1、$f(x)=a^Tx$, $\frac{\partial f}{\partial x}=a$; $f(x)=x^Ta$, $\frac{\partial f}{\partial x}=a$

2、最小二乘：$\min f(x)=\|Ax-b\|_2^2$

$\|Ax-b\|_2^2=(Ax-b)^T(Ax-b)$

$=x^TA^TAx-x^TA^Tb-b^TAx+b^Tb$

$\frac{\partial f}{\partial x}=2A^TAx-2A^Tb+0$

$\Rightarrow x^*=(A^TA)^{-1}A^Tb$ $\quad \frac{\partial^2 f}{\partial x^2}=2A^TA\geq 0$

3、$A^TA\geq 0$

$x^TA^TAx=\|Ax\|_2^2\geq 0$

4、Affine 仿射集

① 通过 x_1,x_2 的直线可以定义为下述形式：

$x=\theta x_1+(1-\theta)x_2\quad(\theta\in R)$

② Affine set：contains the line through any two distinct points in the set

e.g. $\{x|Ax=b\}$

5、line segment 线段

$x=\theta x_1+(1-\theta)x_2\quad 0\leq\theta\leq 1$

6、凸集：convex set

$x_1,x_2\in C$，$0\leq\theta\leq 1\Rightarrow \theta x_1+(1-\theta)x_2\in C$

③ 凸组合：$x=\theta_1x_1+\cdots+\theta_kx_k$

$\theta_1+\cdots+\theta_k=1$，$\theta_i\geq 0$

④ 凸包：包含集合最小的凸集

7、凸锥：convex cone

锥组合：$x=\theta_1x_1+\theta_2x_2(\theta_1\geq 0,\theta_2\geq 0)$

凸锥：所有点的锥组合仍在其中。

8、hyperplane 超平面 $\{x|a^Tx=b\}\ (a\neq 0)$

halfspace 半空间 $\{x|a^Tx\leq b\}\ (a\neq 0)$

9、(Euclidean) ball

$B(x_c,r)=\{x|\|x-x_c\|_2\leq r\}=\{x_c+ru|\|u\|_2\leq 1\}$

10、ellipsoid 椭圆体

$\{x|(x-x_c)^TP^{-1}(x-x_c)\leq 1\}$ with $P\in S_{++}^n$

$\Rightarrow\{x_c+Au|\|u\|_2\leq 1\}$, $A=P^{\frac12}$, $u=P^{-\frac12}(x-x_c)$

11、范数球：$\{x|\|x-x_c\|\leq r\}$ 都是凸集

范数锥：$\{(x,t)|\|x\|\leq t\}$ =阿锥

12、多面体 polyhedra

$Ax\leq b$, $Cx=d$

13、半正定锥 Postive semidefinite cone

① S^n：对称矩阵 S_+^n：对称半正定 S_{++}^n对称正定

② $S_+^n=\{x\in S^n|x\geq 0\}\ x\in S_+^n\Leftrightarrow z^Txz\geq 0\ \forall z$.

凸锥

e.g. $\begin{bmatrix}x & y\\ y & z\end{bmatrix}\in S_+^n\Leftrightarrow xz-y^2\geq 0$

14、保持集合凸性的操作

①定义：$x_1,x_2\in C$，$0\leq\theta\leq 1\Rightarrow\theta x_1+(1-\theta)x_2\in C$

②映射：

(1)仿射函数：原像是凸集 $\Leftrightarrow$ 像是凸集

e.g. $\{x|x^Tx\leq(c^Tx)^2,c^Tx\geq 0\}$ with $P\in S_+^n$

P半正定 $\Rightarrow P=P^{\frac12}P^{\frac12}$，$x^Tx=(P^{\frac12}x)^T(P^{\frac12}x)=\|P^{\frac12}x\|_2^2$

$f(x)=\begin{pmatrix}P^{\frac12}\\ c^T\end{pmatrix}x+Ax=\begin{pmatrix}y\\ t\end{pmatrix}$ 仿射函数

$\|y\|_2^2\leq t^2\|_2^2$ 一=阿锥，凸集

故原像是凸集。

(2)透视函数：$P(x,t)=\frac{x}{t}$，$dom f=\{x|t|t>0\}$

像和原像都是凸集。

(3)线性分式函数：$f(x)=\frac{Ax+b}{c^Tx+d}$ $dom f=\{x|c^Tx+d>0\}$

仿射和透视函数复合运算

15、minimum element 全局最小 $y\in S\Rightarrow x\preceq_K y$

minimal element $y\in S$, $y\preceq_K x\Rightarrow y=x$

$x\preceq_K y$，即 $y-x\in K$

16、Dual cones 对偶锥 $K^*=\{y|y^Tx\geq 0$ for all $x\in K\}$

e.g. $K=R_+^n$, $K^*=R_+^n$; $K=S_+^n$, $K^*=S_+^n$

1、凸函数 dom f is convex set

$f(\theta x+(1-\theta)y)\leq\theta f(x)+(1-\theta)f(y)$

2、阿锥

$f:R^n\to R$ is convex iff $g:R\to R$, $g(t)=f(x+tv)$

$dom g=\{t|x+tv\in dom f\}$ is convex in t for any $x\in dom f$, $v\in R^n$

3、一阶条件

f is differentiable. if $dom f$ is open and the gradient $\nabla f(x)$ exists.

当 $f(y)\geq f(x)+\nabla f(x)(y-x)$ for all $x,y\in dom f$

在凸集下，f 是凸函数

4、二阶条件

f 是凸函数，当且仅当 $\nabla^2f(x)\geq 0$ for all $x\in dom f$

5、Epigraph and sublevel set

α-sublevel set of $f:R^n\to R$

$C_\alpha=\{x\in dom f|f(x)\leq\alpha\}$

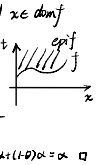
凸函数的 sublevel set 是凸集，反之不成立

证：$f(\theta x_1+(1-\theta)x_2)\leq\alpha$. $\forall x_1,x_2\in C_\alpha$

$f(\theta x_1+(1-\theta)x_2)\leq\theta f(x_1)+(1-\theta)f(x_2)\leq\theta\alpha+(1-\theta)\alpha=\alpha$ □

epi $f=\{(x,t)\in R^{n+1}|x\in dom f, f(x)\leq t\}$

f is convex function iff epi f is a convex set



6、Jensen's inequality： $f(Ez)\leq Ef(z)$

prob$(z=x)=\theta$ prob$(z=y)=1-\theta$

$f(\theta x+(1-\theta)y)\leq\theta f(x)+(1-\theta)f(y)$

7、保持凸性的操作

①非负求和

②仿射函数复合

$f(Ax+b)$ is convex if f is convex

any norm of affine function $f(x)=\|Ax+b\|$ 凸

③逐点最最大

$f_1,\cdots f_n$ are convex, $f(x)=\max\{f_1(x),\cdots f_n(x)\}$ 凸

④上确界

If $f(x,y)$ is convex in x for each $y\in A$, then $g(x)=\sup_{y\in A}f(x,y)$ is convex

