复变函数与积分变换试题(一)解答

一、填空题

1. $-\frac{1}{25}, -\frac{32}{25}$.　2. $\left(\frac{1}{2}+\frac{\sqrt{3}}{2}\right)+\mathrm{i}\left(\frac{1}{2}-\frac{\sqrt{3}}{2}\right)$.　3. $\frac{2}{3}\pi$.

4. $\cos\frac{\pi}{3}+\mathrm{i}\sin\frac{\pi}{3}$.　5. $\{z=x+\mathrm{i}y: y>0, x^2+(y-\sqrt{3})^2>2^2\}$.

二、单项选择题

1. A　2. B　3. A　4. C　5. D

三、解：由 $(3+6\mathrm{i})x+(5-9\mathrm{i})y=6-7\mathrm{i}$，得

$$(3x+5y)+\mathrm{i}(6x-9y)=6-7\mathrm{i}$$

$$\begin{cases}3x+5y=6\\6x-9y=-7\end{cases}$$

解得 $x=\frac{1}{3}, y=1$.

四、证：左边 $=|1-\bar{z}_1z_2|^2-|z_1-z_2|^2$

$$\begin{aligned}&=(1-\bar{z}_1z_2)(1-z_1\bar{z}_2)-(z_1-z_2)(\bar{z}_1-\bar{z}_2)\\&=1-z_1\bar{z}_2-\bar{z}_1z_2+|z_1|^2|z_2|^2\\&\quad-(|z_1|^2-z_1\bar{z}_2-\bar{z}_1z_2+|z_2|^2)\\&=1+|z_1|^2|z_2|^2-|z_1|^2-|z_2|^2\\&=(1-|z_1|^2)(1-|z_2|^2)\end{aligned}$$

五、证：设 $z_1=r_1e^{\mathrm{i}\theta_1}, z_2=r_2e^{\mathrm{i}\theta_2}$. 由 $\frac{z_1}{z_2}=\frac{r_1}{r_2}e^{\mathrm{i}(\theta_1-\theta_2)}>0$，知 $\theta_1=\theta_2$.

于是　$|z_1+z_2|=|(r_1+r_2)e^{\mathrm{i}\theta_1}|=r_1+r_2$

$$|z_1|+|z_2|=|r_1e^{\mathrm{i}\theta_1}|+|r_2e^{\mathrm{i}\theta_2}|=r_1+r_2$$

即

$$|z_1+z_2|=|z_1|+|z_2|$$

六、解：$r=(1+\sin 1)^2+\cos^2 1=2+2\sin 1$

$$\cos\theta=\frac{1+\sin 1}{2+2\sin 1}=\frac{1}{2}$$

$$\sin\theta=\frac{\cos 1}{2+2\sin 1}=\sqrt{1-\cos^2\theta}=\frac{\sqrt{3}}{2}$$

故

$$\begin{aligned}z=re^{\mathrm{i}\theta}=r(\cos\theta+\mathrm{i}\sin\theta)&=(2+2\sin 1)\left(\cos\frac{\pi}{3}+\mathrm{i}\sin\frac{\pi}{3}\right)\\&=(2+2\sin 1)e^{\mathrm{i}\frac{\pi}{3}}\end{aligned}$$

七、解：由于 $1+\mathrm{i}=\sqrt{2}e^{\mathrm{i}\frac{\pi}{4}}, 1-\mathrm{i}=\sqrt{2}e^{-\mathrm{i}\frac{\pi}{4}}$，故由 $(1+\mathrm{i})^n=(1-\mathrm{i})^n$ 可得

$$2^{\frac{n}{2}}e^{\mathrm{i}\frac{n\pi}{4}}=2^{\frac{n}{2}}e^{-\mathrm{i}\frac{n\pi}{4}}$$

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$$e^{i\frac{n\pi}{4}}=e^{-i\frac{n\pi}{4}}$$

$$e^{i\frac{n\pi}{2}}=1$$

可知 $\frac{n\pi}{2}=2k\pi, n=4k, k=0,\pm1,\pm2,\cdots$.

八、解：设 $z=x+iy$，代入原方程得

$$x^2-y^2+i2xy-4i(x+iy)-4+9i=0$$

$$x^2-y^2+4y-4+i(2xy-4x+9)=0$$

等价于

$$\begin{cases}x^2-y^2+4y-4=0\\2xy-4x+9=0\end{cases}$$

$$\begin{cases}x^2-(y-2)^2=0\\2x(y-2)+9=0\end{cases}$$

$$\begin{cases}x^2=(y-2)^2 & (1)\\2x(y-2)+9=0 & (2)\end{cases}$$

解之，得 $x=\pm(y-2)$. 将 $x=y-2$ 代入(2)，无解. 将 $x=2-y$ 代入(2)，解得 $x=\pm\frac{3\sqrt{2}}{2}$，再代入(1)，解得 $y=\frac{4\mp3\sqrt{2}}{2}$. 经检验原方程有解

$$z_1=\frac{1}{2}[3\sqrt{2}+i(4-3\sqrt{2})] \quad 和 \quad z_2=\frac{1}{2}[-3\sqrt{2}+i(4+3\sqrt{2})]$$

九、证：由 $|z_1|=\lambda|z_2|$，知 $z_1\overline{z}_1=\lambda^2z_2\overline{z}_2$. 从而

$$\overline{z}_1(z_1-\lambda^2z_2)=\lambda^2z_2\overline{z}_2-\lambda^2\overline{z}_1z_2=\lambda^2z_2(\overline{z}_2-\overline{z}_1)$$

对上两边取模得

$$|z_1||(z_1-\lambda^2z_2)|=|\overline{z}_1(z_1-\lambda^2z_2)|=|\lambda^2z_2(\overline{z}_2-\overline{z}_1)|$$
$$=\lambda^2|z_2||z_1-z_2|$$

于是
$$|z_1-\lambda^2z_2|=\lambda^2\left|\frac{z_2}{z_1}\right||z_1-z_2|=\lambda|z_1-z_2|$$

十、证：$\left|\frac{z-z_1}{z-z_2}\right|=k\Leftrightarrow|z-z_1|=k|z-z_2|$.

由九题的结论知

$$|(z-z_1)-k^2(z-z_2)|=k|(z-z_1)-(z-z_2)|$$
$$\Leftrightarrow|z(1-k^2)-(z_1-k^2z_2)|=k|z_1-z_2|$$
$$\Leftrightarrow\left|z-\frac{z_1-k^2z_2}{1-k^2}\right|=k\frac{|z_1-z_2|}{|1-k^2|}$$

可见

$$z_0=\frac{z_1-k^2z_2}{1-k^2}, \rho=\frac{k|z_1-z_2|}{|1-k^2|}$$

十一、证：设 $z=re^{i\theta}$，则 $|z|=r$，于是

$$\lim_{\substack{z\to0\\ \arg z=\theta}}\frac{z}{|z|}=\lim_{r\to0}\frac{re^{i\theta}}{r}=e^{i\theta}$$

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由 θ 在 $(-\pi,\pi]$ 内的任意性，可知 $\lim\limits_{z\to 0}\dfrac{z}{|z|}$ 不存在.

十二、证：设 $a_n=\alpha_n+\mathrm{i}\beta_n, n=1,2,\cdots, a=\alpha+\mathrm{i}\beta$. 由 $\lim\limits_{n\to\infty}a_n=a$，则可知 $\lim\limits_{n\to\infty}\alpha_n=\alpha$ 和 $\lim\limits_{n\to\infty}\beta_n=\beta$，从而

$$\lim_{n\to\infty}\frac{\alpha_1+\alpha_2+\cdots+\alpha_n}{n}=\alpha$$

$$\lim_{n\to\infty}\frac{\beta_1+\beta_2+\cdots+\beta_n}{n}=\beta$$

于是

$$\begin{aligned}\lim_{n\to\infty}\frac{a_1+a_2+\cdots+a_n}{n}&=\lim_{n\to\infty}\left(\frac{\alpha_1+\alpha_2+\cdots+\alpha_n}{n}+\mathrm{i}\,\frac{\beta_1+\beta_2+\cdots+\beta_n}{n}\right)\\&=\alpha+\mathrm{i}\beta=a\end{aligned}$$

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复变函数与积分变换试题(二)解答

一、填空题

1. $y=\frac{1}{2}$ 上,$\mathbb{C}$ 上无.　2. $e^{u}\sin v+C$.　3. 1,−3,−3.

4. $e^{-2k\pi}[\cos(2\sqrt{3}k\pi)+i\sin(2\sqrt{3}k\pi)]$, $k\in\mathbb{Z}$.　5. $\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}$.

二、单项选择题

1. D　2. C　3. D　4. D　5. B

三、证:
$$\lim_{\substack{x\to0\\y=kx}}\frac{f(z)-f(0)}{z}=\lim_{x\to0}\frac{x(x^2+k^2x^2)(kx-ix)}{(x^2+k^4x^4)(x+ikx)}$$
$$=\lim_{x\to0}\frac{x^4(1+k^2)(k-i)}{x^3(1+k^4x^2)(1+ik)}$$
$$=\lim_{x\to0}\frac{x(1+k^2)(k-i)}{(1+k^4x^2)(1+ik)}$$
$$=0$$

但
$$\lim_{\substack{z\to0\\x=y^2}}\frac{f(z)-f(0)}{z}=\lim_{y\to0}\frac{y^2(y^4+y^2)(y-iy^2)}{2y^4(y^2+iy)}$$
$$=\lim_{y\to0}\frac{y^5(y^2+1)(1-iy)}{2y^5(y+i)}$$
$$=\lim_{y\to0}\frac{(y^2+1)(1-iy)}{2(y+i)}$$
$$=\frac{1}{2i}\neq0$$

故 $f'(0)$ 不存在.

四、解:设 $z=x+iy$,则 $f(z)=z\mathrm{Re}\,z=(x+iy)x=x^2+ixy$

令 $u=x^2, v=xy$. 显然 u,v 可微,又令

$$\frac{\partial u}{\partial x}=2x=\frac{\partial v}{\partial y}=x$$
$$\frac{\partial u}{\partial y}=0=-\frac{\partial v}{\partial x}=-y$$

解得 $x=y=0$,由函数可导的充要条件知,在 $\mathbb{C}$ 上. 函数 $f(z)$ 仅在原点处可导,故 $f(z)$ 在 $\mathbb{C}$ 上处处不解析.

五、解:设 $z=x+iy$,故

$$f(z)=\frac{1+z}{1-z}=\frac{1+x+iy}{(1-x)-iy}=\frac{(1+x+iy)[(1-x)+iy]}{(1-x)^2+y^2}$$
$$=\frac{1-x^2-y^2+i2y}{(1-x^2)+y^2}$$

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$f(z)$ 的实部 $u=\dfrac{1-x^2-y^2}{(1-x)^2+y^2}$，虚部 $v=\dfrac{2y}{(1-x)^2+y^2}$，又

$$\frac{\partial u}{\partial x}=\frac{-2x[(1-x)^2+y^2]+2(1-x)(1-x^2-y^2)}{[(1-x^2)+y^2]^2}$$

$$=\frac{2[(1-x)^2-y^2]}{[(1-x)^2+y^2]^2}$$

$$\frac{\partial u}{\partial y}=\frac{-2y[(1-x)^2+y^2]-2y(1-x^2-y^2)}{[(1-x^2)+y^2]^2}=\frac{-4(1-x)y}{[(1-x)^2+y^2]^2}$$

$$\frac{\partial v}{\partial x}=\frac{2(1-x)2y}{[(1-x)^2+y^2]^2}=\frac{4(1-x)y}{[(1-x)^2+y^2]^2}$$

$$\frac{\partial v}{\partial y}=\frac{2[(1-x)^2+y^2]-2y\cdot 2y}{[(1-x)^2+y^2]^2}=\frac{2[(1-x)^2-y^2]}{[(1-x)^2+y^2]^2}$$

可见在 $\mathbb{C}\setminus\{1\}$ 上有 $\dfrac{\partial u}{\partial x}=\dfrac{\partial v}{\partial y},\dfrac{\partial u}{\partial y}=-\dfrac{\partial v}{\partial x}$. 即 $f(z)=\dfrac{1+z}{1-z}$ 在 $\mathbb{C}$ 上除去 $z=1$ 处处满足 C-R 条件.

六、解：由 $2xyu+(x^2-y^2)v+2xy(x^2-y^2)=0$，知

$$\mathrm{Im}\left[z^2f(z)+\frac{1}{2}z^4\right]=0$$

而函数 $z^2f(z)+\dfrac{1}{2}z^4$ 解析，由 C-R 条件知 $\mathrm{Re}\left(z^2f(z)+\dfrac{z^4}{2}\right)=k(k\in\mathbb{R})$，于是

$$z^2f(z)+\frac{1}{2}z^4=k$$

从而

$$f(z)=-\frac{z^2}{2}+\frac{k}{z^2}$$

其中 k 是实数.

七、证：由于 $f(z)=u+\mathrm{i}v$ 解析，故其实、虚部 u,v 均为调和函数. 注意到

$$\frac{\partial(u\cdot v)}{\partial x}=\frac{\partial u}{\partial x}v+u\frac{\partial v}{\partial x}$$

$$\frac{\partial^2(u\cdot v)}{\partial x^2}=\frac{\partial^2u}{\partial x^2}v+2\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}+u\frac{\partial^2v}{\partial x^2}$$

同理可求

$$\frac{\partial^2(u\cdot v)}{\partial y^2}=\frac{\partial^2u}{\partial y^2}v+2\frac{\partial u}{\partial y}\frac{\partial v}{\partial y}+u\frac{\partial^2v}{\partial y^2}$$

于是

$$\Delta(u\cdot v)=\frac{\partial^2(u\cdot v)}{\partial x^2}+\frac{\partial^2(u\cdot v)}{\partial y^2}$$

$$=\left(\frac{\partial^2u}{\partial x^2}+\frac{\partial^2u}{\partial y^2}\right)v+2\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\frac{\partial v}{\partial y}\right)+u\left(\frac{\partial^2v}{\partial x^2}+\frac{\partial^2v}{\partial y^2}\right)$$

$$=2\left[\frac{\partial v}{\partial y}\left(-\frac{\partial u}{\partial y}\right)+\frac{\partial u}{\partial y}\frac{\partial v}{\partial y}\right]=0$$

即 $u\cdot v$ 是 D 内的调和函数.

