k-1} - \bar{x}_{k-1})(x_{k-1} - \bar{x}_{k-1})^T F_{k-1}^T + w_{k-1}w_{k-1}^T + \\
& F_{k-1}(x_{k-1} - \bar{x}_{k-1})w_{k-1}^T + w_{k-1}(x_{k-1} - \bar{x}_{k-1})^T F_{k-1}^T
\end{aligned}$$

# Discrete-time Lyapunov equation

- the term $(x_{k-1} - \bar{x}_{k-1})$ is uncorrelated with $w_{k-1}$ (provided that $x_0$ is uncorrelated with $w_k, k = 0, 1, 2, \ldots$)
- The covariance matrix:

$$P_k = E[(x_k - \bar{x}_k)(\cdots)^T] = F_{k-1} P_{k-1} F_{k-1}^T + Q_{k-1}$$

This is called a discrete-time Lyapunov equation, or a Stein equation, which is fundamental in the derivation of the Kalman filter.

# Steady-state solution of the discrete-time Lyapunov equation

Consider the equation $P = FPF^T + Q$ where $F$ and $Q$ are real matrices. Denote by $\lambda_i(F)$ the eigenvalues of the $F$ matrix.

- A unique solution $P$ exists iff $\lambda_i(F) \cdot \lambda_j(F) \neq 1$ for all $i, j$. The unique solution is symmetric.
- If $F$ is stable then the discrete-time Lyapunov equation has a solution $P$ that is unique and symmetric:

$$P = \sum_{i=0}^{\infty} F^i Q (F^T)^i$$

# Solution of the linear system

$$x_k = F_{k,0}x_0 + \sum_{i=0}^{k-1}(F_{k,i+1}w_i + F_{k,i+1}G_i u_i)$$

State transition matrix of the system:

$$F_{k,i} = \begin{cases} F_{k-1}F_{k-2}\cdots F_i & k > i \\ I & k = i \\ 0 & k < i \end{cases}$$

# Property of the solution

- the state $x_k$ is a linear combination of $x_0, \{w_i\}$ and $\{u_i\}$.
- if the input sequence $\{u_i\}$ is known, $x_0$ and $w_i$ are unknown but are Gaussian random variables, then $x_k$ is itself a Gaussian random variable.
- we have $x_k \sim \mathcal{N}(\bar{x}_k, P_k)$, i.e., a Gaussian random variable is completely characterized by its mean and covariance.

# Example

A linear system describing the population of a predator $x(1)$ and that of its prey $x(2)$ can be written as

$$\begin{aligned} x_{k+1}(1) &= x_k(1) - 0.8x_k(1) + 0.4x_k(2) + w_k(1) \\ x_{k+1}(2) &= x_k(2) - 0.4x_k(1) + u_k + w_k(2) \end{aligned}$$

- the predator population causes itself to decrease because of overcrowding
- the prey population causes the predator population to increase
- the prey population decreases due to the predator population
- the prey population increases due to an external food supply $u_k$
- the populations are also subject to random disturbances due to environmental factors

# Example

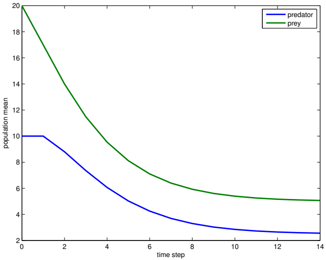
State-space form:

$$x_{k+1} = \begin{bmatrix} 0.2 & 0.4 \\ -0.4 & 1 \end{bmatrix} x_k + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_k + w_k$$

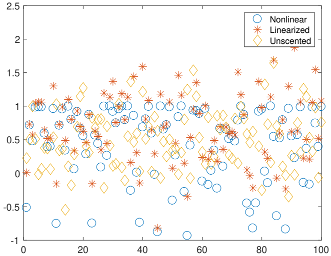
$$w_k \sim N(0, Q) \qquad Q = \mathrm{diag}(1, 2)$$

Assume $\bar{x}_0 = [10, 20]^T$, $P_0 = \mathrm{diag}(40, 40)$ and $u_k = 1$, we obtain the two means and the two diagonal elements of the covariance matrix.