⑤符推复合 $f(x)=h(g(x))$

$f'(x)=h''(g(x))g'(x)^2+h'(g(x))g''(x)\geq 0$

f is convex if $\begin{cases}g\text{ 凸}, h\text{ 凸}, \tilde{x}\text{ 非减}\\ g\text{ 凹}, h\text{ 凹}, \tilde{x}\text{ 非增}\end{cases}$

Vector composition： $f(x)=h(g_1(x),\cdots,g_k(x))$

$f''(x)=g'(x)^T\nabla^2h(g(x))g'(x)+\nabla h(g(x))^Tg''(x)$

⑥下确界

If $f(x,y)$ is convex in (x,y) and C is a convex set, then $g(x)=\inf_{y\in C}f(x,y)$ is convex

证明：Schur complement 舒尔补

$f(x,y)=x^TAx+2x^TBy+y^TCy$ with $\begin{bmatrix}A & B\\ B^T & C\end{bmatrix}\geq 0$, $C>0$

minimizing over y gives $g(x)=\inf_y f(x,y)=x^T(A-BC^{-1}B^T)x$

g is convex, hence Schur complement $A-BC^{-1}B^T\geq 0$

证明：$\nabla f_y=2B^Tx+2Cy=0\Rightarrow y=-C^{-1}B^Tx$

$\Rightarrow g(x)=x^T(A-BC^{-1}B^T)x$

⑦透视函数

$f:R^n\to R$, $g:R^n\to R$

$g(x,t)=tf(\frac{x}{t})$, $dom g=\{(x,t)|\frac{x}{t}\in dom f,t>0\}$

g is convex if f is convex

8、共轭函数（the conjugate function）

$f^*(y)=\sup_{x\in dom f}(y^Tx-f(x))$, f^* is convex even if f is not

eg. $f(x)=-\log x$

$f^*(y)=\sup_{x>0}(xy+\log x)=\begin{cases}-1-\log(-y), & y<0\\ \infty, & \text{otherwise}\end{cases}$

$(xy+\log x)'_x=y+\frac1x=0\Rightarrow x=-\frac1y$ $\quad (xy+\log x)''_{x^2}=-\frac{1}{x^2}<0$.

e.g. $f(x)=\frac12x^TQx$ with $Q\in S_{++}^n$

$f^*(y)=\sup_x(y^Tx-\frac12x^TQx)=\frac12y^TQ^{-1}y$

$x=Q^{-1}y$ 为极大值点

9、Quasiconcave functions 准凹函数

$f:R^n\to R$ is 准凹 if $dom f$ is convex and the sublevel sets $S_\alpha=\{x\in dom f|f(x)\geq\alpha\}$ are convex for all α

f 准凹，$-f$ 准凸

两个准凹函数的和不一定是准凹函数

10、准凹性质：

① 准凹 f：

$\forall\theta\geq 0$ 当 $\Rightarrow f(\theta x+(1-\theta)y)\leq\max\{f(x),f(y)\}$

② 一阶条件：可微的 f 在定义域是凸集的情况下，准凹 iff $f(y)\leq f(x)\Rightarrow\nabla f(x)^T(y-x)\leq 0$)

11、Log-concave and log-convex functions

a positive function f is log-concave if $\log f$ is concave： $f(x)=e^{\log f(x)}$

$f(\theta x+(1-\theta)y)\geq f(x)^\theta f(y)^{1-\theta}$ for $0\leq\theta\leq 1$

性质：twice differentiable f with convex domain is log-concave iff

$f(x)\nabla^2f(x)\leq\nabla f(x)\nabla f(x)^T$ for all $x\in dom f$

小优化问题的标准形式

$\min f_0(x)$

s.t. $f_i(x)\leq 0$, $i=1\cdots m$ $\quad p^*=\infty$, infeasible

$h_i(x)=0$, $i=1\cdots p$ $\quad p^*=-\infty$, unbounded below

最优值：$p^*=\inf\{f_0(x)|f_i(x)\leq 0, h_i(x)=0\}$

2、凸优化问题（标准形式）

$\min f_0(x)$ s.t. $f_i(x)\leq 0$, $a_i^Tx=b_i$ $\Leftrightarrow$ $\min f_0(x)$ s.t. $f(x)\leq 0$, $Ax=b$

f_0,f_i 凸. 若 f_i 为准凸则为准凸优化.

凸优化问题的局部最优解为全局最优

3、最优性条件

x 是最优 iff feasible and $\nabla f_0(x)^T(y-x)\geq 0$ for all feasible y

4、无约束问题：若 x 最优，那么 $x\in dom f_0$, $\nabla f_0(x)=0$.

等式约束问题：$\min f_0(x)$ s.t. $Ax=b$. exists a v

$\Leftrightarrow x\in dom f_0$, $Ax=b$, $\nabla f_0(x)+A^Tv=0$.

非负象限最小化：$\min f_0(x)$ s.t. $x\geq 0$

$x\in dom f_0$, $x\geq 0$. $\begin{cases}\nabla f_0(x)_i\geq 0, & x_i=0\\ \nabla f_0(x)_i=0, & x_i>0\end{cases}$

5、等降等式约束 $\min f_0(x)$ s.t. $f_i(x)\leq 0$, $Ax=b$

$Ax=b\Rightarrow x=x_0+Fz$. 问题变为 $\min f_0(Fz+x_0)$ s.t. $f_i(Fz+x_0)\leq 0$

引入等式约束：$\min f_0(A_0x+b_0)$ s.t. $f_i(A_ix+b_i)\leq 0$

$\Leftrightarrow\min_{x,y_i}f_0(y_0)$ s.t. $f_i(y_i)\leq 0$, $y_i=A_ix+b_i$

对好线性不等式引入松弛变量：

$a_i^Tx\leq b_i\Leftrightarrow a_i^Tx+s_i=b_i$, $s_i\geq 0$

epigraph form： $\min t$ s.t. $f_0(x)-t\leq 0$, $f_i(x)\leq 0$, $Ax=b$.