八、解：

$$\frac{\partial(u+v)}{\partial x}=3x^2+6xy-3y^2-2=\frac{\partial v}{\partial y}-\frac{\partial u}{\partial y} \quad (1)$$

$$\frac{\partial(u+v)}{\partial y}=-3y^2+3x^2-6xy-2=\frac{\partial u}{\partial y}+\frac{\partial v}{\partial y} \quad (2)$$

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(1)＋(2)得

$$2\frac{\partial v}{\partial y}=6(x^2-y^2)-4$$

$$\frac{\partial v}{\partial y}=3(x^2-y^2)-2$$

$$v=\int\frac{\partial v}{\partial y}\mathrm{d}y=\int[3(x^2-y^2)-2]\mathrm{d}y=3\left(x^2y-\frac{y^3}{3}\right)-2y+C(x)$$

(2)－(1)得

$$2\frac{\partial u}{\partial y}=-12xy\quad \frac{\partial u}{\partial y}=-6xy$$

根据 C-R 条件

$$\frac{\partial v}{\partial x}=6xy+C'(x)=-\frac{\partial u}{\partial y}=6xy$$

知 $C'(x)=0, C(x)=C(C\in\mathbb{R})$. 将 $v=3x^2y-y^3-2y+C$ 代入原式得

$$\begin{aligned}u&=(x^3-y^3+3x^2y-3xy^2-2x-2y)-v\\&=(x^3-y^3+3x^2y-3xy^2-2x-2y)-3x^2y+y^3+2y-C\\&=x^3-3xy^2-2x-C\end{aligned}$$

故

$$\begin{aligned}f(z)=(1+\mathrm{i})^2&=(x^3-3xy^2-2x-C)+\mathrm{i}(3x^2y-y^3-2y+C)\\&=z^3-2z+C(1-\mathrm{i})\end{aligned}$$

九、解：1. $\frac{\partial u}{\partial x}=af'(ax+by),\frac{\partial^2 u}{\partial x^2}=a^2f''(ax+by)$

$$\frac{\partial u}{\partial y}=bf'(ax+by),\frac{\partial^2 u}{\partial y^2}=b^2f''(ax+by)$$

由 $a^2+b^2\neq0$，故

$$\Delta u=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=(a^2+b^2)f''(ax+by)=0$$

知

$$f''(ax+by)=0$$

从而

$$f'(ax+by)=C_1$$

$$u=f(ax+by)=C_1(ax+by)+C_2$$

这里 C_1 和 C_2 是任意实常数.

2. 令 $t=\frac{y}{x}$，则

$$\frac{\partial u}{\partial x}=-\frac{y}{x^2}f'\left(\frac{y}{x}\right),\frac{\partial^2 u}{\partial x^2}=\frac{2y}{x^3}f'\left(\frac{y}{x}\right)+\frac{y^2}{x^4}f''\left(\frac{y}{x}\right)$$

$$\frac{\partial u}{\partial y}=\frac{1}{x}f'\left(\frac{y}{x}\right),\frac{\partial^2 u}{\partial y^2}=\frac{1}{x^2}f''\left(\frac{y}{x}\right)$$

由于 u 是调和函数，故

$$0=\Delta u=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=f''\left(\frac{y}{x}\right)\left(\frac{1}{x^2}+\frac{y^2}{x^4}\right)+f'\left(\frac{y}{x}\right)\frac{2y}{x^3}$$

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令 $t=\frac{y}{x}$，得常微分方程

$$f''(t)(1+t^2)+f'(t)2t=0 \Leftrightarrow \frac{f''(t)}{f'(t)}=-\frac{2t}{1+t^2}$$

积分得

$$\ln f'(t)=-\ln(1+t^2)+\ln C_1 \Leftrightarrow f'(t)=\frac{C_1}{1+t^2}$$

再积分得

$$f(t)=C_1\arctan t+C_2$$

从而

$$u=f\left(\frac{y}{x}\right)=C_1\arctan\frac{y}{x}+C_2, C_1, C_2\in\mathbf{R}$$

十、解：1. $\exp(2-\mathrm{i})=\mathrm{e}^{2-\mathrm{i}}=\mathrm{e}^2\mathrm{e}^{-\mathrm{i}}=\mathrm{e}^2(\cos 1-\mathrm{i}\sin 1)$

2. $\ln(3-\sqrt{3}\mathrm{i})=\ln|3-\sqrt{3}\mathrm{i}|+\mathrm{i}\arg(3-\sqrt{3}\mathrm{i})$

$$=\ln(2\sqrt{3})+\mathrm{i}\arctan\left(\frac{-\sqrt{3}}{3}\right)$$

$$=\ln(2\sqrt{3})-\mathrm{i}\arctan\frac{1}{\sqrt{3}}$$

$$=\ln(2\sqrt{3})-\mathrm{i}\frac{\pi}{6}$$

3. $\mathrm{Arctan}(2+3\mathrm{i})=\frac{1}{2\mathrm{i}}\mathrm{Ln}\left[\frac{1+\mathrm{i}(2+3\mathrm{i})}{1-\mathrm{i}(2+3\mathrm{i})}\right]$

$$=\frac{1}{2\mathrm{i}}\mathrm{Ln}\left(\frac{-1+\mathrm{i}}{2-\mathrm{i}}\right)=\frac{1}{2\mathrm{i}}\mathrm{Ln}\left(\frac{-3+\mathrm{i}}{5}\right)$$

$$=\frac{1}{2\mathrm{i}}\left\{\frac{1}{2}\ln\left(\frac{2}{5}\right)+\mathrm{i}\left[\left(\pi-\arctan\frac{1}{3}\right)+2k\pi\right]\right\}$$

$$=\frac{1}{2}\left[(2k+1)\pi-\arctan\frac{1}{3}\right]-\frac{\mathrm{i}}{4}\ln\left(\frac{2}{5}\right), k\in\mathbf{Z}$$

4. $\sec(1+\mathrm{i})=\frac{1}{\cos(1+\mathrm{i})}$

$$\cos(1+\mathrm{i})=\frac{1}{2}\left[\mathrm{e}^{\mathrm{i}(1+\mathrm{i})}+\mathrm{e}^{-\mathrm{i}(1+\mathrm{i})}\right]=\cos 1\,\mathrm{ch}\,1-\mathrm{i}\sin 1\,\mathrm{sh}\,1$$

$$\sec(1+\mathrm{i})=\frac{\cos 1\,\mathrm{ch}1+\mathrm{i}\sin 1\,\mathrm{sh}1}{\cos^2 1\,\mathrm{sh}^2 1+\sin^2 1\,\mathrm{sh}^2 1}=\frac{\cos 1\,\mathrm{ch}1+\mathrm{i}\sin 1\,\mathrm{sh}1}{\cos^2 1+\mathrm{ch}^2 1}$$

5. $(1+\mathrm{i})^{1-\mathrm{i}}=\mathrm{e}^{(1-\mathrm{i})\mathrm{Ln}(1+\mathrm{i})}=\mathrm{e}^{(1-\mathrm{i})(\ln|1+\mathrm{i}|+\mathrm{i}[\arg(1+\mathrm{i})+2k\pi])}$

$$=\mathrm{e}^{(1-\mathrm{i})\left[\ln 2\sqrt{2}+\mathrm{i}\left(\frac{\pi}{4}+2k\pi\right)\right]}$$

$$=\mathrm{e}^{\left[\ln\sqrt{2}+\left(\frac{\pi}{4}+2k\pi\right)+\mathrm{i}\left(\frac{\pi}{4}+2k\pi-\ln\sqrt{2}\right)\right]}$$

$$=\sqrt{2}\,\mathrm{e}^{\frac{\pi}{4}+2k\pi}\left[\cos\left(\frac{\pi}{4}-\frac{\ln 2}{2}\right)+\mathrm{i}\sin\left(\frac{\pi}{4}-\frac{\ln 2}{2}\right)\right], k\in\mathbf{Z}$$

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复变函数与积分变换试题(三)解答

一、填空题

1. $-\frac{4}{3}+\frac{19}{3}i, -\frac{4}{3}+i\frac{22}{3}$.　2. $8\pi i, 0$.　3. $0, 2\pi i, 0$.　4. 0.　5. $-\pi i$.

二、单项选择题

1. C　2. D　3. C　4. D　5. B

三、解：$\Gamma=\Gamma_1\cup\Gamma_2$，而 $\Gamma_1: z=t, -1\leqslant t\leqslant 1$，$\Gamma_2: z=e^{it}, 0\leqslant t\leqslant\pi$. 于是

$$\begin{aligned}\oint_\Gamma |z|\bar{z}dz &= \oint_{\Gamma_1\cup\Gamma_2}|z|\bar{z}dz\\ &=\int_{\Gamma_1}|z|\bar{z}dz+\int_{\Gamma_2}|z|\bar{z}dz\\ &=\int_{-1}^{1}|t|tdt+i\int_0^\pi 1e^{-it}e^{it}dt\\ &=i\int_0^\pi 1dt=i\pi\end{aligned}$$

四、证：设 Γ_1 和 Γ_2 分别是落在 C 的内部的围绕 $z=0$ 和 $z=a$ 的逆时针方向的闭曲线，且 Γ_1 和 Γ_2 彼此互不相交、互不包含. 由多连通区域的 Cauchy 定理，知

$$\begin{aligned}\frac{1}{2\pi i}\oint_\Gamma\frac{f(z)}{z^2(z-a)}dz &= \frac{1}{2\pi i}\oint_{\Gamma_1}\frac{f(z)}{z^2(z-a)}dz+\frac{1}{2\pi i}\oint_{\Gamma_2}\frac{f(z)}{z^2(z-a)}dz\\ &=\frac{1}{2\pi i}\oint_{\Gamma_1}\frac{\frac{f(z)}{z-a}}{z^2}dz+\frac{1}{2\pi i}\oint_{\Gamma_2}\frac{\frac{f(z)}{z^2}}{z-a}dz\\ &=\left[\frac{f(z)}{z-a}\right]'\Bigg|_{z=0}+\frac{f(z)}{z^2}\Bigg|_{z=a}\\ &=\frac{(z-a)f'(z)-f(z)}{(z-a)^2}\Bigg|_{z=0}+\frac{f(a)}{a^2}\\ &=\frac{-af'(0)-f(0)}{a^2}+\frac{f(a)}{a^2}\\ &=\frac{f(a)}{a^2}-\frac{f(0)}{a^2}-\frac{f'(0)}{a}\end{aligned}$$

五、解：$\frac{1}{2\pi i}\oint_{|z|=1}\left[2+\left(z+\frac{1}{2}\right)f(z)\right]\frac{dz}{z}$

$$\begin{aligned}&=\frac{1}{\pi i}\oint_{|z|=1}\frac{1}{z}dz+\frac{1}{2\pi i}\oint_{|z|=1}f(z)dz+\frac{1}{4\pi i}\oint_{|z|=1}\frac{f(z)}{z}dz\\ &=\frac{1}{\pi i}2\pi i+0+\frac{1}{2}f(0)\\ &=2+\frac{1}{2}=\frac{5}{2}\end{aligned}$$

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六、解：i) 当 0 落在 Γ 的内部，1 落在 Γ 的外部时，有

$$\oint_{\Gamma}\frac{e^z}{z(1-z)^3}dz=2\pi i\left.\frac{e^z}{(1-z)^3}\right|_{z=0}=2\pi i$$

ii) 当 1 落在 Γ 的内部，0 落在 Γ 的外部时，有

$$\oint_{\Gamma}\frac{e^z}{z(1-z)^3}dz=-\pi i\left.\left(\frac{e^z}{z}\right)''\right|_{z=1}$$

$$=-\pi i\left.\left(\frac{ze^z-e^z}{z^2}\right)'\right|_{z=1}$$

$$=-\pi i\left.\frac{z^3e^z-2ze^z(z-1)}{z^4}\right|_{z=1}$$

$$=-\pi ie$$

iii) 当 0 和 1 均落在 Γ 的内部时，利用多连通区域上的 Cauchy 定理及 i) 和 ii) 的结果，得

$$\oint_{\Gamma}\frac{e^z}{z(1-z)^3}dz=\oint_{\Gamma_1}\frac{e^z}{z(1-z)^3}dz+\oint_{\Gamma_2}\frac{e^z}{z(1-z)^3}dz$$

$$=2\pi i-\pi ie=\pi i(1-e)$$

iv) 当 0 和 1 均在 Γ 的外部时，由 Cauchy 定理，得

$$\oint_{\Gamma}\frac{e^z}{z(1-z)^3}dz=0$$

七、解：设 Γ_1、Γ_2 和 Γ_3 落在 $|z|=3$ 内且分别围绕 0，-1 和 2 的逆时针方向的闭简单曲线，它们彼此互不相交. 由 Cauchy 定理，有

$$\oint_{|z|=3}\frac{dz}{z^3(z+1)(z-2)}=\left(\oint_{\Gamma_1}+\oint_{\Gamma_2}+\oint_{\Gamma_3}\right)\frac{dz}{z^3(z+1)(z-2)}$$

$$=\pi i\left.\left(\frac{1}{(z+1)(z-2)}\right)''\right|_{z=0}+$$

$$2\pi i\left(\left.\frac{1}{z^3(z-2)}\right|_{z=-1}+\left.\frac{1}{z^3(z+1)}\right|_{z=2}\right)$$

$$=-\frac{3}{4}\pi i+\frac{2}{3}\pi i+\frac{1}{12}\pi i=0$$

八、证：当 z 落在圆周 $|z-1|=2$ 时，有

$$\left|\frac{z+1}{z-1}\right|=\frac{|(z-1)+2|}{|z-1|}\leqslant\frac{|z-1|+2}{2}=2$$

于是由复积分的性质，可知

$$\left|\oint_{|z-1|=2}\frac{z+1}{z-1}dz\right|\leqslant\oint_{|z-1|=2}\left|\frac{z+1}{z-1}\right|ds\leqslant 2\oint_{|z-1|=2}1ds$$

$$=2\cdot 2\pi\cdot 2=8\pi$$

九、证：由于 $f(z)$ 在 $|z-a|<R$ 内解析，则对于 $0<r<R$，$f(z)$ 在 $|z-a|\leqslant\Gamma$ 上解析. 于是由 Cauchy 积分公式，有

$$\oint_{|z-a|=r}\frac{f(z)}{z-a}dz=2\pi if(a),0<r<R$$

从而令 $r\to 0$，有

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$$\lim_{r\to 0}\oint_{|z-a|=r}\frac{f(z)}{z-a}\mathrm{d}z=2\pi\mathrm{i}f(a)$$

十、证：由 Cauchy 积分公式，知

$$f'(0)=\frac{1}{2\pi\mathrm{i}}\oint_{|z|=1}\frac{f(z)}{z^2}\mathrm{d}z$$

又根据假设条件及积分性质，得

$$\begin{aligned}|f'(0)|&=\left|\frac{1}{2\pi\mathrm{i}}\oint_{|z|=1}\frac{f(z)}{z^2}\mathrm{d}z\right|\\&\leqslant\frac{1}{2\pi}\oint_{|z|=1}\left|\frac{f(z)}{z^2}\right|\mathrm{d}s\\&\leqslant\frac{1}{2\pi}\oint_{|z|=1}\cdot 1\mathrm{d}s\\&=\frac{1}{2\pi}\cdot 2\pi\cdot 1\\&=1\end{aligned}$$

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复变函数与积分变换试题(四)解答

一、填空题

1. $\lim\limits_{n\to\infty} a_n = a_0, \lim\limits_{n\to\infty} b_n = b_0$.　2. 1，$|z|<1$ 内，$|z|\leqslant r, 0<r<1$.

