1. Convex set

   (1) Show that a polyhedron $\{x \in R^n : Ax \le b\}$ for some $A \in R^{m\times n}$, $b \in R^m$, is convex.

   (2) Consider a convex function $f : R^n \to R$. Prove that the set

   $$\{(x,t) \mid f(x) \le t, x \in R^n, t \in R\}$$

   is convex.

   (3) Show that $\{x \in R_+^n \mid \prod_{i=1}^n x_i \ge 1\}$ is convex. (Hint: If $a,b \ge 0$ and $0 \le \theta \le 1$, than $a^\theta b^{(1-\theta)} \le \theta a + (1-\theta)b$)

   (4) Show that the set $\{\mathrm{x} \mid \|x-a\|_2 \le \theta \|x-b\|_2\}$,where $a \ne b$ and $0 \le \theta \le 1$, is convex.

2. Convex function

(1) Prove that that the entropy function, defined as

$$f(x) = -\sum_{i=1}^{n} x_i \log(x_i)$$

with $dom(f) = \{\mathrm{x} \in \mathrm{R}_{++}^n : \sum_{i=1}^n x_i = 1\}$, is strictly concave.

(2) Show that $f(x_1, x_2) = \dfrac{1}{x_1 x_2}$ on R_{++}^2 is convex.

(3) Show that $f(X) = tr(X^{-1})$ is convex on $\mathrm{dom}\, f = S_{++}^n$

3. Dual problem

(1) Formulate the dual problems of the following problems with one inequality constraint

$$\begin{aligned} \min \quad & c^T x \\ s.t. \quad & f(x) \le 0 \end{aligned}$$

(2) Find the dual problem of the following general Linear programming

$$\begin{aligned} \min \quad & c^T x \\ s.t. \quad & Gx \le h \\ & Ax = b \end{aligned}$$

4. KKT condition

   (1) Give the KKT conditions of the following optimization problem

$$\begin{aligned} \min \quad & x_1^2 + x_2^2 \\ s.t. \quad & (x_1 - 1)^2 + (x_2 - 1)^2 \le 1 \\ & (x_1 - 1)^2 + (x_2 + 1)^2 \le 1 \end{aligned}$$

(2) Consider the equality constrained least square problem

$$\begin{aligned} \min \quad & \|Ax - b\|_2^2 \\ s.t. \quad & Gx = h \end{aligned}$$

Where $A \in R^{m \times n}$ with $\mathrm{rank} A = n$ and $G \in R^{p \times n}$ with $\mathrm{rank} G = p$

Give the KKT conditions and derive expressions for the primal solution x^* and the dual solution v^*.

5. Gradient and Newton Descent

Consider the optimization

$$\min_{x_1, x_2} \; e^{x_1 + 3x_2 - 0.1} + e^{x_1 - 3x_2 - 0.1} + e^{-x_1 - 0.1}$$

Write a code to solve this optimization using the gradient method and Newton method with the backtracking parameters α=0.1 and β=0.6, draw $f(x_k)$ verses k for k =0, 1, 2……,50.