$$\begin{vmatrix} 1 & 2 & 1 & 1 \\ 0 & 2 & 1 & 11 \\ -2 & -1 & 4 & 4 \\ -2 & -1 & 1 & 10 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 2 & 1 & 1 \\ 0 & 2 & 1 & 11 \\ -2 & -1 & 4 & 4 \\ 0 & 0 & -3 & 6 \end{vmatrix}$$

消

$$= \begin{vmatrix} 1 & 2 & 1 & 1 \\ 0 & 2 & 1 & 11 \\ 0 & 3 & 6 & 6 \\ 0 & 0 & -3 & 6 \end{vmatrix} \leftarrow 提3$$

$$= 3\begin{vmatrix} 1 & 2 & 1 & 1 \\ 0 & 2 & 1 & 11 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & -3 & 6 \end{vmatrix} \leftarrow 消$$

$$= 3\begin{vmatrix} 1 & 2 & 1 & 1 \\ 0 & 0 & -3 & 7 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & -3 & 6 \end{vmatrix}$$

$$= -3\begin{vmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & -3 & 7 \\ 0 & 0 & -3 & 6 \end{vmatrix} \leftarrow 消$$

$$= -3\begin{vmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & -3 & 7 \end{vmatrix}$$

$$\begin{vmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 2 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & n-1 \\ n & 0 & 0 & \cdots & 0 \end{vmatrix}$$

硬解一

$$= \sum (-1)^{t(p_1 p_2 \cdots p_n)} a_{1p_1} a_{2p_2} \cdots a_{np_n}$$

$$= (-1)^{n-1} \times 1 \times 2 \times 3 \times \cdots \times n$$

$$= (-1)^{n-1} n!$$

法二 $D = a_{n1} A_{n1} = n(-1)^{n+1} \begin{vmatrix} 1 & \cdots & 0 \\ 0 & 2 & \\ & & \ddots \end{vmatrix}$

$$= n(-1)^{n+1}(n-1)!$$

$$= (-1)^{n+1} n!$$

$$\begin{vmatrix} x & a & \cdots & a \\ a & x & \cdots & a \\ \vdots & \vdots & & \vdots \\ a & a & \cdots & x \end{vmatrix}$$

$$= \begin{vmatrix} x+(n-1)a & a & \cdots & a \\ x+(n-1)a & x & \cdots & a \\ \vdots & \vdots & & \vdots \end{vmatrix}$$

$$\begin{vmatrix} 0 & 0 & 0 & -1 \end{vmatrix}$$

$$= -9.$$

$$\begin{vmatrix} x+(n-1)a & a & \cdots & x \end{vmatrix}$$

$$= (x+(n-1)a)\begin{vmatrix} 1 & a & \cdots & a \\ 1 & x & \cdots & a \\ \vdots & \vdots & & \vdots \\ 1 & a & \cdots & x \end{vmatrix}$$

$$= (x+(n-1)a)\begin{vmatrix} 1 & 0 & \cdots & 0 \\ 1 & x-a & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 1 & 0 & \cdots & x-a \end{vmatrix}$$

$$= (x+(n-1)a)(x-a)^{n-1}$$

$$\begin{vmatrix} 1 & 2 & 3 & 4 & \cdots & n \\ 1 & 1 & 2 & 3 & \cdots & n-1 \\ 1 & x & 1 & 2 & \cdots & n-2 \\ 1 & x & x & 1 & \cdots & n-3 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 1 & x & x & x & \cdots & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 1 & 1 & 1 & \cdots & 1 \\ 0 & 1-x & 1 & 1 & \cdots & 1 \\ 0 & 0 & 1-x & 1 & \cdots & 1 \\ 0 & 0 & 0 & 1-x & \cdots & 1 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 1 & x & x & x & \cdots & 1 \end{vmatrix} \leftarrow$$

$$= \begin{vmatrix} 0 & 1 & 1 & 1 & \cdots & 1 \\ 0 & 1-x & 1 & 1 & \cdots & 1 \\ 0 & 0 & 1-x & 1 & \cdots & 1 \\ 0 & 0 & 0 & 1-x & \cdots & 1 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 1 & 0 & 0 & 0 & \cdots & 1-x \end{vmatrix}$$ ←变消

$$= \begin{vmatrix} 0 & 1 & 1 & 1 & \cdots & 1 \\ 0 & 0 & x & x & \cdots & x \\ 0 & 0 & 1-x & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 1 & 0 & 0 & 0 & \cdots & 1-x \end{vmatrix}$$ ←提x

依次类推

$$= x^{n-2}\begin{vmatrix} 0 & 1 & 1 & 1 & \cdots & 1 \\ 0 & 0 & 1 & 1 & \cdots & 1 \\ 0 & 0 & 0 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & 0 & \cdots & 1-x \end{vmatrix}$$

$$= (-1)^{n+1} x^{n-2}$$

$$\begin{vmatrix} a_0 & -1 & 0 & \cdots & 0 & 0 \\ a_1 & x & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ a_{n-2} & 0 & 0 & \cdots & x & -1 \\ a_{n-1} & 0 & 0 & \cdots & 0 & x \end{vmatrix}$$

$$= a_0 (-1)^{1+1} \begin{vmatrix} x & -1 & \cdots & 0 & 0 \\ 0 & x & \cdots & 0 & 0 \\ \vdots & & & \vdots & \vdots \\ 0 & 0 & \cdots & x & -1 \\ 0 & 0 & \cdots & 0 & x \end{vmatrix} + \cdots + a_{n-1} (-1)^{n+1} \begin{vmatrix} -1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ x & -1 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & x & -1 \end{vmatrix}$$