# Example




(a) State mean




(b) State covariance

# Example

Steady-state values:

$$\bar{x} = (I - F)^{-1} G u$$

$$= [2.5, 5]^T$$

$$P \sim \begin{bmatrix} 2.88 & 3.08 \\ 3.08 & 7.96 \end{bmatrix}$$

# When the process noise is multiplied by some matrix

Another expression for $x_k$:

$$x_k = F_{k-1}x_{k-1} + G_{k-1}u_{k-1} + L_{k-1}\tilde{w}_{k-1}, \;\; \tilde{w}_k \sim \mathcal{N}(0, \tilde{Q}_k) \qquad (2)$$

As the rightmost term of the above equation has a covariance given by

$$\begin{aligned} E[(L_{k-1}\tilde{w}_{k-1})(L_{k-1}\tilde{w}_{k-1})^T] &= L_{k-1}E(\tilde{w}_{k-1}\tilde{w}_{k-1}^T)L_{k-1}^T \\ &= L_{k-1}\tilde{Q}_{k-1}L_{k-1}^T \end{aligned}$$

Therefore, Equation (2) is equivalent to the equation

$$x_k = F_{k-1}x_{k-1} + G_{k-1}u_{k-1} + w_{k-1}, w_k \sim \mathcal{N}(0, L_k\tilde{Q}_kL_k^T)$$

# Contents

- Discrete-time systems
- Sampled-data systems
- Continuous-time systems

# Definition

- A sampled-data system is a system whose dynamics are described by a continuous-time differential equation, but the input only changes at discrete time instants
- we are interested in obtaining the mean and covariance of the state only at discrete time instants
- The continuous-time dynamics are described as

$$\dot{x} = Ax + Bu + w$$

- the solution of $x(t)$ at some arbitrary time, say $t_k$, is given as

$$x(t_k) = e^{A(t_k - t_{k-1})} x(t_{k-1}) + \int_{t_{k-1}}^{t_k} e^{A(t_k - \tau)} [Bu(\tau) + w(\tau)] d\tau$$

# Transformation to discrete-time propagation

Assume $u(t) = u_{k-1}$ for $t \in [t_{k-1}, t_k]$, $\Delta t = t_k - t_{k-1}$, $x_k = x(t_k)$ and $u_k = u(t_k)$, we have

$$x_k = e^{A\Delta t} x_{k-1} + \left[ \int_{t_{k-1}}^{t_k} e^{A(t_k - \tau)} B d\tau \right] u_{k-1} + \int_{t_{k-1}}^{t_k} e^{A(t_k-\tau)} w(\tau) d\tau$$

Define $F_{k-1}$ and $G_{k-1}$ as

$$F_{k-1} = e^{A\Delta t}$$

$$G_{k-1} = \int_{t_{k-1}}^{t_k} e^{A(t_k-\tau)} B d\tau$$

then

$$x_k = F_{k-1} x_{k-1} + G_{k-1} u_{k-1} + \int_{t_{k-1}}^{t_k} e^{A(t_k-\tau)} w(\tau) d\tau$$

# Propagation of the state mean

prerequisite: $w(t)$ is zero-mean

$$\begin{aligned}\bar{x}_k &= E(x_k) \\ &= F_{k-1}\bar{x}_{k-1} + G_{k-1}u_{k-1}\end{aligned}$$

# Covariance of the state

prerequisite: $w(t) \sim \mathcal{N}(0, Q_c(t))$, besides,

$E[w(t)w^T(\tau)] = Q_c(t)\delta(t-\tau)$.