最小化一些变量：$\min f_0(x_1,x_2)$ s.t. $f_i(x_1)\leq 0$

$\Leftrightarrow\min\tilde{f}_0(x_1)$ s.t. $f_i(x_1)\leq 0$ $\tilde{f}_0(x_1)=\inf_{x_2}f_0(x_1,x_2)$

6、准凸优化问题

if f_0 准凸，存在函数 $\phi_t(x)$ 满足：固定 t 之后，$\phi_t(x)$ 是凸的，且有 $f_0(x)\leq t\Leftrightarrow\phi_t(x)\leq 0$

7、LP：$\min c^Tx+d$ s.t. $Gx\leq h$, $Ax=b$

e.g. 多面体的切比雪夫中心

$P=\{x|a_i^Tx\leq b_i\}$ is center of largest inscribed ball $B=\{x_c+u|\|u\|_2\leq r\}$. 最大内切球

$a_i^Tx\leq b_i$ for all $x\in B$ iff

$\sup\{a_i^T(x_c+u)|\|u\|_2\leq r\}=a_i^Tx_c+r\|a_i\|_2\leq b_i$;

$u=r\frac{a_i}{\|a_i\|_2}$, 用到 $|a^Tb|\leq\|a\|_2\|b\|_2$

$\Leftrightarrow\max r$ s.t. $a_i^Tx_c+r\|a_i\|_2\leq b_i$;

8、线性分式规划 $\min f(x)$ s.t. $Gx\leq h$, $Ax=b$

$f_0(x)=\frac{c^Tx+d}{e^Tx+f}$, $dom f(x)=\{x|e^Tx+f>0\}$

令 $y=\frac{x}{e^Tx+f}$, $z=\frac{1}{e^Tx+f}$ $\Leftrightarrow$ $\min c^Ty+dz$

s.t. $Gy\leq hz$, $Ay=bz$

$e^Ty+fz=1$, $z\geq 0$

11、二阶锥规划 (SOCP)

$\min f^Tx$ s.t. $\|A_ix+b_i\|_2\leq c_i^Tx+d_i$, $Fx=g$

12、几何规划

monomial function 单项式：

$f(x)=cx_1^{a_1}x_2^{a_2}\cdots x_n^{a_n}$ $dom f=R_{++}^n$

with $c>0$, $\alpha_i\in R$

posynomial function：sum of monomials

$f(x)=\sum_{k=1}^K c_kx_1^{a_{1k}}\cdots x_n^{a_{nk}}$, $dom f=R_{++}^n$

GP： $\min f_0(x)$, s.t. $f_i(x)\leq 1$, $h_i(x)=1$

with $f_i(x)$ posynomial, $h_i(x)$ monomial

$y_i=\log x_i\Leftrightarrow x_i=e^{y_i}$

$f(x)=cx_1^{a_1}\cdots x_n^{a_n}$ transforms to

$=ce^{a_1y_1+\cdots+a_ny_n}$

$\log f(e^{y_1},\cdots,e^{y_n})=a^Ty+b$ $\quad (b=\log c)$

13、半定规划 Semidefinite program

$\min c^Tx$ s.t. $x_1\tilde{F}_1+x_2\tilde{F}_2+\cdots+x_n\tilde{F}_n+G\leq 0$ $\quad Ax=b$

with $F_i,G\in S^k$

舒尔补：$\begin{bmatrix}A & B\\ C & D\end{bmatrix}\succeq 0\Rightarrow A-BC^{-1}B\geq 0$

$\Rightarrow C-B^TA^{-1}B\geq 0$

LP and equivalent SDP

LP: $\min c^Tx$ s.t. $Ax\leq b$ $\Leftrightarrow$ SDP: $\min c^Tx$ s.t. $diag(Ax-b)\leq 0$

最小化特征根：$\min\lambda_{max}(A(x))$

$\Leftrightarrow$ SDP: $\min t$ s.t. $A(x)\leq tI$

$\lambda_{max}(A)\leq t\Leftrightarrow A\leq tI$

最小化特征范数：

$\min\|A(x)\|_2=(\lambda_{max}(A(x)^TA(x)))^{\frac12}$

$\Leftrightarrow\min t$ s.t. $\begin{bmatrix}tI & A(x)\\ A(x)^T & tI\end{bmatrix}\succeq 0$

即通过 $\|A\|_2\leq t$

$\Leftrightarrow A^TA\leq t^2I$, $t\geq 0\Leftrightarrow\begin{bmatrix}tI & A(x)\\ A(x)^T & tI\end{bmatrix}\succeq 0$

4、对偶问题

原问题：$\min c^Tx$ s.t. $Ax=b$, $x\geq 0$

对偶问题：$\max -b^Tv$ s.t. $A^Tv+c\geq 0$

假设凸优化的最优值为 d^*,

弱对偶：$d^*\leq p^*$，强对偶：$d^*=p^*$

e.g.原： $\min c^Tx$ s.t. $Ax\leq b$.

$g(\lambda)=\inf_x(c^Tx+\lambda^T(Ax-b))=\begin{cases}-b^T\lambda & A^T\lambda+c=0\\ -\infty\end{cases}$

对偶： $\max -b^T\lambda$ s.t. $A^T\lambda+c=0$, $\lambda\geq 0$

e.g.原： $\min x^TPx$ s.t. $Ax\leq b$

$g(\lambda)=\inf_x(x^TPx+\lambda^T(Ax-b))=-\frac14\lambda^TAP^{-1}A^T\lambda-b^T\lambda$

对偶： $\max -\frac14\lambda^TAP^{-1}A^T\lambda-b^T\lambda$ s.t. $\lambda\geq 0$

5、互补松弛条件

假设强对偶：x^* 原问题最优，(λ^*,ν^*) 对偶最优

那么：$\lambda_i^*>0\Rightarrow f_i(x^*)=0$；$f_i(x^*)<0\Rightarrow\lambda_i^*=0$.

6、KKT条件：

① 原问题的约束：$f_i(x)\leq 0$, $h_i(x)=0$

② 对偶的约束：$\lambda\geq 0$

③ 互补松弛：$\lambda_if_i(x)=0$

④ 拉格朗日函数关于 x 的梯度：

$\nabla_xL(x,\lambda,\nu)=\nabla f_0(x)+\sum\lambda_i\nabla f_i(x)+\sum\nu_i\nabla h_i(x)=0$

强对偶成立，x,λ,ν 最优，则必满足 KKT 条件

7、凸优化问题的 KKT 条件

$\tilde{x},\tilde{\lambda},\tilde{\nu}$ 对于一个凸优化问题满足 KKT 条件，那么

$f_0(\tilde{x})=L(\tilde{x},\tilde{\lambda},\tilde{\nu})$, $g(\tilde{\lambda},\tilde{\nu})=L(\tilde{x},\tilde{\lambda},\tilde{\nu})$

强对偶 $\quad$ 互补松弛性

$\Rightarrow f_0(\tilde{x})=g(\tilde{\lambda},\tilde{\nu})$

1、梯度下降法 Gradient descent method

① 强凸：f is strongly convex on S if there exists an $m>0$ such that $\nabla^2f(x)\geq mI$ for all $x\in S$

② $x^{(k+1)}=x^{(k)}+t^{(k)}\Delta x^{(k)}$ with $f(x^{(k+1)})<f(x^{(k)})$

③ $\Delta x=-\nabla f(x)$

2、最深下降法 Steepest descent method

① 归一化最深下降方向 (at x, for norm $\|\cdot\|$):