3. 3.　4. $|b|$ R.　5. $1<|z-2|<2, \dfrac{1}{(z-3)(4-z)}$.

二、单项选择题

1. D　2. B　3. C　4. B　5. C

三、证：当 $0<|z|<1$ 时，因为

$$\begin{aligned}
|e^z-1| &= \left|z+\frac{z^2}{2!}+\cdots+\frac{z^n}{n!}+\cdots\right| \\
&= |z|\left|1+\frac{z}{2!}+\cdots+\frac{z^{n-1}}{n!}+\cdots\right| \\
&\leqslant |z|\left(1+\frac{|z|}{2!}+\cdots+\frac{|z|^{n-1}}{n!}+\cdots\right) \\
&< |z|\left(1+\frac{1}{2!}+\cdots+\frac{1}{n!}+\cdots\right) \\
&= |z|\left[1+\left(\frac{1}{2!}+\cdots+\frac{1}{n!}+\cdots\right)\right] \qquad (1)
\end{aligned}$$

$$\begin{aligned}
|e^z-1| &\geqslant \left(1-\left|\frac{z}{2!}+\cdots+\frac{z^{n-1}}{n!}+\cdots\right|\right) \\
&\geqslant |z|\left[1-\left(\frac{|z|}{2!}+\cdots+\frac{|z|^{n-1}}{n!}+\cdots\right)\right] \\
&\geqslant |z|\left[1-\left(\frac{1}{2!}+\cdots+\frac{1}{n!}+\cdots\right)\right] \qquad (2)
\end{aligned}$$

又因为

$$\begin{aligned}
&\frac{1}{2!}+\frac{1}{3!}+\cdots+\frac{1}{n!}+\cdots \\
&= \frac{1}{2}+\frac{1}{2}\left(\frac{1}{3}+\frac{1}{4\cdot 3}+\cdots+\frac{1}{n\cdot(n-1)\cdot\cdots\cdot 3}+\cdots\right) \\
&< \frac{1}{2}+\frac{1}{2}\left(\frac{1}{3}+\frac{1}{3^2}+\cdots+\frac{1}{3^{n-2}}+\cdots\right) \\
&= \frac{1}{2}+\frac{1}{2}\cdot\frac{1}{3}\cdot\frac{1}{1-\frac{1}{3}}=\frac{3}{4} \qquad (3)
\end{aligned}$$

将式(3)代入(1)和(2)，得

$$\frac{1}{4}|z|<|e^z-1|<\frac{7}{4}|z|$$

四、解：1. 由于函数 $\dfrac{e^z}{1-z}$ 仅在 $z=1$ 不解析，故其在原点 $z_0=0$ 处的 Taylor 展开式的收敛半径

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$R=1$，且

$$\frac{e^z}{1-z}=e^z\cdot\frac{1}{1-z}$$
$$=\left(1+z+\cdots+\frac{z^n}{n!}+\cdots\right)(1+z+\cdots+z^n+\cdots)$$
$$=1+2z+\cdots+\left(1+1+\frac{1}{2!}+\cdots+\frac{1}{n!}\right)z^n+\cdots,\ |z|<1$$

2. 由于函数 $\frac{z}{z^2+3z+2}$ 的不解析点为 $z_1=-1$ 和 $z_2=-2$，故其在 $z_0=2$ 处的 Taylor 展开式的收敛半径 $R=3$，且

$$\frac{z}{z^2+3z+2}=\frac{-1}{z+1}+\frac{2}{z+2}=\frac{-1}{3+z-2}+\frac{2}{4+z-2}$$
$$=-\frac{1}{3}\frac{1}{1-\left(-\frac{z-2}{3}\right)}+\frac{1}{2}\frac{1}{1-\left(-\frac{z-2}{4}\right)}$$
$$=-\frac{1}{3}\sum_{n=0}^{\infty}(-1)^n\frac{(z-2)^n}{3^n}+\frac{1}{2}\sum_{n=0}^{\infty}(-1)^n\frac{(z-2)^n}{4^n}$$
$$=\sum_{n=0}^{\infty}(-1)^n\left(\frac{1}{2\cdot 4^n}-\frac{1}{3^{n+1}}\right)(z-2)^n,\ |z-2|<3$$

五、证：由解析函数的 Taylor 展开式的系数公式，得

$$\frac{1}{2\pi i}\oint_{|\xi|=r}f(\xi)\frac{\xi^{n+1}-z^{n+1}}{\zeta-z}\frac{d\xi}{\xi^{n+1}}$$
$$=\frac{1}{2\pi i}\oint_{|\xi|=r}f(\xi)(\xi^n+\xi^{n-1}z+\cdots+\xi z^{n-1}+z^n)\frac{d\xi}{\xi^{n+1}}$$
$$=\frac{1}{2\pi i}\oint_{|\xi|=r}\frac{f(\xi)}{\xi}d\xi+\left(\frac{1}{2\pi i}\oint_{|\xi|=r}\frac{f(\xi)}{\xi^2}d\xi\right)z+\cdots+\left(\frac{1}{2\pi i}\oint_{|\xi|=r}\frac{f(\xi)}{\xi^{n+1}}d\xi\right)z^n$$
$$=a_0+a_1z+\cdots+a_nz^n$$
$$=S_n(z)$$

六、解：1. 当 $0<|z|<1$ 时，有

$$\frac{1}{z(1-z)^2}=\frac{1}{z}\left(\frac{1}{1-z}\right)'=\frac{1}{2}(1+z+\cdots+z^n+\cdots)'$$
$$=\frac{1}{z}(1+2z+\cdots+nz^{n-1}+\cdots)$$
$$=\frac{1}{z}+2+3z+\cdots+nz^{n-2}+\cdots$$

当 $0<|z-1|<1$，有

$$\frac{1}{z(1-z)^2}=\frac{1}{(z-1)^2}\frac{1}{1+z-1}=\frac{1}{(z-1)^2}\sum_{n=0}^{\infty}(-1)^n(z-1)^n$$
$$=\frac{1}{(z-1)^2}-\frac{1}{z-1}+1-(z-1)+(z-2)^2-\cdots$$

2. 当 $0<|z-i|<1$ 时，有

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$$\frac{1}{z^2(z-\mathrm{i})}=\frac{-1}{z-\mathrm{i}}\left(\frac{1}{\mathrm{i}+z-\mathrm{i}}\right)'=-\frac{1}{z-\mathrm{i}}\left[\frac{1}{1+\frac{z-\mathrm{i}}{\mathrm{i}}}\right]'\frac{1}{\mathrm{i}}$$

$$=-\frac{1}{\mathrm{i}}\ \frac{1}{z-\mathrm{i}}\left[\frac{1}{1-\mathrm{i}(z-\mathrm{i})}\right]'$$

$$=-\frac{1}{\mathrm{i}}\ \frac{1}{z-\mathrm{i}}[1+\mathrm{i}(z-\mathrm{i})+\cdots+\mathrm{i}^n(z-\mathrm{i})^n+\cdots]'$$

$$=-\frac{1}{\mathrm{i}}\ \frac{1}{z-\mathrm{i}}[1+2\mathrm{i}^2(z-\mathrm{i})+\cdots+n\mathrm{i}^n(z-\mathrm{i})^{n-1}+\cdots]$$

$$=\frac{\mathrm{i}}{z-\mathrm{i}}-2\mathrm{i}-3\mathrm{i}^2(z-\mathrm{i})-\cdots-n\mathrm{i}^{n-1}(z-\mathrm{i})^{n-2}+\cdots$$

3.当 $1<|z|<\infty$ 时

$$e^{\frac{1}{1-z}}=1+\frac{1}{1-z}+\frac{1}{2!}\left(\frac{1}{1-z}\right)^2+\frac{1}{3!}\left(\frac{1}{1-z}\right)^3+\cdots$$

$$=1-\frac{1}{z}\ \frac{1}{1-\frac{1}{z}}+\frac{1}{2!}\left(-\frac{1}{z}\right)^2\left[\frac{1}{1-\frac{1}{z}}\right]^2+\frac{1}{3!}\left(-\frac{1}{z}\right)^3\left[\frac{1}{1-\frac{1}{z}}\right]^3+\cdots$$

$$=1-\frac{1}{z}\left(1+\frac{1}{z}+\frac{1}{z^2}+\cdots\right)+\frac{1}{2!}\ \frac{1}{z^2}\left(1+\frac{2}{z}+\frac{3}{z^2}+\cdots\right)-\frac{1}{3!}\ \frac{1}{z^3}\left(1+\frac{3}{z}+\frac{6}{z^2}+\cdots\right)$$

$$=1-\frac{1}{z}+\left(-1+\frac{1}{2!}\right)\frac{1}{z^2}+\left(-1+1-\frac{1}{3!}\right)\frac{1}{z^3}+\cdots$$

$$=1-\frac{1}{z}-\frac{1}{2}\ \frac{1}{z^2}-\frac{1}{6}\ \frac{1}{z^3}+\cdots$$

七、解:

$$\oint_\Gamma\left(\sum_{n=-2}^{\infty}z^n\right)\mathrm{d}z=\sum_{n=-2}^{\infty}\oint_\Gamma z^n\mathrm{d}z$$
$$=\oint_\Gamma z^{-1}\mathrm{d}z$$
$$=2\pi\mathrm{i}$$

八、证:当 $|z|<R$ 时,注意到

$$|(\mathrm{Re}\,a_n)z^n|=|\mathrm{Re}\,a_n||z|^n\leqslant|a_n||z|^n=|a_nz^n|$$

由于级数 $\sum_{n=0}^{\infty}|a_nz^n|$ 收敛,故由比较判别法知,级数 $\sum_{n=0}^{\infty}|(\mathrm{Re}\,a_n)z^n|$ 收敛,即级数 $\sum_{n=0}^{\infty}(\mathrm{Re}\,a_n)z^n$ 在 $|z|<R$ 内处处收敛且绝对收敛,故其收敛半径大于或等于 R.

九、解:由于

$$e^{z^2}=1+z^2+\frac{z^4}{2!}+\frac{z^6}{3!}+\cdots$$

$$\cos z=1-\frac{z^2}{2!}+\frac{z^4}{4!}-\frac{z^6}{6!}+\cdots$$

故

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$$
\begin{array}{r|l}
1+z^2+\dfrac{z^4}{2!}+\dfrac{z^6}{3!}+\cdots & 1-\dfrac{z^2}{2!}+\dfrac{z^4}{4!}-\cdots \\
\underline{1-\dfrac{z^2}{2!}+\dfrac{z^4}{4!}-\dfrac{z^6}{6!}+\cdots} & \overline{1+\dfrac{3}{2}z^2+\dfrac{29}{24}z^4+\cdots} \\
\dfrac{3}{2}z^2+\dfrac{11}{24}z^4+\dfrac{121}{720}z^6+\cdots & \\
\underline{\dfrac{3}{2}z^2-\dfrac{3}{4}z^4+\dfrac{1}{16}z^6+\cdots} & \\
\dfrac{29}{24}z^4+\dfrac{76}{720}z^6+\cdots &
\end{array}
$$

即

$$\frac{e^{z^2}}{\cos z}=1+\frac{3}{2}z^2+\frac{29}{24}z^4+\cdots,\ |z|<\frac{\pi}{2}$$

十、证：由 Laurent 展开式的系数公式，知

$$
\begin{aligned}
a_n&=\frac{1}{2\pi i}\oint_{|z|=1}\frac{f(z)}{z^{n+1}}dz=\frac{1}{2\pi i}\oint_{|z|=1}\frac{\cos\left(z+\dfrac{1}{z}\right)}{z^{n+1}}dz\\
&\xlongequal{z=e^{i\theta}}\frac{1}{2\pi i}\int_0^{2\pi}\frac{\cos(e^{i\theta}+e^{-i\theta})}{(e^{i\theta})^{n+1}}ie^{i\theta}d\theta\\
&=\frac{1}{2\pi}\int_0^{2\pi}\cos(2\cos\theta)e^{-in\theta}d\theta\\
&=\frac{1}{2\pi}\int_0^{2\pi}\cos(2\cos\theta)\cdot\cos n\theta d\theta\\
&\quad-i\frac{1}{2\pi}\int_0^{2\pi}\cos(2\cos\theta)\cdot\sin n\theta d\theta
\end{aligned}
$$

注意到

$$\int_0^{2\pi}\cos(2\cos\theta)\cdot\sin n\theta d\theta=\int_{-\pi}^{\pi}\cos(2\cos\theta)\cdot\sin n\theta d\theta$$

且被积函数 $\cos(2\cos\theta)\cdot\sin n\theta$ 为积分变量 θ 的奇函数，故

$$\int_0^{2\pi}\cos(2\cos\theta)\cdot\sin n\theta d\theta=0$$

从而

$$a_n=\frac{1}{2\pi}\int_0^{2\pi}\cos(2\cos\theta)\cdot\cos n\theta d\theta,n=0,\pm1,\pm2,\cdots$$

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复变函数与积分变换试题(五)解答

一、填空题

1.可去. 2.6 阶极. 3.可去. 4.$\frac{1}{24}$. 5.0. 6.0.

二、单项选择题

1.B 2.B 3.C 4.A 5.D 6.A

三、解:1.$z=1$ 是 $e^{\frac{z}{1-z}}$ 的本性奇点;

2.$z=0$ 是$\frac{1}{z(z^2+1)^2}$的一阶极点,$z=\pm i$ 是$\frac{1}{z(z^2+1)^2}$的二阶极点;

3.$z=0$ 是$\frac{1-\cos z}{z^2}$的可去奇点;

4.$z=1$ 是 $e^{\frac{1}{z-1}}$ 的本性奇点.