$$
\begin{aligned}
P_k &= E[(x_k - \bar{x}_k)(x_k - \bar{x}_k)^T] \\
&= E\left[\left(F_{k-1}x_{k-1} + G_{k-1}u_{k-1} + \int_{t_{k-1}}^{t_k} e^{A(t_k-\tau)}w(\tau)d\tau - \bar{x}_k\right)(\cdots)^T\right] \\
&= F_{k-1}P_{k-1}F_{k-1}^T + E\left[\left(\int_{t_{k-1}}^{t_k} e^{A(t_k-\tau)}w(\tau)d\tau\right)(\cdots)^T\right] \\
&= F_{k-1}P_{k-1}F_{k-1}^T + \int_{t_{k-1}}^{t_k}\int_{t_{k-1}}^{t_k} e^{A(t_k-\tau)}E[w(\tau)w^T(\alpha)]e^{A^T(t_k-\alpha)}d\tau d\alpha \\
&= F_{k-1}P_{k-1}F_{k-1}^T + \int_{t_{k-1}}^{t_k} e^{A(t_k-\tau)}Q_c(\tau)e^{A^T(t_k-\tau)}d\tau
\end{aligned}
$$

# Covariance of the state

Define

$$Q_{k-1} = \int_{t_{k-1}}^{t_k} e^{A(t_k-\tau)} Q_c(\tau) e^{A^T(t_k-\tau)} d\tau$$

we have

$$P_k = F_{k-1} P_{k-1} F_{k-1}^T + Q_{k-1}$$

For small values of $(t_k - t_{k-1})$ we have

$$e^{A(t_k-\tau)} \approx I \text{ for } \tau \in [t_{k-1}, t_k]$$

$$Q_{k-1} \approx Q_c(t_k)\Delta t$$

# Example

Suppose we have a first-order, continuous-time dynamic system (e.g. the behaviour of the current through a series $RL$ circuit that is driven by a random voltage $w(t)$, where $f = -R/L$) given by the equation

$$\dot{x} = fx + w$$

$$E[w(t)w(t+\tau)] = q_c\delta(\tau)$$

where $w(t)$ is zero-mean noise.

- suppose we are interested in obtaining the mean and covariance of the state $x(t)$ every $\Delta t$ time units, i.e., $t_k - t_{k-1} = \Delta t$
- for this simple scalar example, we can explicitly calculate $Q_{k-1}$ as

$$Q_{k-1} = \frac{q_c}{2f}[\exp(2f\Delta t) - 1]$$

# Example

Expanding $Q_{k-1}$ in a Taylor series around $\Delta t = 0$ results:

$$
\begin{aligned}
Q_{k-1} &= \frac{q_c}{2f}[\exp(2f\Delta t) - 1] \\
&\approx \frac{q_c}{2f}\left[\left(1 + 2f\Delta t + \frac{(2f\Delta t)^2}{2!}\right) - 1\right] \\
&\approx \frac{q_c}{2f}[1 + 2f\Delta t - 1] \\
&= q_c \Delta t
\end{aligned}
$$

# Example

The sampled mean of the state is computed as (noting that the control input is zero)

$$\begin{aligned}\bar{x}_k &= F_{k-1}\bar{x}_{k-1} + G_{k-1}u_{k-1} \\ &= \exp[f(t_k - t_{k-1})]\bar{x}_{k-1} + 0 \\ &= \exp(f\Delta t)\bar{x}_{k-1} \\ &= \exp(kf\Delta t)\bar{x}_0\end{aligned}$$

- If $f > 0$ (i.e., the system is unstable) then the mean $\bar{x}_k$ will increase without bound (unless $\bar{x}_0 = 0$)
- If $f < 0$ (i.e., the system is stable) then the mean $\bar{x}_k$ will decay to zero regardless of the value of $\bar{x}_0$

# Example

The sampled covariance of the state is computed as

$$\begin{aligned} P_k &= F_{k-1} P_{k-1} F_{k-1}^T + Q_{k-1} \\ &\approx (1 + 2f\Delta t) P_{k-1} + q_c \Delta t \\ P_k - P_{k-1} &= (2f P_{k-1} + q_c)\Delta t \end{aligned}$$

- assume $f < 0$, when $P_{k-1} = -q_c/2f$, $P_k$ reaches steady state, i.e., $P_k - P_{k-1} = 0$
- if $f \geq 0$, then $P_k - P_{k-1}$ will always be greater than 0, which means that $\lim_{k\to\infty} P_k = \infty$