$\Delta x_{nsd}=\arg\min\{\nabla f(x)^Tv|\|v\|=1\}$

for small v, $f(x+v)\approx f(x)+\nabla f(x)^Tv$

② (unnormalized) steepest descent direction

$\Delta x_{sd}=\|\nabla f(x)\|_*\Delta x_{nsd}$

satisfies $\nabla f(x)^T\Delta x_{sd}=-\|\nabla f(x)\|_*^2$

③ steep descent method

general descent method with $\Delta x=\Delta x_{sd}$

convergence properties similar to gradient descent

3、Newton step

$\Delta x_{nt}=-\nabla^2f(x)^{-1}\nabla f(x)$

解释：$\hat{f}(x+v)=f(x)+\nabla f(x)^Tv+\frac12v^T\nabla^2f(x)v$

二阶近似函数的最小值

$\nabla f(x+v)\approx\nabla\hat{f}(x+v)=\nabla f(x)+\nabla^2f(x)v=0$

搜索梯度=0

牛顿下降量：$\lambda(x)=(\nabla f(x)^T\nabla^2f(x)^{-1}\nabla f(x))^{\frac12}$

a measure of the proximity of x to x^*

终止条件 $\varepsilon>0$, 判断 $\frac{\lambda^2}{2}\leq\varepsilon$

4、等式约束

① 目标函数是二次函数

$\min\frac12x^TPx+q^Tx+r$ s.t. $Ax=b$

optimality condition： $\begin{bmatrix}P & A^T\\ A & 0\end{bmatrix}\begin{bmatrix}x^*\\ v^*\end{bmatrix}=\begin{bmatrix}-q\\ b\end{bmatrix}$

② Newton step

$\begin{bmatrix}\nabla^2f(x) & A^T\\ A & 0\end{bmatrix}\begin{bmatrix}\Delta x_{nt}\\ w\end{bmatrix}=\begin{bmatrix}-\nabla f(x)\\ 0\end{bmatrix}$

w 一般特别复杂

③ 由不可行解出发

$\begin{bmatrix}\nabla^2f(x) & A^T\\ A & 0\end{bmatrix}\begin{bmatrix}\Delta x_{nt}\\ w\end{bmatrix}=-\begin{bmatrix}\nabla f(x)\\ Ax-b\end{bmatrix}$

$\begin{bmatrix}\nabla^2f(x) & A^T\\ A & 0\end{bmatrix}\begin{bmatrix}\Delta x_{nt}\\ \Delta\omega\end{bmatrix}=-\begin{bmatrix}\nabla f(x)+A^Tv\\ Ax-b\end{bmatrix}$

其中 $w=v+\Delta v_{nt}$

1、变分法：

$J=h(x(t_f),t_f)+\int_{t_0}^{t_f}g(x(t),u(t),t)dt$

$\dot{x}(t)=a(x(t),u(t),t)$

$H(x,u,p,t)=g(x,u,t)+p^T(t)\cdot a(x,u,t)$

即哈密顿函数 $H=g+p^Ta$

$\delta J_a=0$ 的必要条件是

① 状态方程：$\dot{x}=a(x,u,t)$

② 协态方程：$\dot{p}=-H_x^T$

③ 控制方程：$H_u=0$

④ 边界条件(t_f 变化)：$h_x(t_f)+H(t_f)=0$

对称性：$\dot{x}=a(x,u,t)=(\frac{\partial H}{\partial p})^T$, $\dot{p}=-(\frac{\partial H}{\partial x})^T$

如果 $x_i(t_f)$ 是自由的，还需满足横截条件：

$p_i(t_f)=\frac{\partial h}{\partial x_i}(t_f)$

e.g. Simple double integrator system starting at $y(0)=10$, $\dot{y}(0)=0$, must drive to origin $y(t_f)=\dot{y}(t_f)=0$ to minimize the cost ($t_f>0$) $J=\frac12\alpha t_f^2+\frac12\int_0^{t_f}bu^2(t)dt$

Define the dynamics with $x_1=y$, $x_2=\dot{y}$ so that

$\dot{x}(t)=Ax(t)+Bu(t)$ $\quad A=\begin{bmatrix}0&0\\1&0\end{bmatrix}$, $B=\begin{bmatrix}0\\1\end{bmatrix}$

with $p(t)=[p_1(t),p_2(t)]^T$, define the Hamiltonian.

$H=g+p^T(t)a=\frac12bu^2+p^T(t)(Ax(t)+Bu(t))$

$=\frac12bu^2(t)+p_1(t)x_2(t)+p_2(t)u(t)$

The necessary conditions are then that:

$\dot{p}=-H_x^T\rightarrow\dot{p}_1=-\frac{\partial H}{\partial x_1}=0\rightarrow p_1(t)=c_1$

$\dot{p}_2=-\frac{\partial H}{\partial x_2}=-p_1\rightarrow p_2(t)=-c_1t+c_2$

$H_u=bu+p_2=0\rightarrow u=-\frac{p_2}{b}=-\frac{c_2}{b}+\frac{c_1}{b}t$

Now impose the boundary conditions

$H(t_f)+h_x(t_f)=\frac12bu^2(t_f)+p_1(t_f)x_2(t_f)$

$+p_2(t_f)u(t_f)+\alpha t_f=0$

$=\frac12bu^2(t_f)+(-bu(t_f))u(t_f)+\alpha t_f$

$=-\frac12bu^2(t_f)+\alpha t_f=0\rightarrow t_f=\frac{1}{2\alpha b}(-c_2+c_1t_f)^2$

Now go back to the state equations:

$x_2(0)=0$, $x_1(0)=10$ $\quad\Rightarrow\quad c_1=\frac{120b}{t_f^3}$, $c_2=\frac{60b}{t_f^2}$

$x_2(t_f)=0$, $x_1(t_f)=0$ $\quad\quad c_3=0$, $c_4=10$

e.g. $\dot{x}=-x(t)+u(t)$, $x(0)=1$, $x(t_f)=0$.

$J=\frac12\int_0^{t_f}(x^2+u^2)dt$

$x(0)-x(t_f)$给定，不需横截条件

$H=\frac12(x^2+u^2)+\lambda(-x+u)$

协态方程：$\dot{\lambda}=-\frac{\partial H}{\partial x}=-x+\lambda$

状态方程：$\frac{\partial H}{\partial u}=u+\lambda=0$ $\Rightarrow u=-\lambda$

得 $\begin{cases}\dot{x}(t)=-x(t)-\lambda(t)\\ \dot{\lambda}(t)=-x(t)+\lambda(t)\end{cases}$

拉氏变换：

$\begin{cases}sX(s)-x(0)=-X(s)-\lambda(s)\\ s\lambda(s)-\lambda(0)=-X(s)+\lambda(s)\end{cases}$

$\Rightarrow X(s)=\frac{x(0)-\lambda(s)}{s+1}$

$\Rightarrow(s^2-2)\lambda(s)=(s+1)\lambda(0)-x(0)$

$\lambda(s)=\frac{(s+1)\lambda(0)-x(0)}{s^2-2}$

$=\frac{s+1}{(s+\sqrt2)(s-\sqrt2)}\lambda(0)-\frac{1}{(s+\sqrt2)(s-\sqrt2)}x(0)$

$\Rightarrow\lambda(t)=\frac{1}{2\sqrt2}(e^{-\sqrt2t}-e^{\sqrt2t})\lambda(0)$

$+\frac{1}{2\sqrt2}[(\sqrt2-1)e^{\sqrt2t}+(\sqrt2+1)e^{-\sqrt2t}]x(0)$

$X(s)=\frac{s-1}{(s+\sqrt2)(s-\sqrt2)}x(0)-\frac{1}{(s+\sqrt2)(s-\sqrt2)}\lambda(0)$