四、解:1.$\operatorname{Res}\left[\frac{z+1}{z^2-2z},0\right]=\lim\limits_{z\to 0}z\frac{z+1}{z^2-2z}=-\frac{1}{2}$

$$\operatorname{Res}\left[\frac{z+1}{z^2-2z},2\right]=\lim_{z\to 2}(z-2)\frac{z+1}{z^2-2z}=\frac{3}{2}$$

2.由于

$$\begin{aligned}
\sin\frac{z}{z+1}&=\sin\left(1-\frac{1}{z+1}\right)\\
&=\sin 1\cos\left(\frac{1}{z+1}\right)-\cos 1\sin\left(\frac{1}{z+1}\right)\\
&=\sin 1\left[1-\frac{1}{2!}\left(\frac{1}{z+1}\right)^2+\frac{1}{4!}\left(\frac{1}{z+1}\right)^4-\cdots\right]\\
&\quad-\cos 1\left[\frac{1}{z+1}-\frac{1}{3!}\left(\frac{1}{z+1}\right)^3+\frac{1}{5!}\left(\frac{1}{z+1}\right)^5-\cdots\right]
\end{aligned}$$

可见

$$\operatorname{Res}\left[\sin\frac{1}{z+1},-1\right]=-\cos 1$$

3. $\operatorname{Res}\left[\frac{z}{\cos z},\frac{\pi}{2}+k\pi\right]=\lim\limits_{z\to\frac{\pi}{2}+k\pi}\left[z-\left(\frac{\pi}{2}+k\pi\right)\right]\frac{z}{\cos z}$

$$=\lim_{z\to\frac{\pi}{2}+k\pi}\frac{z}{\dfrac{\cos\left[z-\left(\frac{\pi}{2}+k\pi\right)+\left(\frac{\pi}{2}+k\pi\right)\right]}{z-\left(\frac{\pi}{2}+k\pi\right)}}$$

$$=\lim_{z\to\frac{\pi}{2}+k\pi}\frac{z}{-\dfrac{\sin\left[z-\left(\frac{\pi}{2}+k\pi\right)\right]\cdot\sin\left(\frac{\pi}{2}+k\pi\right)}{z-\left(\frac{\pi}{2}+k\pi\right)}}$$

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$$=(-1)^{k+1}\left(\frac{\pi}{2}+k\pi\right),k=0,\pm1,\pm2,\cdots$$

五、解：1. $f(z)=\exp\left(\frac{1}{z^2}\right)=1+\frac{1}{z^2}+\frac{1}{2!}\frac{1}{z^4}+\cdots,0<|z|<\infty$. 故

$$\text{Res}\left[\exp\left(\frac{1}{z^2}\right),\infty\right]=-a_{-1}=0$$

2. $$\text{Res}\left[\frac{2z}{3+z^2},\infty\right]=-\text{Res}\left[\frac{2\frac{1}{z}}{3+\frac{1}{z^2}}\frac{1}{z^2},0\right]$$

$$=-\text{Res}\left[\frac{2}{z(1+3z^2)},0\right]$$

$$=-\lim_{z\to0}z\frac{2}{z(1+3z^2)}$$

$$=-2$$

3. $f(z)=\cos z-\sin z$

$$=1-\frac{z^2}{2!}+\frac{z^4}{4!}-\cdots-\left(z-\frac{z^3}{3!}+\frac{z^5}{5!}-\cdots\right)$$

$$=1-z-\frac{z^2}{2!}+\frac{z^3}{3!}+\frac{z^4}{4!}-\cdots,0<|z|<\infty$$

故 $\text{Res}[\cos z-\sin z,\infty]=-a_{-1}=0$.

六、解：1. $\oint_C\frac{2\cos z}{(e+e^{-1})(z-i)^3}dz=2\pi i\text{Res}\left[\frac{2\cos z}{(e+e^{-1})(z-i)^3},i\right]$

$$=2\pi i\frac{1}{2!}\lim_{z\to i}\frac{d^2}{dz^2}(z-i)^3\frac{2\cos z}{(e+e^{-1})(z-i)^3}$$

$$=\pi i\frac{2}{e+e^{-1}}\lim_{z\to i}\frac{d^2}{dz^2}\cos z$$

$$=\pi i\frac{2}{e+e^{-1}}(-\cos z)\big|_{z=i}$$

$$=\pi i\frac{2}{e+e^{-1}}(-\cos i)\ |=-\pi i$$

2. $f(z)=\frac{1-\cos z}{z^m}=\frac{1}{z^{m-2}}\left(\frac{1}{2!}-\frac{z^2}{4!}+\frac{z^4}{6!}-\cdots\right),0<|z|<\infty$.

可见 $z=0$ 为 $f(z)$ 的 $m-2$ 阶极点. 当 $m\leqslant2$ 时，$\text{Res}\left[\frac{1-\cos z}{z^m},0\right]=a_{-1}=0$；当 $m=2n>2$ 时，$\text{Res}\left[\frac{1-\cos z}{z^m},0\right]=a_{-1}=0$；当 $m=2n+1>2$ 时

$$\text{Res}\left[\frac{1-\cos z}{z^m},0\right]=a_{-1}=\frac{(-1)^{n-1}}{(2n)!}$$

于是

$$\oint_C\frac{1-\cos z}{z^m}dz=2\pi i\left[\frac{1-\cos z}{z^m},0\right]=\begin{cases}0,m\leqslant2\\0,m=2n>2\\\frac{(-1)^{n-1}2\pi i}{(2n)!},m=2n+1>2\end{cases}$$

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3. $\oint_C \frac{1}{(z+i)^{10}(z-1)(z-3)}dz = 2\pi i\{\text{Res}[f(z),-i]+\text{Res}[f(t),1]\}$

$= -2\pi i\{\text{Res}[f(z),3]+\text{Res}[f(z),\infty]\}$

$= -2\pi i\left\{\lim_{z\to 3}\frac{1}{(z+i)^{10}(z-1)} - \text{Res}\left[\frac{1}{\left(\frac{1}{z}+i\right)^{10}\left(\frac{1}{z}-1\right)\left(\frac{1}{z}-3\right)}\frac{1}{z^2},0\right]\right\}$

$= 2\pi i\left\{\frac{1}{2(3+i)^{10}} - \text{Res}\left[\frac{z^{10}}{(1+iz)^{10}(1-z)(1-3z)},0\right]\right\}$

$= -\frac{\pi i}{(3+i)^{10}}$

4. $f(z)=\frac{1}{1+z^4}$ 在 C 内以 $z_1=e^{i\frac{\pi}{4}}$ 和 $z_2=e^{i\frac{7\pi}{4}}$ 为一阶极点，且

$$\text{Res}\left[\frac{1}{1+z^4},e^{i\frac{\pi}{4}}\right]=\frac{1}{4z^3}\bigg|_{z=e^{i\frac{\pi}{4}}}=\frac{1}{2\sqrt{2}(-1+i)}$$

$$\text{Res}\left[\frac{1}{1+z^4},e^{i\frac{7\pi}{4}}\right]=\frac{1}{4z^3}\bigg|_{z=e^{i\frac{7\pi}{4}}}=\frac{1}{-2\sqrt{2}(1+i)}$$

故

$$\oint_C \frac{dz}{1+z^4}=2\pi i\left\{\text{Res}\left[\frac{1}{1+z^4},e^{i\frac{\pi}{4}}\right]+\text{Res}\left[\frac{1}{1+z^4},e^{i\frac{7\pi}{4}}\right]\right\}=-\frac{\pi i}{\sqrt{2}}$$

七、解：1. $\int_0^{2\pi}\frac{d\theta}{5+3\sin\theta}=\oint_{|z|=1}\frac{1}{5+3\frac{z^2-1}{2iz}}\frac{1}{iz}=\oint_{|z|=1}\frac{2}{3z^2+10iz-3}dz$

$$=2\cdot 2\pi i\text{Res}\left[\frac{1}{3z^2+10iz-3},-\frac{i}{3}\right]$$

$$=4\pi i\lim_{z\to-\frac{i}{3}}\left(z+\frac{i}{3}\right)\frac{1}{3z^2+10iz-3}$$

$$=\frac{\pi}{2}$$

2. $\int_{-\infty}^{\infty}\frac{\cos x}{x^2+4x+5}dx=\text{Re}\left[\int_{-\infty}^{\infty}\frac{e^{ix}}{x^2+4x+5}dx\right]$

$$=\text{Re}\left\{2\pi i\text{Res}\left[\frac{e^{iz}}{z^2+4z+5},-2+i\right]\right\}$$

$$=\text{Re}\left(2\pi i\lim_{z\to-2+i}\frac{e^{iz}}{2z+4}\right)$$

$$=\pi e^{-1}\cos 2$$

3. $\int_0^{\infty}\frac{x\sin x}{1+x^2}dx=\frac{1}{2}\int_{-\infty}^{\infty}\frac{x\sin x}{1+x^2}dx$

$$=\frac{1}{2}\text{Im}\int_{-\infty}^{\infty}\frac{x}{1+x^2}e^{ix}dx$$

$$=\frac{1}{2}\text{Im}\left\{2\pi i\text{Res}\left[\frac{z}{1+z^2}e^{iz},i\right]\right\}$$

$$=\frac{1}{2}\text{Im}\left(2\pi i\lim_{z\to i}\frac{z}{1+z^2}e^{iz}\right)$$

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$$=\frac{1}{2}\mathrm{Im}\left(2\pi\mathrm{i}\,\frac{\mathrm{i}}{2\mathrm{i}}\mathrm{e}^{-1}\right)$$

$$=\pi\mathrm{e}^{-1}$$

八、证：由题设知

$$f(z)=\frac{a_{-n}}{(z-a)^n}+\cdots+\frac{a_{-1}}{z-a}+a_0+a_1(z-a)+\cdots,0<|z-a|<r$$

于是

$$g(z)=(z-a)^n f(z)=a_{-n}+\cdots+a_{-1}(z-a)^{n-1}+a_0(z-a)^n+\cdots$$

若令 $g(a)=a_{-n}$，则 $g(z)$ 在 $z=a$ 处是解析的. 于是

$$\oint_C f(z)\mathrm{d}z=2\pi\mathrm{i}\mathrm{Res}[f(z),a]$$

$$=2\pi\mathrm{i}\,\frac{1}{(n-1)!}\lim_{z\to a}\frac{\mathrm{d}^{n-1}}{\mathrm{d}z^{n-1}}(z-a)^n f(z)$$

$$=2\pi\mathrm{i}\,\frac{1}{(n-1)!}\lim_{z\to a}\frac{\mathrm{d}^{n-1}}{\mathrm{d}z^{n-1}}g(z)$$

$$=2\pi\mathrm{i}\,\frac{1}{(n-1)!}\lim_{z\to a}g^{(n-1)}(z)$$

$$=\frac{2\pi\mathrm{i}}{(n-1)!}g^{(n-1)}(a)$$

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复变函数与积分变换试题(六)解答

一、填空题

1. $\frac{3}{4}\pi i[\delta(\omega+1)-\delta(\omega-1)]-\frac{1}{4}\pi i[\delta(\omega+3)-\delta(\omega-3)]$.

2. $\frac{1}{|a|}e^{-i\frac{t_0}{a}\omega}F\left(\frac{\omega}{a}\right)$.　　3. $e^{-i\omega\tau}\left[\frac{1}{i\omega}+\pi\delta(\omega)\right]$.

4. $-\frac{3}{2}e^{-|t|}$.　　5. $2\pi\delta(\omega)$.　　6. e^{-3}.

二、单项选择题

1.A　2.C　3.B　4.A　5.D　6.B

三、解:由于 $\mathscr{F}[e^{-|t|}]=\frac{2}{1+\omega^2}$

$$\mathscr{F}[e^{-|t|}\cos t]=\frac{1}{2}\left[\frac{2}{1+(\omega+1)^2}+\frac{2}{1+(\omega-1)^2}\right]$$
$$=\frac{2(\omega^2+2)}{\omega^4+4}$$

于是由 $f(t)=e^{-|t|}\cos t$ 有积分形式的表达式

$$e^{-|t|}\cos t=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{2(\omega^2+i)}{\omega^4+4}e^{i\omega t}d\omega$$
$$=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\omega^2+2}{\omega^4+4}\cos\omega t\,d\omega$$
$$=\frac{2}{\pi}\int_{0}^{\infty}\frac{\omega^2+2}{\omega^4+4}\cos\omega t\,d\omega$$

因此

$$\int_0^{\infty}\frac{\omega^2+2}{\omega^4+4}\cos\omega t\,d\omega=\frac{\pi}{2}e^{-|t|}\cos t$$

四、解:1. $F(\omega)=\mathscr{F}[f(t)]=\int_{-\infty}^{\infty}f(t)e^{-i\omega t}dt=\int_{-1}^{1}(1-t^2)e^{-i\omega t}dt$

$$=2\int_0^1(1-t^2)\cos\omega t\,dt=2\left[\int_0^1\cos\omega t\,dt-\int_0^1 t^2\cos\omega t\,dt\right]$$
$$=\frac{4}{\omega^3}(\sin\omega-\omega\cos\omega)$$

2. $\mathscr{F}[u(t)e^{-\beta t}]=\frac{1}{\beta+i\omega}$

$$\mathscr{F}[tu(t)e^{-\beta t}]=-\frac{1}{i}\mathscr{F}[(-it)u(t)e^{-\beta t}]=i\left(\frac{1}{\beta+i\omega}\right)'$$
$$=i\frac{-i}{(\beta+i\omega)^2}=\frac{1}{(\beta+i\omega)^2}$$

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$$\mathscr{F}[tu(t)e^{-\beta t}\sin\omega_0 t]=\frac{i}{2}\left\{\frac{1}{[\beta+i(\omega+\omega_0)]^2}-\frac{1}{[\beta+i(\omega-\omega_0)]^2}\right\}$$

3. 因为 $\mathscr{F}[\operatorname{sgn} t]=\frac{2}{i\omega}$，利用对称性质知 $\operatorname{sgn}(-\omega)=\frac{1}{2\pi}\mathscr{F}\left[\frac{2}{it}\right]=-\mathscr{F}\left[\frac{i}{\pi t}\right]$，于是由卷积定理，有

$$\begin{aligned}F(\omega)&=\mathscr{F}\left[\cos\omega_0 t+i\frac{1}{\pi t}*\cos\omega_0 t\right]\\&=\mathscr{F}[\cos\omega_0 t]+\mathscr{F}\left[\frac{i}{\pi t}*\cos\omega_0 t\right]\\&=\pi[\delta(\omega+\omega_0)+\delta(\omega-\omega_0)][1-\operatorname{sgn}(-\omega)]\\&=\pi[\delta(\omega+\omega_0)+\delta(\omega-\omega_0)](1+\operatorname{sgn}\omega)=2\pi\delta(\omega-\omega_0)\end{aligned}$$