$$D = \begin{vmatrix} 1 & x_1 + a_1 & x_1^2 + b_1 x_1 + b_2 & x_1^3 + c_1 x_1^2 + c_2 x_1 + c_3 \\ 1 & x_2 + a_1 & x_2^2 + b_1 x_2 + b_2 & x_2^3 + c_1 x_2^2 + c_2 x_2 + c_3 \\ 1 & x_3 + a_1 & x_3^2 + b_1 x_3 + b_2 & x_3^3 + c_1 x_3^2 + c_2 x_3 + c_3 \\ 1 & x_4 + a_1 & x_4^2 + b_1 x_4 + b_2 & x_4^3 + c_1 x_4^2 + c_2 x_4 + c_3 \end{vmatrix}$$

依此类推

$$= \begin{vmatrix} 1 & x_1 & x_1^2 & x_1^3 \\ 1 & x_2 & x_2^2 & x_2^3 \\ 1 & x_3 & x_3^2 & x_3^3 \\ 1 & x_4 & x_4^2 & x_4^3 \end{vmatrix} = \prod_{1 \le j < i \le 4} (x_i - x_j)$$

$$D_n = \begin{vmatrix} 2 & 1 & & & \\ 1 & 2 & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & 2 & 1 \\ & & & 1 & 2 \end{vmatrix}$$

$n = k+1$

$$D_{k+1} = 2\begin{vmatrix} 2 & 1 & & & \\ 1 & 2 & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & 2 & 1 \\ & & & 1 & 2 \end{vmatrix}_{k\text{阶}} + (-1)\begin{vmatrix} 1 & & & & \\ 1 & 2 & 1 & & \\ & 1 & 2 & 1 & \\ & & \ddots & \ddots & \\ & & & 2 & 1 \\ & & & 1 & 2 \end{vmatrix}_{k\text{阶}}$$

$$= 2D_k + (-1)\begin{vmatrix} 2 & 1 & & & \\ 1 & 2 & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & 2 & 1 \\ & & & 1 & 2 \end{vmatrix}_{k-1\text{阶}}$$

$$= 2D_k - D_{k-1}$$
$$= 2(k+1) - k$$
$$= k+2$$
$$= (k+1)+1$$

$$D_n = \begin{vmatrix} 5 & 3 & & & \\ 2 & 5 & 3 & & \\ & 2 & \ddots & \ddots & \\ & & \ddots & \ddots & 3 \\ & & & 2 & 5 \end{vmatrix}$$

$$= 5D_{n-1} - 2\begin{vmatrix} 3 & 0 & & & \\ 2 & 5 & 3 & & \\ & 2 & 5 & 3 & \\ & & & & 3 \\ & & & 2 & 5 \end{vmatrix}$$

$$D_n = 5D_{n-1} - 6D_{n-2}$$

$$D_n - 3D_{n-1} = 2(D_{n-1} - 3D_{n-2}) = \cdots = 2^{n-3}(D_3 - 3D_2)$$

$$= 2^{n-2}(D_2 - 3D_1) = 2^n$$

$$D_n = 2^n + 3D_{n-1}$$

$$= 2^n + 3\,2^{n-1} + \cdots + 3^{n-3}2^3 + 3^{n-2}\cdot 2^2 + 3^{n-1}\cdot 2 + 3^n$$

$$= 3^{n+1} - 2^{n+1}$$

$$\sum_{p_1\cdots p_n}\begin{vmatrix} a_{1p_1} & \cdots & a_{1p_n} \\ \vdots & & \vdots \\ a_{np_1} & \cdots & a_{np_n} \end{vmatrix} = 0$$

说明：对任1排列 $p_i \cdots p_n \cdots p_j$

将 p_n 归位，需 $t(p_i \cdots p_n \cdots p_j)$ 次

将所有元素归位后 原式的系数

$\Rightarrow \sum (-1)^{t(p_i \cdots p_n \cdots p_j)}$ $\Rightarrow$ 由行列式的定义

$$= \begin{vmatrix} 1 & \cdots & 1 \\ \vdots & & \vdots \\ 1 & \cdots & 1 \end{vmatrix} = 0$$

原式 $= a \times 0 = 0$

$$D_n = \begin{vmatrix} 1+a_1 & 1 & 1 & \cdots & 1 \\ 1 & 1+a_2 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & 1 & 1 & \cdots & 1+a_n \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & 1+a_1 & 1 & \cdots & 1 \\ 0 & 1 & 1+a_2 & \cdots & 1 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 1 & 1 & \cdots & 1+a_n \end{vmatrix}$$

← 加边

加边

$$= \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ -1 & a_1 & 0 & \cdots & 0 \\ -1 & 0 & a_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ -1 & 0 & 0 & \cdots & a_n \end{vmatrix}$$

$$\overline{\overline{\begin{array}{c} C_1 + C_2 \times \frac{1}{a_1} \\ C_1 + C_3 \times \frac{1}{a_2} \\ \cdots \\ C_1 + C_j \times \frac{1}{C_{j-1}} \end{array}}} \begin{vmatrix} 1 + \frac{1}{a_1} + \cdots + \frac{1}{a_n} & 1 & 1 & \cdots & 1 \\ 0 & a_1 & 0 & \cdots & 0 \\ 0 & 0 & a_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & a_n \end{vmatrix}$$

$$= \prod_{i=1}^{n} a_i \left(1 + \sum_{j=1}^{n} \frac{1}{a_j}\right)$$