$x(t)=\frac{1}{4\sqrt2}[(\sqrt2+1)e^{-\sqrt2t}+(\sqrt2-1)e^{\sqrt2t}]x(0)$

$+\frac{1}{2\sqrt2}(e^{\sqrt2t}-e^{-\sqrt2t})\lambda(0)$

$x(0)=1$, $x(t_f)=0$ $\Rightarrow\lambda(0)=\frac{\sqrt2-1}{e^{\sqrt2t_f}-e^{-\sqrt2t_f}}$

$\Rightarrow\lambda(t)\Rightarrow u(t)$

e.g. $\dot{x}_1(t)=x_2(t)$ $\quad x_1(0)=1$ $\quad x_1(1)=0$

$\dot{x}_2(t)=u(t)$ $\quad x_2(0)=1$ $\quad x_2(1)$ 自由

$J(u)=\frac12\int_0^1u^2(t)dt$ 取极小值

$x(1)$ 自由，横截条件 因 $h(x(t_f),t_f)=0$,

所以 $p_2(1)=\frac{\partial h}{\partial x_2(1)}=0$. $H=\frac12u^2+p_1x_2+p_2u$

$\dot{p}_1=-\frac{\partial H}{\partial x_1}=0$, $\dot{p}_2=-\frac{\partial H}{\partial x_2}=-p_1$, $\frac{\partial H}{\partial u}=0$

即 $u+p_2=0$. 得 $u^*(t)=-p_2(t)$

1. $C=\{x\in\mathbb{R}^n \mid \frac{\|x-a\|_2}{\|x-b\|_2}\le\alpha,\ \|x-a\|_2\le\|x-b\|_2\}$

$=\{x\in\mathbb{R}^n \mid \|x-a\|_2\le\alpha\|x-b\|_2,\ \|x-a\|_2^2\le\|x-b\|_2^2\}$

$=\begin{cases}\phi & \alpha<0\\ \{x\in\mathbb{R}^n \mid \|x-a\|_2^2\le\alpha^2\|x-b\|_2^2\}, & 0\le\alpha\le1\\ \{x\in\mathbb{R}^n \mid \|x-a\|_2\le\|x-b\|_2\}, & \alpha>1\end{cases}$

① C 为 ϕ 时，是凸集

② $0\le\alpha<1$ 时，

$\|x-a\|_2\le\alpha\|x-b\|_2\Rightarrow\|x-a\|_2^2\le\alpha^2\|x-b\|_2^2$

$x^Tx-2a^Tx+a^Ta\le\alpha^2(x^Tx-2b^Tx+b^Tb)$

$(1-\alpha^2)x^Tx+2(\alpha^2b^T-a^T)x+a^Ta-\alpha^2b^Tb\le0$

令 $f(x)=(1-\alpha^2)x^Tx+2(\alpha^2b^T-a^T)x+a^Ta-\alpha^2b^Tb$

$0\le\alpha<1\Rightarrow(1-\alpha^2)\ge0$

$\nabla^2f(x)=(1-\alpha^2)I\ge0$，故 $f(x)$ 是凸函数

$f(x)\le0$ 为 $f(x)$ 的 0-sublevel set

故 $\{x\mid f(x)\le0\}$ 是凸集

③ $\alpha>1$ 时，

$C=\{x\mid\|x-a\|_2\le\|x-b\|_2\}=\{x\mid2(b-a)^Tx+a^Ta-b^Tb\le0\}$

C是个半空间，故是凸集

2、

① $f:S^n\to\mathbb{R}$ with $f(X)=\log\det X$, $\mathrm{dom}X=S^n_{++}$

$g(t)=f(X+tV)=\log\det(X+tV)$

$=\log[\det(X^{\frac12})\cdot\det(I+tX^{-\frac12}VX^{-\frac12})\cdot\det(X^{\frac12})]$

$=\log\det X+\log\det(I+tX^{-\frac12}VX^{-\frac12})$

$=\log\det X+\sum_{i=1}^n\log(1+t\lambda_i)$

λ_i 为 $X^{-\frac12}VX^{-\frac12}$ 的特征向量

$g'(t)=\sum_{i=1}^n\frac{\lambda_i}{1+t\lambda_i}$

$g''(t)=-\sum_{i=1}^n\frac{\lambda_i^2}{(1+t\lambda_i)^2}\le0$

g is concave, hence f is concave

② $g(\lambda,\nu)=\inf_x(\sum_{i=1}^n\lambda_ix_i+\sum_{i=1}^p\nu_i(x^Ta_i))$ with $\lambda,\nu\in\mathbb{R}^m$

$g(\lambda,\nu)=-\sup_x(-\sum_{i=1}^n\lambda_ix_i-\sum_{i=1}^p\nu_i(x^Ta_i))$

$=-\sup_x(-x^T\lambda-x^T\mathrm{diag}(\nu)x)$ $\quad x=(\lambda,\nu)$, $y=x$

$=-\sup_{x\in\mathrm{dom}f}f(x,\lambda,\nu)$ $\quad g(x)=\inf_y f(x,y)$

$f(x,\lambda,\nu)$ 是 λ 和 ν 的线性函数

故 $f(x,\lambda,\nu)$ 是凸函数

$\because -g(\lambda,\nu)=\sup_{x\in\mathrm{dom}f}f(x,\lambda,\nu)$ is convex $\quad -g(x)=\sup_y(-f(x,y))$

$g(\lambda,\nu)$ is concave.

③ $f(x)=-\sum_{i=1}^n\log(b_i-a_i^Tx)$ on $\mathbb{R}^n$

$y_i=b_i-a_i^Tx$, $y=b-Ax$.