4. $\mathscr{F}[tu(t)]=\frac{1}{\omega^2}+\pi i\delta'(\omega)$

$$\mathscr{F}[e^{i\omega_0 t}tu(t)]=\frac{1}{(\omega-\omega_0)^2}+\pi i\delta'(\omega-\omega_0)$$

五、解：1. $f_1*f_2=\int_{-\infty}^{\infty}u(\tau)e^{-(t-\tau)}u(t-\tau)d\tau$

$$\begin{aligned}&=\int_0^t e^{-(t-\tau)}d\tau\\&=e^{-t}\int_0^t e^{\tau}d\tau=1-e^{-t}\end{aligned}$$

2. $f_1*f_2(t)=\int_{-\infty}^{\infty}e^{-a\tau}u(\tau)\sin(t-\tau)u(t-\tau)d\tau$

$$\begin{aligned}&=\int_0^t e^{-a\tau}\sin(t-\tau)d\tau\\&=\int_0^t e^{-a\tau}\frac{e^{i(t-\tau)}-e^{-i(t-\tau)}}{2i}d\tau\\&=\frac{1}{2i}\left(\frac{e^{it}-e^{-at}}{a+i}-\frac{e^{-it}-e^{-at}}{a-i}\right)\\&=\frac{a\sin t-\cos t+e^{-at}}{a^2+1}\end{aligned}$$

六、解：1. $\int_{-\infty}^{\infty}\frac{1-\cos t}{t^2}dt=2\int_{-\infty}^{\infty}\frac{\sin^2\frac{t}{2}}{t^2}dt=\int_{-\infty}^{\infty}\frac{\sin^2\frac{t}{2}}{\left(\frac{t}{2}\right)^2}d\left(\frac{t}{2}\right)$

$$\xlongequal{x=\frac{t}{2}}\int_{-\infty}^{\infty}\left(\frac{\sin x}{x}\right)^2dx=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left|\mathscr{F}\left[\frac{\sin t}{t}\right]\right|^2d\omega$$

又

$$\begin{aligned}\mathscr{F}\left[\frac{\sin t}{t}\right]&=\int_{-\infty}^{\infty}\frac{\sin t}{t}e^{-i\omega t}dt=2\int_0^{\infty}\frac{\sin t}{t}\cos\omega t\,dt\\&=\int_0^{\infty}\frac{\sin[(1+\omega)t]+\sin[(1-\omega)t]}{t}dt\\&=\begin{cases}\pi, & |\omega|<1\\0, & \text{其他}\end{cases}\end{aligned}$$

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从而

$$\int_{-\infty}^{\infty}\frac{1-\cos t}{t^2}\mathrm{d}t=\frac{1}{2\pi}\int_{-1}^{1}\pi^2\mathrm{d}\omega=\pi$$

2. $$\int_{-\infty}^{\infty}\frac{\sin^4 t}{t^2}\mathrm{d}t=\int_{-\infty}^{\infty}\frac{\sin^2 t-\frac{1}{4}\sin^2 2t}{t^2}\mathrm{d}t$$

$$=\int_{-\infty}^{\infty}\left(\frac{\sin t}{t}\right)^2\mathrm{d}t-\frac{1}{2}\int_{-\infty}^{\infty}\left(\frac{\sin 2t}{2t}\right)^2\mathrm{d}2t$$

$$=\int_{-\infty}^{\infty}\left(\frac{\sin t}{t}\right)^2\mathrm{d}t-\frac{1}{2}\int_{-\infty}^{\infty}\left(\frac{\sin x}{x}\right)^2\mathrm{d}x$$

$$=\frac{1}{2}\int_{-\infty}^{\infty}\left(\frac{\sin t}{t}\right)^2\mathrm{d}t=\frac{1}{2}\,\frac{1}{2\pi}\int_{-\infty}^{\infty}\left|\mathscr{F}\left[\frac{\sin t}{t}\right]\right|^2\mathrm{d}\omega$$

$$=\frac{1}{4\pi}\int_{-1}^{1}\pi^2\mathrm{d}\omega=\frac{\pi}{2}$$

七、解：设 $Y(\omega)=\mathscr{F}[y(t)]$，并注意到

$$\int_{-\infty}^{\infty}\frac{y(\tau)}{(t-\tau)^2+a^2}\mathrm{d}\tau=y(t)*\frac{1}{t^2+a^2}$$

对原方程两端同时取 Fourier 变换并利用卷积定理，得到

$$Y(\omega)=\mathscr{F}\left[\frac{1}{t^2+b^2}\right]/\mathscr{F}\left[\frac{1}{t^2+a^2}\right]$$

而

$$\mathscr{F}\left[\frac{1}{t^2+a^2}\right]=\int_{-\infty}^{\infty}\frac{1}{t^2+a^2}e^{-i\omega t}\mathrm{d}t$$

$$=\begin{cases}2\pi i\mathrm{Res}\left[\dfrac{e^{i\omega z}}{z^2+a^2},ai\right]=\dfrac{\pi}{a}e^{-a\omega},\omega>0\\[2ex] 2\pi i\mathrm{Res}\left[\dfrac{e^{-i\omega z}}{z^2+a^2},ai\right]=\dfrac{\pi}{a}e^{a\omega},\omega<0\end{cases}$$

于是可得

$$Y(\omega)=\frac{\frac{\pi}{b}e^{-b|\omega|}}{\frac{\pi}{a}e^{-a|\omega|}}=\frac{a}{b}e^{-(b-a)|\omega|}$$

从而

$$y(t)=\mathscr{F}^{-1}[Y(\omega)]=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{a}{b}e^{-(b-a)|\omega|}e^{i\omega t}\mathrm{d}\omega$$

$$=\frac{a}{2\pi b}\left[\int_{-\infty}^{0}e^{[(b-a)+it]\omega}\mathrm{d}\omega\int_{0}^{\infty}e^{[-(b-a)+it]\omega}\mathrm{d}\omega\right]$$

$$=\frac{a(b-a)}{\pi b[t^2+(b-a)^2]}$$

八、解：令 $X(\omega)=\mathscr{F}[x(t)]$，则对原方程两边同时取 Fourier 变换，得到

$$i\omega X(\omega)-\left[\frac{X(\omega)}{i\omega}+\pi X(0)\delta(\omega)\right]=\frac{2}{1+\omega^2}$$

将 $X(0)=\int_{-\infty}^{\infty}x(t)\mathrm{d}t=0$ 代入上述方程并化简，有

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$$X(\omega)=-\frac{2\omega\mathrm{i}}{(1+\omega^2)^2}$$

于是

$$x(t)=\mathscr{F}^{-1}[X(\omega)]=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{-2\omega\mathrm{i}}{(1+\omega^2)^2}\mathrm{e}^{\mathrm{i}\omega t}\mathrm{d}\omega$$
$$=-\frac{\mathrm{i}}{\pi}\int_{-\infty}^{\infty}\frac{\omega}{(1+\omega^2)^2}\mathrm{e}^{\mathrm{i}\omega t}\mathrm{d}\omega$$

当 $t>0$ 时

$$x(t)=-\frac{\mathrm{i}}{\pi}2\pi\mathrm{i}\mathrm{Res}\left[\frac{\omega}{(1+\omega^2)^2}\mathrm{e}^{\mathrm{i}\omega t},\mathrm{i}\right]$$
$$=\frac{t}{2}\mathrm{e}^{-t}$$

当 $t<0$ 时

$$x(t)=-\frac{\mathrm{i}}{\pi}\int_{-\infty}^{\infty}\frac{\omega}{(1+\omega^2)^2}\mathrm{e}^{\mathrm{i}\omega t}\mathrm{d}\omega$$
$$\xlongequal{\omega=-\tau}-\frac{\mathrm{i}}{\pi}\int_{\infty}^{-\infty}\frac{-\tau}{(1+\tau^2)^2}\mathrm{e}^{\mathrm{i}(-t)\tau}(-\mathrm{d}\tau)$$
$$=\frac{\mathrm{i}}{\pi}\int_{-\infty}^{\infty}\frac{\tau\mathrm{e}^{\mathrm{i}(-t)\tau}}{(1+\tau^2)^2}\mathrm{d}\tau$$
$$=\frac{\mathrm{i}}{\pi}2\pi\mathrm{i}\mathrm{Res}\left[\frac{\tau}{(1+\tau^2)^2}\mathrm{e}^{\mathrm{i}(-t)\tau},\mathrm{i}\right]=\frac{t}{2}\mathrm{e}^{t}$$

当 $t=0$ 时

$$x(0)=-\frac{\mathrm{i}}{\pi}\int_{-\infty}^{\infty}\frac{\omega}{(1+\omega^2)^2}\mathrm{d}\omega=0$$

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复变函数与积分变换试题(七)解答

一、填空题

1. $\frac{1}{a}F\left(\frac{s}{a}\right)e^{-\frac{b}{a}s}$.　2. $\frac{2s^3-24s}{(s^2+2^2)^3}$, $\mathrm{Re}s>0$.　3. $\frac{\pi}{2}-\arctan\frac{s}{3}$.

4. $\frac{\sin t}{t}$.　5. $\frac{2(1-\cos t)}{t}$.　6. $t+(t-3)u(t-2)$.

二、单项选择题

1.C　2.B　3.A　4.C　5.D　6.A

三、解:1. $\mathscr{L}\left[\frac{2}{t}(1-\cos at)\right]=\int_s^{\infty}\mathscr{L}[2(1-\cos at)]ds$

$$=\int_s^{\infty}2\left(\frac{1}{s}-\frac{s}{s^2+a^2}\right)ds$$

$$=[2\ln s-\ln(s^2+a^2)]\Big|_s^{\infty}=\ln\left(1+\frac{a^2}{s^2}\right)$$

2. $\mathscr{L}[u(3t)]=\frac{1}{3}\frac{1}{\frac{s}{3}}=\frac{1}{s}$

$$\mathscr{L}[u(3t-5)]=\mathscr{L}\left\{u\left[3\left(t-\frac{5}{3}\right)\right]\right\}$$

$$=e^{-\frac{5}{3}s}\mathscr{L}[u(3t)]=e^{-\frac{5}{3}s}\frac{1}{s}=\frac{e^{-\frac{5}{3}s}}{s},\mathrm{Re}s>0$$

3. 由于 $\mathscr{L}[t^n]=\frac{n!}{s^{n+1}}$ 及位移性质,有

$$\mathscr{L}[t^ne^{at}]=\frac{n!}{(s-a)^{n+1}},\mathrm{Re}(s-a)>0$$

4. $\mathscr{L}\left[\frac{e^{bt}-e^{at}}{t}\right]=\int_s^{\infty}\mathscr{L}[e^{bt}-e^{at}]ds$

$$=\int_s^{\infty}\left(\frac{1}{s-b}-\frac{1}{s-a}\right)ds=\left(\ln\frac{s-b}{s-a}\right)\Big|_s^{\infty}=\ln\frac{s-a}{s-b}$$

四、解:1. $f(t)=\mathscr{L}^{-1}\left[\frac{s^3+5s^2+9s+7}{(s+1)(s+2)}\right]=\mathscr{L}^{-1}\left[s+2+\frac{s+3}{(s+1)(s+2)}\right]$

$$=\delta'(t)+2\delta(t)+\mathscr{L}^{-1}\left[\frac{2}{s+1}-\frac{1}{s+2}\right]$$

$$=\delta'(t)+2\delta(t)+2e^{-t}-e^{-2t}$$

2. $f(t)=\mathscr{L}^{-1}\left[\frac{s^2+2s-1}{s(s-1)^2}\right]=\mathscr{L}^{-1}\left[\frac{-1}{s}+\frac{2}{s-1}+\frac{2}{(s-1)^2}\right]$

$$=-u(t)+2e^t$$

3. $f(t)=\mathscr{L}^{-1}\left[\ln\frac{s^2-1}{s^2}\right]=-\frac{1}{t}\left[\left(\ln\frac{s^2-1}{s^2}\right)'\right]$

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$$=-\frac{1}{t}\mathscr{L}^{-1}\left[\frac{2s}{s^2-1}-\frac{2}{s}\right]=-\frac{1}{t}\mathscr{L}^{-1}\left[\frac{1}{s-1}+\frac{1}{s+1}-\frac{2}{3}\right]$$

$$=-\frac{1}{t}(e^{t}+e^{-t}-2)=\frac{2}{t}(1-\text{ch } t)$$

4. $f(t)=\mathscr{L}^{-1}\left[\frac{s^2}{(s^2+a^2)^2}\right]=\mathscr{L}^{-1}\left[\frac{s}{s+a^2}\cdot\frac{s}{s^2+a^2}\right]$

$$=(\cos at)*(\cos at)$$

$$=\int_0^t \cos a\tau\cos[a(t-\tau)]\mathrm{d}\tau$$

$$=\frac{1}{2}\int_0^t[\cos at+\cos(2a\tau-at)]\mathrm{d}\tau$$

$$=\frac{1}{2}\left(t\cos at+\frac{\sin at}{a}\right)$$

五、解：1. $\mathscr{L}\left[\frac{e^{-3t}-e^{-6t}}{t}\right]=\int_s^{\infty}\mathscr{L}[e^{-3t}-e^{-6t}]\mathrm{d}s$

$$=\int_s^{\infty}\left(\frac{1}{s+3}-\frac{1}{s+6}\right)\mathrm{d}s=\ln\left(\frac{s+6}{s+3}\right)$$

或

$$\int_0^{\infty}\frac{e^{-3t}-e^{-6t}}{t}e^{-st}\mathrm{d}t=\ln\left(\frac{s+6}{s+3}\right)$$

令 $s\to 0^+$，得 $\int_0^{\infty}\frac{e^{-3t}-e^{-6t}}{t}\mathrm{d}t=\ln 2$.

2. $\mathscr{L}[t\sin t]=\int_0^{\infty}t\sin t e^{-st}\mathrm{d}t$

$$=-\frac{\mathrm{d}}{\mathrm{d}s}\mathscr{L}[\sin t]=-\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{1}{s^2+1}\right)=\frac{2s}{(s^2+1)^2}$$

令 $s=3$，得 $\int_0^{\infty}te^{-3t}\sin t\mathrm{d}t=\frac{2s}{(s^2+1)^2}\Big|_{s=3}=\frac{3}{50}$.