# Contents

- Discrete-time systems
- Sampled-data systems
- Continuous-time systems

Consider the continuous-time system

$$\dot{x} = Ax + Bu + w$$

where $u(t)$ is a known control input and $w(t)$ is zero-mean noise with a covariance of

$$E[w(t)w^T(\tau)] = Q_c\delta(t-\tau)$$

Taking the mean:

$$\dot{\bar{x}} = A\bar{x} + Bu$$

We can use the equation

$$P_k = F_{k-1} P_{k-1} F_{k-1}^T + Q_{k-1}$$

that describes the covariance of a sampled-data system and taking the limit as $\Delta t = t_k - t_{k-1} \to 0$. As

$$\begin{aligned} F &= e^{A\Delta t} \\ &= I + A\Delta t + \frac{(A\Delta t)^2}{2!} + \cdots \end{aligned}$$

For small values of $\Delta t$, this can be approximated as

$$F \approx I + A\Delta t$$

Thus we obtain

$$
\begin{aligned}
P_k &\approx (I + A\Delta t)P_{k-1}(I + A\Delta t)^T + Q_{k-1} \\
&= P_{k-1} + AP_{k-1}\Delta t + P_{k-1}A^T\Delta t + AP_{k-1}A^T(\Delta t)^2 + Q_{k-1}
\end{aligned}
$$

Subtracting $P_{k-1}$ from both sides and dividing by $\Delta t$ gives

$$
\frac{P_k - P_{k-1}}{\Delta t} = AP_{k-1} + P_{k-1}A^T + AP_{k-1}A^T\Delta t + \frac{Q_{k-1}}{\Delta t} \qquad (3)
$$

Recall that for small $\Delta t$

$$
Q_{k-1} \approx Q_c(t_k)\Delta t
$$

Taking the limit of Equation (3) as $\Delta t$ goes to zero gives the continuous-time Lyapunov equation

$$
\dot{P} = AP + PA^T + Q_c
$$

# Continuous-time Lyapunov equation

Conditions under which the continuous-time Lyapunov equation has a steady-state solution, i.e.,

$$AP + PA^T + Q_c = 0$$

- A unique solution $P$ exists iff $\lambda_i(A) + \lambda_j(A) \neq 0, \forall i, j$. This unique solution is symmetric.
- If $A$ is stable, then there is a unique and symmetric $P$

$$P = \int_0^\infty e^{A^T \tau} Q_c e^{A\tau} d\tau$$

- If $A$ is stable and $Q_c$ is positive (semi) definite, then the unique solution $P$ is symmetric and positive (semi) definite
- $\cdots\cdots$

# Example

Suppose we have the first-order, continuous-time dynamic system given by the equation

$$\dot{x} = fx + w$$
$$E[w(t)w(t+\tau)] = q_c\delta(\tau)$$

where $w(t)$ is zero-mean noise.

# Example: The mean

The equation for the continuous-time propagation of the mean of state is

$$\dot{\bar{x}} = f\bar{x}$$

Solving this equation for $\bar{x}(t)$ gives

$$\bar{x}(t) = \exp(ft)\bar{x}(0)$$

- The mean will increase without bound if $f > 0$ (i.e., if the system is unstable)
- The mean will asymptotically tend to zero if $f < 0$ (i.e., if the system is stable)

# Example: The covariance

The equation for the continuous-time propagation of the covariance of the state is

$$\dot{P} = 2fP + q_c$$

Solving this equation for $P(t)$ gives

$$P(t) = \left( P(0) + \frac{q_c}{2f} \right) \exp(2ft) - \frac{q_c}{2f}$$

- The covariance will increase without bound if $f > 0$ (i.e., if the system is unstable)
- The covariance will asymptotically tend to $-q_c/2f$ if $f < 0$ (i.e., if the system is stable)

# Example: Steady-state solution

The steady-state value of $P$ can also be computed (provided that $f < 0$) as

$$
\begin{aligned}
P &= \int_0^\infty e^{2f\tau} q_c d\tau \\
&= \frac{q_c}{2f} e^{2f\tau} |_0^\infty \\
&= -\frac{q_c}{2f}
\end{aligned}
$$

Compare the results with those of the previous example.