$f(y)=-\sum_{i=1}^n\log y_i$

$f(y)$ 为凸函数，y 为仿射函数

故 $f(x)=-\sum_{i=1}^n\log(b_i-a_i^Tx)$ 也为凸函数

④ $f(x,y)=\frac{x^2}{y}$

$\frac{\partial f}{\partial x}=\frac{2x}{y}$, $\frac{\partial f}{\partial y}=-\frac{x^2}{y^2}$

$\nabla^2f(x,y)=\begin{bmatrix}\frac{2}{y} & -\frac{2x}{y^2}\\ -\frac{2x}{y^2} & \frac{2x^2}{y^3}\end{bmatrix}=\frac{2}{y^3}\begin{bmatrix}y^2 & -xy\\ -xy & x^2\end{bmatrix}=\frac{2}{y^3}\begin{bmatrix}y\\ -x\end{bmatrix}\begin{bmatrix}y\\ -x\end{bmatrix}^T\ge0$

convex for $y>0$

⑤ $\log-\sum-\exp$: $f(x)=\log\sum_{i=1}^n\exp x_i$ is convex

$\frac{\partial f}{\partial x_i}=\frac{\exp x_i}{\sum_{i=1}^n\exp x_i}$, $\frac{\partial^2f}{\partial x_i\partial x_j}=-\frac{\exp x_i\exp x_j}{(\sum_{i=1}^n\exp x_i)^2}$

$\frac{\partial^2f}{\partial x_i^2}=-\frac{\exp x_i(\sum\exp x_i)-(\exp x_i)^2}{(\sum\exp x_i)^2}$

$\nabla^2f(x)=\frac{1}{1^Tz}\mathrm{diag}(z)-\frac{1}{(1^Tz)^2}zz^T$ $(z_i=\exp x_i)$

要证 $\nabla^2f(x)\ge0$，即证 $v^T\nabla^2f(x)v\ge0$ for all v：

$v^T\nabla^2f(x)v=\frac{(\sum z_iv_i^2)(\sum z_i)-(\sum v_iz_i)^2}{(\sum z_i)^2}\ge0$

由 Cauchy-Schwarz 不等式：$(\sum v_iz_i)^2\le(\sum z_iv_i^2)(\sum z_i)$

$|x^Ty|\le\|x\|_2\|y\|_2$.

⑥ $f(x)=\frac1x\int_0^xf(t)dt$ 是凸的

$f'(x)=-\frac{1}{x^2}\int_0^xf(t)dt+\frac{f(x)}{x}$

$f''(x)=\frac{2}{x^3}\int_0^xf(t)dt-2\frac{f(x)}{x^2}+\frac{f'(x)}{x}$

$=\frac{2}{x^3}\int_0^x(f(t)-f(x)-f'(x)(t-x))dt$

因为 f 是凸函数，$f(t)\ge f(x)+f'(x)(t-x)$

for all $x,t\in\mathrm{dom}f\Rightarrow f''(x)\ge0$

⑦ 利用一阶条件来证明二阶条件：即 $\nabla^2f(x)\ge0$

一阶条件：$f'(x)(y-x)\le f(y)-f(x)\le f(y)(y-x)$

假设 $y>x\Rightarrow\frac{f'(y)-f'(x)}{y-x}\ge0\Rightarrow f''(x)\ge0$

若假设 $f''(z)\ge0$，并且 $x<y$

$\Rightarrow0\le\int_x^yf''(z)(y-z)dz=(f'(z)(y-z))\Big|_{z=x}^{z=y}+\int_x^yf'(z)dz$

$=-f'(x)(y-x)+f(y)-f(x)\Rightarrow$

$f(y)\ge f(x)+f'(x)(y-x)$，故 f 是凸的

3、$\begin{cases}\dot x_1=x_2 & x_1(0)=10,\ x_2(0)=0\\ \dot x_2=u & x_1(4)=0,\ x_2(4)=0\end{cases}$ $\quad t_f=4$

$J=\int_0^4u^2(t)dt$

$\dot x(t)=\begin{bmatrix}\dot x_1(t)\\ \dot x_2(t)\end{bmatrix}=\begin{bmatrix}x_2(t)\\ u(t)\end{bmatrix}=a$

$H=g+p^Ta=u^2(t)+p_1x_2(t)+p_2u(t)$

$\begin{cases}\dot x=a=\begin{bmatrix}x_2(t)\\ u(t)\end{bmatrix}\\ \dot p=-H_x\\ H_u=0\end{cases}\Rightarrow\begin{cases}\begin{bmatrix}\dot p_1(t)\\ \dot p_2(t)\end{bmatrix}=-\begin{bmatrix}0\\ p_1(t)\end{bmatrix}\\ 2u(t)+p_2=0\end{cases}$

$\Rightarrow\begin{cases}\dot p_1=0\Rightarrow p_1=c_1\\ \dot p_2=-p_1\Rightarrow p_2=-c_1t+c_2\end{cases}$

$u(t)=-\frac{p_2}{2}=\frac{c_1}{2}t-\frac{c_2}{2}$

$\dot x_2=u(t)=\frac{c_1}{2}t-\frac{c_2}{2}$ $\quad x_2(t)=\frac14c_1t^2-\frac{c_2}{2}t+c_3$

$\left.\begin{matrix}x_2(0)=0\\ x_2(4)=0\end{matrix}\right\}\Rightarrow\begin{cases}c_3=0\\ c_2=2c_1\end{cases}$ $\quad x_2(t)=\frac{c_1}{4}t^2-c_1t$

$\dot x_1=x_2=\frac{c_1}{4}t^2-c_1t$ $\quad x_1(t)=\frac{c_1}{12}t^3-\frac{c_1}{2}t^2+c_4$

$\left.\begin{matrix}x_1(0)=10\\ x_1(4)=0\end{matrix}\right\}\Rightarrow\begin{cases}c_4=10\\ c_1=\frac{15}{4},\ c_2=2c_1=\frac{15}{2}\end{cases}$

$u(t)=\frac{15}{2}t-\frac{15}{4}$

4、(1) $\min\ -c^Tx+\sum_{i=1}^n y_i\log y_i$

s.t. $Px=y$, $x\ge0$, $e^Tx=1$; $e=[1,\cdots,1]^T$

$L(x,y,\lambda,\nu,w)$

$=-c^Tx+\sum_{i=1}^n y_i\log y_i+\lambda^T(-x)+\nu^T(Px-y)+w(e^Tx-1)$

$\begin{cases}\nabla_xL=0\\ \nabla_{y_i}L=0\end{cases}\Rightarrow\begin{cases}-c-\lambda+P^T\nu+we=0\\ 1+\log y_i-\nu_i=0\end{cases}\Rightarrow y_i=e^{\nu_i-1}$

$g(\lambda,\nu,w)=\begin{cases}\sum_{i=1}^n[e^{\nu_i-1}(\nu_i-1)]-\sum_{i=1}^n\nu_ie^{\nu_i-1}-w\\ =-\sum_{i=1}^ne^{\nu_i-1}-w,\ -c-\lambda+P^T\nu+we=0\\ -\infty,\ \text{otherwise}\end{cases}$

(2) $\min\ x^Tx$ s.t. $Ax=b$

$L(x,\nu)=x^Tx+\nu^T(Ax-b)$ 为梯度为零

$\Rightarrow\nabla_xL(x,\nu)=2x+A^T\nu=0\Rightarrow x=-\frac12A^T\nu$

$g(\nu)=L(-\frac12A^T\nu,\nu)=-\frac14\nu^TAA^T\nu-b^T\nu$

$p^*\ge g(\nu)$

(3) $\min c^Tx$ s.t. $Ax=b$, $x\ge0$

$L(x,\lambda,\nu)=c^Tx+\nu^T(Ax-b)-\lambda^Tx$

$=-b^T\nu+(c+A^T\nu-\lambda)^Tx$

$g(\lambda,\nu)=\inf_xL(x,\lambda,\nu)=\begin{cases}-b^T\nu, & A^T\nu-\lambda+c=0\\ -\infty, & \text{otherwise}\end{cases}$