3. $\int_0^{\infty}te^{-2t}\mathrm{d}t=\mathscr{L}[t]\,|_{s=2}=\frac{\Gamma(2)}{s^2}\Big|_{s=2}=\frac{\Gamma(2)}{4}=\frac{1}{4}$

4. $\int_0^{\infty}\frac{1-\cos t}{t}e^{-t}\mathrm{d}t=\mathscr{L}\left[\frac{1-\cos t}{t}\right]\Big|_{s=1}$

$$=\frac{1}{2}\ln\left(1+\frac{1}{s^2}\right)\Big|_{s=1}=\frac{\ln 2}{2}$$

六、解：设 $Y(s)=\mathscr{L}[y(t)]$，对原方程两边同时取 Laplace 变换，得

$$SY(s)+2\frac{Y(s)}{s}=e^{-s}\frac{1}{s}$$

$$Y(s)\left(\frac{s^2+2}{s}\right)=\frac{e^{-s}}{s}$$

$$Y(s)=\frac{e^{-s}}{s^2+2}$$

$$y(t)=\mathscr{L}^{-1}[Y(s)]=\mathscr{L}^{-1}\left[\frac{e^{-s}}{s^2+2}\right]$$

又

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$$\mathscr{L}^{-1}\left[\frac{1}{s^2+2}\right]=\frac{1}{\sqrt{2}}\sin(\sqrt{2}t)$$

由延迟性质,得

$$y(t)=\mathscr{L}^{-1}[Y(s)]=\frac{1}{\sqrt{2}}\sin[\sqrt{2}(t-1)]u(t-1)$$

七、解:$\mathscr{L}[x(t)]=X(s)$,$\mathscr{L}[y(t)]=Y(s)$,则对原方程两边同时取 Laplace 变换,并代初始条件,得到

$$\begin{cases}S^2Y(s)-S^2X(s)+SX(s)-Y(s)=\dfrac{1}{s-1}-\dfrac{2}{s}\\2S^2Y(s)-S^2X(s)-2SY(s)+X(s)=-\dfrac{1}{s^2}\end{cases}$$

整理之

$$\begin{cases}(s+1)Y(s)-SX(s)=\dfrac{2-s}{s(s-1)^2}\\2SY(s)-(s+1)X(s)=-\dfrac{1}{s^2(s-1)}\end{cases}$$

解得

$$X(s)=\frac{2s-1}{s^2(s-1)^2},Y(s)=\frac{1}{s(s-1)^2}$$

于是

$$\begin{cases}x(t)=\mathscr{L}^{-1}[X(s)]=\mathscr{L}^{-1}\left[\dfrac{2s-1}{s^2(s-1)^2}\right]=\mathscr{L}^{-1}\left[\dfrac{1}{(s-1)^2}-\dfrac{1}{s^2}\right]=t(e^t-1)\\y(t)=\mathscr{L}^{-1}[Y(s)]=\mathscr{L}^{-1}\left[\dfrac{1}{s(s-1)^2}\right]=\mathscr{L}^{-1}\left[\dfrac{1}{s}-\dfrac{1}{s-1}+\dfrac{1}{(s-1)^2}\right]\\\qquad=1-e^t(t-1)\end{cases}$$

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复变函数与积分变换试题(综合一)解答

一、填空题

1. $\mathbb{C}$ 上处,在点 $z=1$,$\mathbb{C}$ 上无. 2. $e^{-\left(\frac{\pi}{4}+2a\pi\right)}\cos\left(\frac{\ln 2}{2}\right)$,$e^{-\left(\frac{\pi}{4}+2k\pi\right)}\sin\left(\frac{\ln 2}{2}\right)$,$k\in\mathbb{Z}$.

3. $\frac{\pi^2 i}{3}$. 4. $\frac{1}{3!}$. 5. $\frac{1}{3}e^{-\frac{2}{3}i\omega}F\left(\frac{\omega}{3}\right)$.

二、单项选择题

1. C 2. C 3. A 4. D 5. C

三、解:$\frac{\partial u}{\partial x}=e^{-y}\cos x$,$\frac{\partial^2 u}{\partial x^2}=-e^{-y}\sin x$

$\frac{\partial u}{\partial y}=-e^{-y}\sin x$,$\frac{\partial^2 u}{\partial y^2}=e^{-y}\sin x$

可见

$$\Delta u=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=e^{-y}(-\sin x+\sin x)=0$$

即 $u=e^{-y}\sin x$ 是调和函数

又

$$\frac{\partial v}{\partial x}=-\frac{\partial u}{\partial y}=e^{-y}\sin x$$

$$v=\int e^{-y}\sin x\,dx=-e^{-y}\cos x+\varphi(y)$$

$$\frac{\partial v}{\partial y}=e^{-y}\cos x+\varphi'(y)=\frac{\partial u}{\partial x}=e^{-y}\cos x$$

$$\varphi'(y)=0 \quad \varphi(y)=k(k\in\mathbb{R})$$

从而

$$\begin{aligned}f(z)&=u+iv=e^{-y}(\sin x-i\cos x)+ik\\&=-ie^{iz}+ik\end{aligned}$$

代入条件 $f(0)=-i$,知 $k=0$,于是

$$f(z)=-ie^{iz}$$

四、解:1. $f(z)=1+2\left(\frac{-1}{z-3}+\frac{3}{z-4}\right)$

$$\begin{aligned}&=1+\frac{-2}{z}\left[\frac{1}{1-\frac{3}{z}}\right]-\frac{3}{2}\left[\frac{1}{1-\frac{z}{4}}\right]\\&=1+\left[-\frac{2}{z}\sum_{n=0}^{\infty}\left(\frac{3}{z}\right)^n\right]-\frac{3}{2}\sum_{n=0}^{\infty}\left(\frac{z}{4}\right)^n\\&=\sum_{n=1}^{\infty}\frac{2\cdot 3^{n-1}}{z^n}-\frac{3}{2}\sum_{n=0}^{\infty}\frac{z^n}{4^n}+1\end{aligned}$$

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$$=\sum_{n=1}^{\infty}\frac{2\cdot 3^{n-1}}{z^n}-\frac{1}{2}-\frac{3}{2}\sum_{n=1}^{\infty}\frac{z^n}{4^n}$$

2. $f(z)=\sin\left(\frac{z}{z-1}\right)=\sin\left(1+\frac{1}{z-1}\right)$

$$=\sin 1\cos\frac{1}{z-1}+\cos 1\sin\frac{1}{z-1}$$

$$=\sin 1\left[1-\frac{1}{2!}\frac{1}{(z-1)^2}+\frac{1}{4!}\frac{1}{(z-1)^4}-\cdots\right]$$

$$+\cos 1\left[\frac{1}{z-1}-\frac{1}{3!}\frac{1}{(z-1)^3}+\frac{1}{5!}\frac{1}{(z-1)^5}-\cdots\right]$$

五、解：$f(z)=1-2a\frac{1}{z+a}=1-2\frac{1}{1+\frac{z}{a}}=1-2\sum_{n=0}^{\infty}(-1)^n\frac{z^n}{a^n}$

而

$$\oint_C\frac{f(z)}{z^{n+1}}\mathrm{d}z=\frac{2\pi\mathrm{i}}{n!}f^{(n)}(0)$$

可知当 $n=0$ 时，有

$$\oint_C\frac{f(z)}{z}\mathrm{d}z=2\pi\mathrm{i}f(0)=-2\pi\mathrm{i}$$

当 $n=1,2,\cdots$ 时，有

$$\oint_C\frac{f(z)}{z^{n+1}}\mathrm{d}z=2\pi\mathrm{i}\left[-\frac{2(-1)^n}{a^n}\right]$$

六、解：1. $\oint_C z(z+\bar{z})\mathrm{d}z=\oint_C(z^2+|z|^2)\mathrm{d}z$

$$=\int_0^{\pi}(\mathrm{e}^{\mathrm{i}\theta})^2\mathrm{i}\mathrm{e}^{\mathrm{i}\theta}\mathrm{d}\theta+\int_0^{\pi}\mathrm{i}\mathrm{e}^{\mathrm{i}\theta}\mathrm{d}\theta$$

$$=\mathrm{i}\int_0^{\pi}\mathrm{e}^{\mathrm{i}3\theta}\mathrm{d}\theta+(\mathrm{e}^{\mathrm{i}\theta})\Big|_0^{\pi}$$

$$=\frac{1}{3}\mathrm{e}^{\mathrm{i}3\theta}\Big|_0^{\pi}-2=-\frac{2}{3}-2=-\frac{8}{3}$$

2. $\oint_C\frac{\cos\pi z}{z^3(z-1)^2}\mathrm{d}z=2\pi\mathrm{i}\left\{\mathrm{Res}\left[\frac{\cos\pi z}{z^3(z-1)^2},0\right]+\mathrm{Res}\left[\frac{\cos\pi z}{z^3(z-1)^2},1\right]\right\}$

$$=2\pi\mathrm{i}\frac{1}{2!}\lim_{z\to 0}\frac{\mathrm{d}^2}{\mathrm{d}z}\left[\frac{\cos\pi z}{(z-1)^2}\right]+2\pi\mathrm{i}\lim_{z\to 1}\frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{\cos\pi z}{z^3}\right)$$

$$=\pi^3\mathrm{i}$$

3. $\oint_C\frac{z^{13}}{(z^2+2)^3(z^2-1)^4}\mathrm{d}z=-2\pi\mathrm{i}\mathrm{Res}\left[\frac{z^{13}}{(z^2+2)^3(z^2-1)^4},\infty\right]$

$$=2\pi\mathrm{i}\mathrm{Res}\left[\frac{1}{z^2}\frac{\frac{1}{z^{13}}}{\left(\frac{1}{z^2}+2\right)^3\left(\frac{1}{z^2}-1\right)^4},0\right]$$

$$=2\pi\mathrm{i}\mathrm{Res}\left[\frac{1}{z}\frac{1}{(1+2z^2)^3(1-z^2)^4},0\right]$$

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$$=2\pi\mathrm{i}\lim_{z\to 0}\frac{1}{(1+2z^2)^3(1-z^2)^4}=2\pi\mathrm{i}$$

4. $\int_{-\infty}^{\infty}\frac{\cos x}{(x^2+1)(x^2+9)}\mathrm{d}x=\mathrm{Re}\int_{-\infty}^{\infty}\frac{\mathrm{e}^{\mathrm{i}x}}{(x^2+1)(x^2+9)}\mathrm{d}x$

$$=\mathrm{Re}\left(2\pi\mathrm{i}\left\{\mathrm{Res}\left[\frac{\mathrm{e}^{\mathrm{i}z}}{(z^2+1)(z^2+a)},\mathrm{i}\right]+\mathrm{Res}\left[\frac{\mathrm{e}^{\mathrm{i}z}}{(z^2+1)(z^2+9)},3\mathrm{i}\right]\right\}\right)$$

$$=\frac{\pi}{12\mathrm{e}}$$

七、解：由 $f(z)$ 和 $g(z)$ 在点 z_0 处解析，故可设

$$f(z)=\sum_{n=0}^{\infty}\alpha_n(z-z_0)^n$$

$$g(z)=\sum_{n=0}^{\infty}\beta_n(z-z_0)^n,\ |z-z_0|<\mathrm{R},\ \mathrm{R}>0$$

其中 $\alpha_n=\frac{f^{(n)}(z_0)}{n!}$，$\alpha_0=f(z_0)\neq 0$；$\beta_n=\frac{g^{(n)}(z_0)}{n!}$，$n=0,1,2,\cdots$，因为 z_0 是 $g(z)$ 的二阶零点，故 $\beta_0=\beta_1=0$，$\beta_2\neq 0$.

$$g(z)=(z-z_0)^2[\beta_2+\beta_3(z-z_0)+\cdots]$$
$$=(z-z_0)^2\varphi(z)$$

其中 $\varphi(z)=\beta_2+\beta_3(z-z_0)+\cdots$ 在 z_0 处解析，$\varphi(z_0)=\beta_2\neq 0$，此时

$$\frac{f(z)}{g(z)}=\frac{f(z)}{(z-z_0)^2\varphi(z)}=\frac{1}{(z-z_0)^2}\frac{f(z)}{\varphi(z)}$$

可见 z_0 是 $\frac{f(z)}{g(z)}$ 的二阶极点.

于是

$$\mathrm{Res}\left[\frac{f(z)}{g(z)},z_0\right]=\lim_{z\to z_0}\frac{\mathrm{d}}{\mathrm{d}z}(z-z_0)^2\frac{f(z)}{g(z)}$$
$$=\lim_{z\to z_0}\frac{f'(z)\varphi(z)-f(z)\varphi'(z)}{\varphi^2(z)}$$
$$=\frac{f'(z_0)\varphi(z_0)-f(z_0)\varphi'(z_0)}{\varphi^2(z_0)}$$

将 $\varphi(z_0)=\beta_2\frac{g''(z)}{2!}=b_2$，$\varphi'(z_0)=\beta_3=\frac{g'''(z_0)}{3!}=b_3$，$f(z_0)=a_0$，$f'(z_0)=a_1$ 代入式，得

$$\mathrm{Res}\left[\frac{f(z)}{g(z)},z_0\right]=\frac{a_1b_2-a_0b_3}{b_2^2}$$

八、解：由于被积函数卷积分变量 θ 的偶函数，故

$$I=\frac{1}{2}\int_{-\pi}^{\pi}\frac{\sin^2 2\theta}{1-2a\cos\theta+a^2}\mathrm{d}\theta=\frac{1}{2}\int_{-\pi}^{\pi}\frac{\frac{1}{2}(1-\cos 6\theta)}{1-2a\cos\theta+a^2}\mathrm{d}\theta$$
$$=\frac{1}{4}\left(\int_0^{2\pi}\frac{\mathrm{d}\theta}{1-2a\cos\theta+a^2}-\int_0^{2\pi}\frac{\cos 6\theta}{1-2a\cos\theta+a^2}\mathrm{d}\theta\right)$$

而

$$\int_0^{2\pi}\frac{\mathrm{d}\theta}{1-2a\cos\theta+a^2}\xlongequal{z=\mathrm{e}^{\mathrm{i}\theta}}\oint_{|z|=1}\frac{1}{1-2a\frac{z^2+1}{2z}+a^2}\frac{\mathrm{d}z}{\mathrm{i}z}$$