$p^*\ge-b^T\nu$ if $A^T\nu-\lambda+c=0$

(4) $\min\|x\|$ s.t. $Ax=b$

$g(\nu)=\inf_x(\|x\|-\nu^TAx+b^T\nu)=\begin{cases}b^T\nu & \|A^T\nu\|_*\le1\\ -\infty & \text{otherwise}\end{cases}$

where $\|v\|_*=\sup_{\|u\|\le1}u^Tv$ 对偶范数

proof: $\inf_x(\|x\|-y^Tx)=0$ if $\|y\|_*\le1$

$-\infty$ otherwise

- if $\|y\|_*\le1$ then $\|x\|-y^Tx\ge0$ for all x with equality if $x=0$
- if $\|y\|_*>1$, choose $x=tu$ where $\|u\|\le1$

$u^Ty=\|y\|_*>1$:

$\|x\|-y^Tx=t(\|u\|-\|y\|_*)\to-\infty$ as $t\to\infty$

$\|tu\|-y^Ttu=t\|u\|-t\|y\|_*$

(5) $\min\ c^Tx$

s.t. $Ax=b$, $-1\le x_i\le1$, $i=1,\cdots,n$

$\Rightarrow e=[1,\cdots,1]^T$

$\Rightarrow\min\ c^Tx$

s.t. $Ax=b$, $-e\le x\le e$

$L(x,\lambda_1,\lambda_2,\nu)=c^Tx+\lambda_1^T(x-e)+\lambda_2^T(-e-x)+\nu^T(Ax-b)$

$=(c+\lambda_1-\lambda_2+A^T\nu)x-\lambda_1^Te-\lambda_2^Te-\nu^Tb$

$\nabla_xL=0\Rightarrow c+\lambda_1-\lambda_2+A^T\nu=0$

$g(\lambda_1,\lambda_2,\nu)=\begin{cases}-\lambda_1^Te-\lambda_2^Te-\nu^Tb, & c+\lambda_1-\lambda_2+A^T\nu=0\\ -\infty, & \text{otherwise}\end{cases}$

(6) $\min\|Ax-b\|_2^2$ s.t. $Gx=h$

$L(x,\nu)=\|Ax-b\|_2^2+\nu^T(Gx-h)$

$=x^TA^TAx+(G^T\nu-2A^Tb)^Tx-\nu^Th$

最小 $x=-\frac12(A^TA)^{-1}(G^T\nu-2A^Tb)$ 对偶函数：

$g(\nu)=-\frac14(G^T\nu-2A^Tb)^T(A^TA)^{-1}(G^T\nu-2A^Tb)-\nu^Th$

KKT 条件：$2A^T(Ax^*-b)+G^T\nu^*=0$, $Gx^*=h$

求解 x^*, ν^*

5、$\min f(x_1,x_2)=x_1^2+x_2^2+x_1x_2-10x_1-4x_2+100$

s.t. $x_1+x_2=8$

$x_2=8-x_1$

$\min g(x_1)=x_1^2+(8-x_1)^2+x_1(8-x_1)-10x_1$

$-4(8-x_1)+100=x_1^2-14x_1+132$

$\nabla_{x_1}g(x_1)=2x_1-14$

$\nabla^2_{x_1}g(x_1)=2$

$\Delta x_1^{(k)}=-\nabla^2g(x_k)^{-1}\nabla g(x_k)=-\frac{2x_1-14}{2}=-x_1+7$

$x_1^{(0)}=0$, $x_1^{(1)}=x_1^{(0)}+\Delta x_1^{(0)}=0+7=7$

$\lambda_2^2=\nabla f(x_1)^T\nabla^2g(x_1)^{-1}\nabla g(x_1)=0$

$\therefore x_1=x_1^{(1)}=7$, $x_2=8-x_1=1$

$\min f(x_1,x_2)=83$

6、(1) 证明 $x^*=(1,\frac12,-1)$ 是下列优化问题的最优解：$\min\ \frac12x^TPx+q^Tx+r$

s.t. $-1\le x_i\le1$, $i=1,2,3$，P,q,r 是已知量

解：$\nabla f_0(x^*)=(-1,0,2)$ 且

$\nabla f_0(x^*)(y-x)=-1(y_1-1)+2(y_3+1)\ge0$

for all y satisfying $-1\le y_i\le1$.

(2) $\min\ c^Tx$ s.t. $l\le x\le u$ where $l\le u$

目标和约束可以分开 The objective is a sum of terms c_ix_i, $x_i\to l_i\le x_i\le u_i$

对 $c_i>0$, then $x_i^*=l_i$; if $c_i<0$, then $x_i^*=u_i$, if $c_i=0$, $x_i\in[l_i,u_i]$ 故 $p^*=l^Tc^+u^Tc^-$ where $c_i^+=\max\{c_i,0\}$, and $c_i^-=\max\{-c_i,0\}$

(3) $\min c^Tx$ s.t. $1^Tx=1$, $x\ge0$

suppose $c_1=c_2=\cdots=c_k<c_{k+1}\le\cdots\le c_n$

$\Rightarrow c^Tx\ge c_1(1^Tx)=c_{\min}$ for all feasible x

$x_1+\cdots+x_k=1$, $x_1\ge0,\cdots,x_k\ge0$, $x_{k+1}\ge0$,

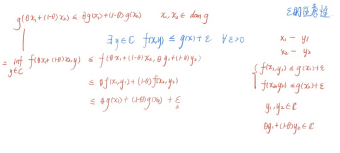
$x_{k+1}=\cdots=x_n=0$, $p^*=c_i=c_{\min}\Rightarrow$

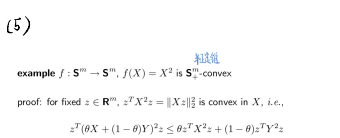
$p^*=\min\{c_1,c_{\min}\}$

(4) $g(x)=\inf_{y\in C}f(x,y)$

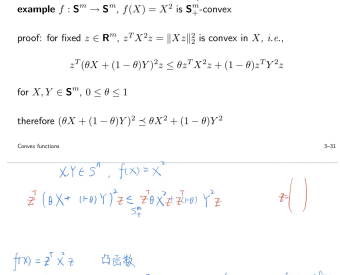
$f(x,y)$ is convex in (x,y)

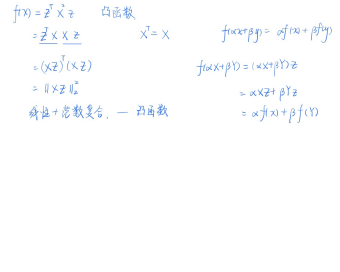
C is a convex set.





(5)





1、障碍函数

$\min\ f_0(x)$ s.t. $f_i(x)\le0$, $Ax=b$

$\Rightarrow\min\ f_0(x)-\frac1t\sum_{i=1}^m\log(-f_i(x))$ s.t. $Ax=b$

logarithmic barrier function

$\phi(x)=-\sum_{i=1}^m\log(-f_i(x))$, $\mathrm{dom}\phi=\{x\mid f_1(x)<0,\cdots,f_m(x)<0\}$

2、变分法

If $J(x(t))=\int_{t_0}^{t_f}f(x(t))dt$

$\delta J(x(t),\delta x)=\int_{t_0}^{t_f}\{\frac{\partial f(x(t))}{\partial x(t)}\}\delta x\,dt+f(x(t_f))\delta t_f-f(x(t_0))\delta t_0$

$=\int_{t_0}^{t_f}f_x(x(t))\delta x\,dt+f(x(t_f))\delta t_f-f(x(t_0))\delta t_0$

$\longrightarrow J(x(t))=\int_{t_0}^{t_f}g(x(t),\dot x(t),t)\,dt$

1、t_0,t_f, $x(t_0)$, $x(t_f)$ 固定

$\delta J(x(t),\delta x)=\int_{t_0}^{t_f}[g_x(x(t),\dot x(t),t)-\frac{d}{dt}g_{\dot x}(x(t),\dot x(t),t)]\delta x(t)dt=0$

$\Rightarrow$ 欧拉方程 $\quad\frac{\partial g(x(t),\dot x(t),t)}{\partial x}-\frac{d}{dt}(\frac{\partial g(x(t),\dot x(t),t)}{\partial\dot x})=0$

2、t_0,t_f, $x(t_0)$ 固定，$x(t_f)$ 可变

$\delta J(x(t),\delta x)=\int_{t_0}^{t_f}[g_x-\frac{d}{dt}g_{\dot x}]\delta x(t)dt+g_{\dot x}(x(t_f),\dot x(t_f),t_f)\delta x(t_f)$

① 欧拉方程：$\frac{\partial g}{\partial x}-\frac{d}{dt}\frac{\partial g}{\partial\dot x}=0$

② $x(t_f)=x_f$ if x_f is given as fixed

if $x(t_f)$ are free, then $\frac{\partial g(x(t),\dot x(t),t)}{\partial\dot x(t_f)}=0$

3、$t_0,x(t_0)$ 固定，t_f free，various constraints on $x(t_f)$

$\delta J(x(t),\delta x)=\int_{t_0}^{t_f}[g_x-\frac{d}{dt}g_{\dot x}]\delta x(t)dt+g_{\dot x}(x(t_f),\dot x(t_f),t_f)\delta x(t_f)$

$+g(x^*(t_f),\dot x^*(t_f),t_f)\delta t_f$

$\delta x_f\simeq\delta x(t_f)+\dot x^*(t_f)\delta t_f$ ⇓

$=\int_{t_0}^{t_f}[g_x-\frac{d}{dt}g_{\dot x}]\delta x(t)dt+g_{\dot x}(x^*(t_f),\dot x^*(t_f),t_f)\delta x_f$

$+[g(x^*(t_f),\dot x^*(t_f),t_f)-g_{\dot x}(x^*(t_f),\dot x^*(t_f),t_f)\dot x^*(t_f)]\delta t_f$

$\longrightarrow\delta x_f$ and δt_f

① 无关

$\begin{cases}g_x-\frac{d}{dt}g_{\dot x}=0\\ g(t_f)-g_{\dot x}(t_f)=0\\ g_{\dot x}(t_f)=0\end{cases}$

② 显式约束：$x(t_f)=\theta(t_f)$ then $\delta x_f=\frac{d\theta}{dt}(t_f)\delta t_f$

$\begin{cases}g_x-\frac{d}{dt}g_{\dot x}=0\\ x(t_f)=\theta(t_f)\\ g_{\dot x}(t_f)[\frac{d\theta}{dt}(t_f)-\dot x^*(t_f)]+g(t_f)=0\end{cases}$

③ 隐式约束：$m(x(t_f),t_f)=0$

$J(x(t),t)=h(x(t_f),t_f)+\int_{t_0}^{t_f}g(x(t),\dot x(t),t)\,dt$

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$J_a(x(t),\nu,t)=\frac{h(x(t_f),t_f)+\nu^Tm(x(t_f),t_f)}{w(x(t_f),\nu,t_f)}+\int_{t_0}^{t_f}g(x(t),\dot x(t),t)\,dt$

$\begin{cases}g_x-\frac{d}{dt}g_{\dot x}=0\\ w_x(t_f)+g_{\dot x}(t_f)=0\\ w_{t_f}+g(t_f)-g_{\dot x}(t_f)\dot x(t_f)=0\end{cases}$

$\delta J_a=[w_x(t_f)+g_{\dot x}(t_f)]\delta x_f+[w_{t_f}+g(t_f)-g_{\dot x}(t_f)\dot x(t_f)]\delta t_f+m^T(t_f)\delta\nu$

$+\int_{t_0}^{t_f}[g_x-\frac{d}{dt}g_{\dot x}]\delta x\,dt$

判断下降法在等式约束中：用二次函数逼近原函数

Analytic center of linear matrix inequality

$$\begin{array}{ll}\text{minimize} & -\log\det X\\ \text{subject to} & \mathbf{tr}(A_iX)=b_i,\quad i=1,\dots,p\end{array}$$

variable $X\in\mathbf{S}^n$ $\quad\delta X$ 一矩阵

optimality conditions $\quad L(X,\nu)=-\log\det(X)+\sum\nu_i(\mathrm{tr}(A_iX)-b_i)$

正定

$$X^\star\succ0,\qquad -(X^\star)^{-1}+\sum_{j=1}^p\nu_j^\star A_j=0,\qquad \mathbf{tr}(A_iX^\star)=b_i,\quad i=1,\dots,p$$

Newton equation at feasible X: $\quad\frac{d\log\det(X)}{dX}=X^{-1}$ $\quad\frac{d\,\mathrm{tr}(AX)}{dX}=A$

$$X^{-1}\Delta XX^{-1}+\sum_{j=1}^p w_jA_j=X^{-1},\qquad \mathbf{tr}(A_i\Delta X)=0,\quad i=1,\dots,p$$

- follows from linear approximation $(X+\Delta X)^{-1}\approx X^{-1}-X^{-1}\Delta XX^{-1}$ 一阶近似
- $n(n+1)/2+p$ variables ΔX, w

拉氏变换、$1\to\frac1s$, $\delta(t)\to1$, $k\to\frac ks$, $t\to\frac{1}{s^2}$, $t^n\to\frac{n!}{s^{n+1}}$,

$e^{at}\to\frac{1}{s+a}$, $te^{-at}\to\frac{1}{(s+a)^2}$, $1-e^{-at}\to\frac{a}{s(s+a)}$, $\frac1Te^{-\frac tT}\to\frac{1}{Ts+1}$