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$$=\frac{1}{i}\oint_{|z|=1}\frac{dz}{(z-a)(1-az)}=\frac{1}{i}2\pi i\text{Res}\left[\frac{1}{(z-a)(1-az)},a\right]$$

$$=2\pi\frac{1}{1-a^2}=\frac{2\pi}{1-a^2}$$

$$\int_0^{2\pi}\frac{\cos 6\theta}{1-2a\cos\theta+a^2}=\int_0^{2\pi}\frac{\cos 6\theta+i\sin 6\theta}{1-2a\cos\theta+a^2}d\theta$$

$$\xlongequal{z=e^{i\theta}}\oint_{|z|=1}\frac{z^6}{1-2a\frac{z^2+1}{2z}+a^2}\frac{dz}{iz}$$

$$=\frac{1}{i}\oint_{|z|=1}\frac{z^6}{(z-a)(1-az)}dz$$

$$=\frac{1}{i}2\pi i\text{Res}\left[\frac{z^6}{(z-a)(1-az)},a\right]=\frac{2\pi a^6}{1-a^2}$$

故

$$I=\int_0^{\pi}\frac{\sin^2 3\theta}{1-2a\cos\theta+a^2}d\theta=\frac{1}{4}\frac{2\pi}{1-a^2}(1-a^6)$$

$$=\frac{\pi}{2}(1+a^2+a^4)$$

九、解：$F(\omega)=\mathscr{F}[f(t)]=\int_{-\infty}^{\infty}e^{-\beta|t|}e^{-i\omega t}$

$$=\int_{-\infty}^{0}e^{(\beta-i\omega)t}dt+\int_0^{\infty}e^{-(\beta+i\omega)t}dt$$

$$=\frac{e^{(\beta-i\omega)t}}{\beta-i\omega}\bigg|_{-\infty}^{0}+\frac{e^{-(\beta+i\omega)t}}{\beta+i\omega}\bigg|_0^{\infty}=\frac{1}{\beta-i\omega}-\frac{1}{\beta+i\omega}=\frac{2\beta}{\beta^2+\omega^2}$$

而

$$f(t)=e^{-\beta|t|}=\mathscr{F}^{-1}[F(\omega)]=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{2\beta}{\beta^2+\omega^2}e^{i\omega t}d\omega$$

$$=\frac{\beta}{\pi}\int_{-\infty}^{\infty}\frac{1}{\beta^2+\omega^2}(\cos\omega t+i\sin\omega t)d\omega$$

$$=\frac{\beta}{\pi}\int_{-\infty}^{\infty}\frac{\cos\omega t}{\beta^2+\omega^2}d\omega=\frac{2\beta}{\pi}\int_0^{\infty}\frac{\cos\omega t}{\beta^2+\omega^2}d\omega$$

即

$$\int_0^{\infty}\frac{\cos\omega t}{\beta^2+\omega^2}d\omega=\frac{\pi}{2\beta}e^{-\beta|t|}$$

十、解：1. $\mathscr{L}\left[\frac{\sin 2t}{t}\right]=\int_s^{\infty}\mathscr{L}[\sin 2t]ds=\int_s^{\infty}\frac{2}{2^2+s^2}ds$

$$=\arctan\frac{s}{2}\bigg|_s^{\infty}=\frac{\pi}{2}-\arctan\left(\frac{s}{2}\right)$$

$$\mathscr{L}\left[\int_0^t\frac{\sin 2\tau}{\tau}d\tau\right]=\frac{1}{s}\mathscr{L}\left[\frac{\sin 2t}{t}\right]=\frac{1}{s}\left[\frac{\pi}{2}-\arctan\left(\frac{s}{2}\right)\right]$$

$$F(s)=\mathscr{L}[f(t)]=\mathscr{L}\left[e^{-5t}\int_0^t\frac{\sin 2\tau}{\tau}d\tau\right]=\frac{1}{s+5}\left[\frac{\pi}{2}-\arctan\left(\frac{s+5}{2}\right)\right]$$

$$=\frac{1}{s+5}\text{arccot}\left(\frac{s+5}{2}\right)$$

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$$2.\ f(t)=\mathscr{L}^{-1}\left[\frac{2s^2+3s+2}{(s+1)(s+3)^3}\right]$$
$$=\mathscr{L}^{-1}\left[\frac{1}{4}\frac{1}{s+1}-\frac{1}{4}\frac{1}{s+3}+\frac{3}{2}\frac{1}{(s+3)^2}-6\frac{1}{(s+3)^3}\right]$$
$$=\frac{1}{4}\mathscr{L}^{-1}\left[\frac{1}{s+1}\right]-\frac{1}{4}\mathscr{L}^{-1}\left[\frac{1}{s+3}\right]$$
$$+\frac{3}{2}\mathscr{L}^{-1}\left[\frac{1}{(s+3)^2}\right]-6\mathscr{L}^{-1}\left[\frac{1}{(s+3)^3}\right]$$
$$=\frac{1}{4}e^{-t}-\frac{1}{4}e^{-3t}+\frac{3}{2}te^{-3t}-3t^2e^{-3t}$$
$$=\frac{1}{4}e^{-t}-e^{-3t}\left(3t^2-\frac{3}{2}t+\frac{1}{4}\right)$$

十一、解：对方程两边同时取 Laplace 变换并代入初值条件，得

$$S^3Y(s)+3S^2Y(s)+3SY(s)+Y(s)=\frac{1}{s}$$
$$(s+1)^3Y(s)=\frac{1}{s}$$
$$Y(s)=\frac{1}{s(s+1)^3}$$

于是该初值问题的解为

$$y(t)=\mathscr{L}^{-1}[Y(s)]=\mathscr{L}^{-1}\left[\frac{1}{s(s+1)^3}\right]$$
$$=\lim_{s\to 0}s\frac{e^{st}}{s(s+1)^3}+\frac{1}{2!}\lim_{s\to -1}\frac{d^2}{ds^2}\left[(s+1)^3\frac{e^{st}}{s(s+1)^3}\right]$$
$$=1+\frac{1}{2}\lim_{s\to -1}\frac{d}{ds}\frac{(ts-1)e^{st}}{s^2}$$
$$=1+\frac{1}{2}\lim_{s\to -1}\frac{[t+t(ts-1)]e^{st}-2s(ts-1)e^{st}}{s^4}$$
$$=1+\frac{1}{2}e^{-t}[t+t(-t-1)+2(-t-1)]$$
$$=1-\frac{e^{-t}}{2}(t^2+2t+2)$$

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复变函数与积分变换试题(综合二)解答

一、解:1. $\cos(1+2i)=\frac{1}{2}[e^{i(1+2i)}+e^{-i(1+2i)}]$

$$=\frac{1}{2}(e^{-2}e^{i}+e^{2}e^{-i})$$

$$=\frac{1}{2}[e^{-2}(\cos 1+i\sin 1)+e^{2}(\cos 1-i\sin t)]$$

$$=\frac{e^{2}+e^{-2}}{2}\cos-i\frac{e^{2}-e^{-2}}{2}\sin 1=\text{ch }2\cos 1-i\text{sh }2\sin 1$$

2. $1^{\sqrt{3}}=e^{\sqrt{3}\text{Ln }1}e^{\sqrt{3}\ln 1}e^{\sqrt{3}i2k\pi}=\cos(2\sqrt{3}k\pi)+i\sin(2\sqrt{3}k\pi),k\in\mathbb{Z}$

3. $\text{Ln}(3+4i)=\ln|3+4i|+i[\arg(3+4i)+2k\pi]$

$$=\ln 5+i(\arctan\frac{4}{3}+2k\pi),k\in\mathbb{Z}$$

4. $\text{Arctan }2i=-\frac{i}{2}\text{Ln}\frac{1+i(2i)}{1-i(2i)}=-\frac{i}{2}\text{Ln}\left(-\frac{1}{3}\right)$

$$=-\frac{i}{2}\left\{\ln\left(\frac{1}{3}\right)+i\left[\arg\left(-\frac{1}{3}\right)+2k\pi\right]\right\}$$

$$=\frac{1}{2}(2k+1)\pi+i\frac{\ln 3}{2},k\in\mathbb{Z}$$

二、解:$\frac{\partial u}{\partial x}=\cos y\cdot e^{x}+[(x+1)\cos y-y\sin y]e^{x}$

$$=[(x+2)\cos y-y\sin y]e^{x}$$

$$\frac{\partial^2 u}{\partial x^2}=\cos y e^{x}+[(x+2)\cos y-y\sin y]e^{x}$$

$$=[(x+3)\cos y-y\sin y]e^{x}$$

$$\frac{\partial u}{\partial y}=[-(x+1)\sin y-\sin y-y\cos y]e^{x}$$

$$=-[(x+2)\sin y+y\cos y]e^{x}$$

$$\frac{\partial^2 u}{\partial y^2}=[-(x+2)\cos y-\cos y+y\sin y]e^{x}$$

$$=-[(x+3)\cos y-y\sin y]e^{x}$$

可见

$$\Delta u=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$$

即 $u(x,y)=[(x+1)\cos y-y\sin y]e^{x}$ 是 $\mathbb{C}$ 上的调和函数,其共轭调和函数

$$u(x,y)=\int_{(0,0)}^{(x,y)}v_x\,dx+v_y\,dy+C$$

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$$=\int_{(0,0)}^{(x,y)} -u_y dx + u_x dy + C$$

$$=\int_{(0,0)}^{(x,y)} [(x+2)\sin y + y\cos y]e^x dx$$

$$+[(x+2)\cos y - y\sin y]e^x dy + C$$

$$=\int_0^y [(x+2)\cos y - y\sin y]e^x dy + C$$

$$=\left[(x+2)\sin y - \int_0^y y\sin y dy\right]e^x + C$$

$$=[(x+2)\sin y + y\cos y - \sin y]e^x + C$$

$$=[(x+1)\sin y + y\cos y]e^x + C$$

函数 $f(z)=u(x,y)+iv(x,y)=[(x+1)\cos y - y\sin y]e^x + i\{[(x+1)\sin y + y\cos y]e^x + C\} = (z+1)e^z + iC$ 在 $\mathbb{C}$ 上处处解析.

三、解：$|f(z)|^2 = u^2 + v^2$，故

$$\frac{\partial \ln(1+|f(z)|^2)}{\partial x} = \frac{2uu'_x + 2vv'_x}{1+|f(z)|^2}$$

$$\frac{\partial^2 \ln(1+|f(z)|^2)}{\partial x^2}$$

$$=\frac{[2(u'_x)^2 + 2(v'_x)^2 + 2(uu''_{x^2} + vv''_{x^2})](1+|f(z)|^2) - (2uu'_x + 2vv'_x)^2}{(1+|f(z)|^2)^2}$$

同理

$$\frac{\partial^2 \ln(1+|f(z)|^2)}{\partial y^2}$$

$$=\frac{[2(u'_y)^2 + 2(v'_y)^2 + 2(uu''_{y^2} + vv''_{y^2})](1+|f(z)|^2) - (2uu'_y + 2vv'_y)^2}{(1+|f(z)|^2)^2}$$

又 $\Delta u = \Delta v = 0$ 和 C-R 条件，有

$$\frac{\partial^2 \ln(1+|f(z)|^2)}{\partial x^2} + \frac{\partial^2 \ln(1+|f(z)|^2)}{\partial y^2}$$

$$=\frac{1}{(1+|f(z)|^2)^2}\{2[(u'_x)^2 + (v'_x)^2 + (u'_y)^2 + (v'_y)^2]\cdot$$

$$(1+|f(z)|^2) - [(uu'_x + vv'_x)^2 + uu'_y + vv'_y)^2]\}$$

$$=\frac{1}{(1+|f(z)|^2)^2}[4|f'(z)|^2(1+|f(z)|^2)$$

$$-4|f'(z)|^2|f(z)|^2]$$

$$=\frac{4|f'(z)|^2}{(1+|f(z)|^2)^2}$$

四、解：1. $\int_C \frac{\bar{z}}{|z|}dz = \int_\pi^0 \frac{\overline{3e^{i\theta}}}{3} 3ie^{i\theta} d\theta = 3i\int_\pi^0 1 d\theta = -3\pi i$

2. $\oint_{|z|=2} \frac{e^z}{z(1-z)^3}dz = 2\pi i\left\{\text{Res}\left[\frac{e^z}{z(1-z)^3}, 0\right] + \text{Res}\left[\frac{e^z}{z(1-z)^3}, 1\right]\right\}$

$$=2\pi i\left[1 + \frac{1}{2!}\lim_{z\to 1}\frac{d^2}{dz^2}(z-1)^3 \frac{e^z}{z(1-z)^3}\right]$$

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$$=2\pi i\left[1+\frac{1}{2}\lim_{z\to 1}\frac{d}{dz}\frac{(z-1)e^z}{z^2}\right]$$

$$=2\pi i\left[1+\frac{1}{2}\lim_{z\to 1}\frac{z^3e^z-2z(z-1)e^z}{z^4}\right]$$

$$=2\pi i\left(1+\frac{e}{2}\right)$$

3. $$\oint_{|z|=1}\frac{z\sin z}{(1-e^z)^3}dz=2\pi i\text{Res}\left[\frac{z\sin z}{(1-e^z)^3},0\right]=2\pi i\lim_{z\to 0}z\frac{z\sin z}{(1-e^z)^3}$$

$$=2\pi i\lim_{x\to 0}\left[-\frac{x^2}{(e^x-1)^2}\frac{\sin x}{e^x-1}\right]=-2\pi i$$

4. $$\oint_{|z|=2}\frac{dz}{(z-3)(z^5-1)}=-2\pi i\left\{\text{Res}\left[\frac{1}{(z-3)(z^5-1)},3\right]+\text{Res}\left[\frac{1}{(z-3)(z^5-1)},\infty\right]\right\}$$

$$=-2\pi i\left\{\frac{1}{242}-\text{Res}\left[\frac{1}{\left(\frac{1}{z}-3\right)\left(\frac{1}{z^5}-1\right)}\frac{1}{z^2},0\right]\right\}$$

$$=-2\pi i\left\{\frac{1}{242}-\text{Res}\left[\frac{z^4}{(1-3z)(1-z^5)},0\right]\right\}$$

$$=-\frac{\pi i}{121}$$

5. $$\int_0^\pi\frac{dx}{3+2\cos x\sin x}=\int_0^\pi\frac{dx}{3+\sin 2x}\xlongequal{\theta=2x}\frac{1}{2}\int_0^{2\pi}\frac{d\theta}{3+\sin\theta}$$

$$\xlongequal{z=e^{i\theta}}\frac{1}{2}\oint_{|z|=1}\frac{1}{3+\frac{z^2-1}{2iz}}\frac{dz}{iz}=\oint_{|z|}\frac{dz}{z^2+6iz-1}$$

$$=2\pi i\text{Res}\left[\frac{1}{z^2+6iz-1},-(3-2\sqrt{2})i\right]$$

$$=2\pi i\cdot 6i=-12\pi$$

6. $$\int_0^\infty\frac{\cos x}{x^2+9}dx=\frac{1}{2}\int_{-\infty}^\infty\frac{\cos x}{x^2+9}dx=\frac{1}{2}\text{Re}\int_{-\infty}^\infty\frac{e^{ix}}{x^2+9}dx$$

$$=\frac{1}{2}\text{Re}\left\{2\pi i\text{Res}\left[\frac{e^{iz}}{z^2+9},3i\right]\right\}$$

$$=\frac{1}{2}\text{Re}2\pi i\frac{e^{-3}}{6i}=\frac{\pi}{6e^3}$$

五、解：由于 $\frac{f(z)}{z^k}=\cdots+\frac{a_{-1}}{z^{k+1}}+\frac{a_0}{z^k}+\cdots+\frac{a_{k-1}}{z}+a_k+a_{k+1}z+\cdots,0<|z|<\infty$.

故 $\text{Res}\left[\frac{f(z)}{z^k},0\right]=a_{k-1}$.

六、解：1. $$\frac{z^2-2z+5}{(z-2)(z^2+1)}=\frac{A}{z-2}+\frac{Bz+C}{z^2+1}$$

$$(z^2+1)A+(Bz+C)(z-2)=z^2-2z+5$$

$$\begin{cases}A+B=2\\-2B+C=-2\\A-2C=5\end{cases}$$

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解得

$$\begin{cases} A=\dfrac{9}{5} \\ B=\dfrac{1}{5} \\ C=-\dfrac{8}{5} \end{cases}$$

故

$$\begin{aligned}\frac{z^2-2z+5}{(z-2)(z^2+1)} &= \frac{9}{5}\frac{1}{z-2}+\frac{1}{5}\frac{z}{z^2+1}-\frac{8}{5}\frac{1}{z^2+1} \\ &= \frac{9}{10}\frac{1}{1-\frac{z}{2}}+\frac{1}{5z}\frac{1}{1+\frac{1}{z^2}}-\frac{8}{5z^2}\frac{1}{1+\frac{1}{z^2}} \\ &= \frac{9}{10}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n+\frac{1}{5}\frac{1}{z}\sum_{n=0}^{\infty}(-1)^n\frac{1}{z^{2n}} \\ &\quad -\frac{8}{5}\frac{1}{z^2}\sum_{n=0}^{\infty}(-1)^n\frac{1}{z^{2n}} \\ &= \frac{1}{5}\sum_{n=0}^{\infty}(-1)^n\frac{1}{z^{2n+1}} \\ &\quad -\frac{8}{5}\sum_{n=0}^{\infty}\frac{(-1)^n}{z^{2(n+1)}}+\frac{9}{10}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n\end{aligned}$$

2. $$\begin{aligned}\cos\frac{z}{z-1} &= \cos\left(1+\frac{1}{z-1}\right)=\cos 1\cos\frac{1}{z-1}-\sin 1\sin\frac{1}{z-1} \\ &= \cos 1\left[1-\frac{1}{2!}\frac{1}{(z-1)^2}+\frac{1}{4!}\frac{1}{(z-1)^4}-\cdots\right] \\ &\quad -\sin 1\left[\frac{1}{z-1}-\frac{1}{3!}\frac{1}{(z-1)^3}+\frac{1}{5!}\frac{1}{(z-1)^5}-\cdots\right]\end{aligned}$$

$$0<|z-1|<\infty$$

七、解：$$\begin{aligned}f(z) &= \frac{1}{(z-1)(z-3)}=\frac{1}{2}\left(\frac{1}{z-3}-\frac{1}{z-1}\right) \\ &= -\frac{1}{6}\frac{1}{1-\frac{z}{3}}+\frac{1}{2}\frac{1}{1-z}=-\frac{1}{6}\sum_{n=0}^{\infty}\left(\frac{z}{3}\right)^n+\frac{1}{2}\sum_{n=0}^{\infty}z^n \\ &= \sum_{n=0}^{\infty}\left(\frac{1}{2}-\frac{1}{2\cdot 3^{n+1}}\right)z^n=\sum_{n=0}^{\infty}\frac{1}{2}\left(1-\frac{1}{3^{n+1}}\right)z^n\end{aligned}$$

而

$$\begin{aligned}\oint_{|z|=2}\frac{f(z)}{z^{n+1}}\mathrm{d}z &= \frac{2\pi i}{n!}f^{(n)}(0)=2\pi i\frac{f^{(n)}(0)}{n!} \\ &= \frac{1}{2}\left(1-\frac{1}{3^{n+1}}\right)2\pi i \\ &= \pi i\left(1-\frac{1}{3^{n+1}}\right)\end{aligned}$$

八、解：$$\frac{1}{(s^2+4s+13)^2}=\frac{1}{9}\frac{3}{(s+2)^2+3^2}\frac{3}{(s+2)^2+3^2}$$

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根据 Laplace 变换的位移性质,知

$$\mathscr{L}^{-1}\left[\frac{3}{(s+2)^2+3^2}\right]=e^{-2t}\sin 3t$$

所以由卷积定理,得

$$\begin{aligned}
\mathscr{L}^{-1}\left[\frac{1}{(s^2+4s+13)^2}\right]&=\frac{1}{9}\mathscr{L}^{-1}\ \frac{1}{9}\left[\frac{3}{(s+2)^2+3^2}\cdot\frac{3}{(s+2)^2+3^2}\right]\\
&=\frac{1}{9}e^{-2t}\sin 3t*e^{-2t}\sin 3t\\
&=\frac{1}{9}\int_0^t e^{-2\tau}\sin 3\tau e^{-2(t-\tau)}\sin 3(t-\tau)d\tau\\
&=\frac{e^{-2t}}{9}\int_0^t \sin 3\tau\sin 3(t-\tau)d\tau\\
&=\frac{e^{-2t}}{9}\int_0^t \frac{1}{2}[\cos(6\tau-3t)-\cos 3t]d\tau\\
&=\frac{e^{-2t}}{18}\left[\frac{\sin(6\tau-3t)}{6}-\tau\cos 3t\right]\Bigg|_0^t\\
&=\frac{e^{-2t}}{18}\left[\frac{\sin 3t}{6}-t\cos 3t+\frac{\sin 3t}{6}\right]\\
&=\frac{e^{-2t}}{54}(\sin 3t-3t\cos 3t)
\end{aligned}$$

九、解:解 $X(\omega)=\mathscr{F}[x(t)]$,对方程两边同时取 Fourier 变换,得

$$i\omega X(\omega)-\frac{4}{i\omega}X(\omega)=\frac{2}{1+\omega^2}$$

$$\begin{aligned}
X(\omega)&=\frac{2}{1+\omega^2}\ \frac{1}{i\omega-\frac{4}{i\omega}}\\
&=\frac{-2i\omega}{(1+\omega^2)(4+\omega^2)}
\end{aligned}$$

于是

$$\begin{aligned}
x(t)=\mathscr{F}^{-1}[X(\omega)]&=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{-2i\omega}{(1+\omega^2)(4+\omega^2)}e^{i\omega t}d\omega\\
&=-\frac{i}{\pi}\int_{-\infty}^{\infty}\frac{\omega e^{i\omega t}}{(1+\omega^2)(4+\omega^2)}d\omega
\end{aligned}$$

当 $t>0$ 时,有

$$\begin{aligned}
x(t)&=-\frac{i}{\pi}2\pi i\left\{\mathrm{Res}\left[\frac{\omega e^{i\omega t}}{(1+\omega^2)(4+\omega^2)},i\right]+\right.\\
&\qquad\left.\mathrm{Res}\left[\frac{\omega e^{i\omega t}}{(1+\omega^2)(4+\omega^2)},2i\right]\right\}\\
&=2\left(\frac{ie^{-t}}{2i\cdot 3}+\frac{2ie^{-2t}}{-3\cdot 4i}\right)=\frac{1}{3}(e^{-t}-e^{-2t})
\end{aligned}$$

当 $t=0$ 时,有

$$x(0)=-\frac{i}{\pi}\int_{-\infty}^{\infty}\frac{\omega}{(1+\omega^2)(4+\omega^2)}d\omega=0$$

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当 $t<0$ 时，有

$$x(t)=-\frac{\mathrm{i}}{\pi}\int_{-\infty}^{\infty}\frac{\omega \mathrm{e}^{\mathrm{i}\omega t}}{(1+\omega^2)(4+\omega^2)}\mathrm{d}\omega \xlongequal{\omega=-u} \frac{\mathrm{i}}{\pi}\int_{-\infty}^{\infty}\frac{u\mathrm{e}^{\mathrm{i}(-t)u}}{(1+u^2)(4+u^2)}\mathrm{d}u$$

$$=\frac{1}{3}(\mathrm{e}^{2t}-\mathrm{e}^{t})$$

故原方程的解为

$$x(t)=\begin{cases}\frac{1}{3}(\mathrm{e}^{-t}-\mathrm{e}^{-2t}),t>0\\ 0,t=0\\ \frac{1}{3}(\mathrm{e}^{2t}-\mathrm{e}^{t}),t<0\end{cases}$$

十、解：1. $\cos^3 t=\cos t\cdot\cos^2 t=\frac{1}{2}\cos t(1+\cos 2t)$

$$=\frac{1}{2}\cos t+\frac{1}{2}\cos t\cos 2t=\frac{1}{2}\cos t+\frac{1}{2}\cdot\frac{1}{2}(\cos t+\cos 3t)$$

$$=\frac{3}{4}\cos t+\frac{1}{4}\cos 3t$$

$$F(s)=\mathscr{L}[\cos^3 t]=\frac{3}{4}\mathscr{L}[\cos t]+\frac{1}{4}\mathscr{L}[\cos 3t]$$

$$=\frac{3}{4}\frac{s}{s^2+1}+\frac{1}{4}\frac{s}{s^2+3^2}$$

2. $F(s)=\mathscr{L}\left[t^2\int_0^t \mathrm{e}^{-4\tau}\sin 2\tau\mathrm{d}\tau\right]=-\frac{\mathrm{d}}{\mathrm{d}s}\left\{\mathscr{L}\left[\int_0^t \mathrm{e}^{-4\tau}\sin 2\tau\mathrm{d}\tau\right]\right\}$

$$=-\frac{\mathrm{d}}{\mathrm{d}s}\left\{\frac{1}{s}\mathscr{L}[\mathrm{e}^{-4t}\sin 2t]\right\}=-\frac{\mathrm{d}}{\mathrm{d}s}\left[\frac{1}{s}\frac{2}{(s+4)^2+2^2}\right]$$

$$=-\frac{-2[(s+4)^2+2^2+2s(s+4)]}{s^2[(s+4)^2+2^2]^2}=\frac{2(3s^2+16s+20)}{s^2[(s+4)^2+2^2]^2}$$

3. $\frac{2s^3-s^2-1}{(s+1)^2(s^2+1)^2}=\frac{1}{2}\frac{1}{s^2+1}-\frac{1}{2}\frac{1-s^2}{(s^2+1)^2}-\frac{1}{(1+s)^2}$

$$=\frac{1}{2}\frac{1}{s^2+1}-\frac{1}{2}\left(\frac{s}{s^2+1}\right)'-\frac{1}{(1+s)^2}$$

$$\mathscr{L}^{-1}\left[\frac{2s^3-s^2-1}{(s+1)^2(s^2+1)^2}\right]=\frac{1}{2}\mathscr{L}^{-1}\left[\frac{1}{s^2+1}\right]-\frac{1}{2}\mathscr{L}^{-1}\left[\left(\frac{s}{s^2+1}\right)'\right]-\mathscr{L}^{-1}\left[\frac{1}{(1+s)^2}\right]$$

$$=\frac{1}{2}\sin t+\frac{1}{2}t\cos t-t\mathrm{e}^{-t}$$

十一、解：设 $X(s)=\mathscr{L}[x(t)]$，对方程两边同时取 Laplace 并代入初值条件，得

$$S^2X(s)-s-1+4SX(s)-4+3X(s)=\frac{1}{s+1}$$

$$X(s)=\frac{1}{s^2+4s+3}\left(\frac{1}{s+1}+s+5\right)=\frac{s^2+6s+6}{(s+1)^2(s+3)}$$

$$=\frac{7}{4}\frac{1}{s+1}+\frac{1}{2}\frac{1}{(s+1)^2}-\frac{3}{4}\frac{1}{s+3}$$

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取 Laplace 逆变换，得初值问题的解

$$\begin{aligned} x(t) &= \mathscr{L}^{-1}[X(s)] = \frac{7}{4}\mathscr{L}^{-1}\left[\frac{1}{s+1}\right] + \frac{1}{2}\mathscr{L}^{-1}\left[\frac{1}{(s+1)^2}\right] - \frac{3}{4}\mathscr{L}^{-1}\left[\frac{1}{s+3}\right] \\ &= \frac{7}{4}e^{-t} + \frac{1}{2}te^{-t} - \frac{3}{4}e^{-3t} \\ &= \frac{1}{4}(2t+7)e^{-t} - \frac{3}{4}e^{-3t} \end{aligned}$$

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复变函数与积分变换试题(综合三)解答

一、填空题

1. $\sqrt{2}\,\mathrm{e}^{-\mathrm{i}\frac{\pi}{12}}$. 2. $\frac{\partial u}{\partial x}+\mathrm{i}\frac{\partial v}{\partial x}$. 3. 1. 4. $1<|z-2|<2$. 5. 本性.

二、单项选择题

1. D 2. A 3. C 4. B 5. A

三、解:

$$u'_x+v'_x=x^2+4xy+y^2+(x-y)(2x+4y)-2 \tag{1}$$

$$u'_y+v'_y=-v'_x+u'_x=-(x^2+4xy+y^2)+(x-y)(4x+2y)-2 \tag{2}$$

(1)+(2)得

$$u'_x=\frac{1}{2}[(x-y)6(x+y)-4]=3(x^2-y^2)-2$$

$$u=\int u'_x\mathrm{d}x=x^3-3xy^2-2x+\varphi(y)$$

(1)−(2)得

$$v'_x=x^2+4xy+y^2-(x-y)^2=6xy$$

根据 C-R 条件,有

$$6xy=v'_x=-u'_y=6xy+\varphi'(y)$$

$$\varphi'(y)=0,\varphi(y)=C,\ C\in\mathbb{R}$$

$$u=x^3-3xy^2-2x+C$$

$$\begin{aligned}v&=(x-y)(x^2+4xy+y^2)-2(x+y)-u\\&=(x-y)(x^2+4xy+y^2)-2(x+y)-x^3+3xy^2+2x-C\\&=-y^3+3x^2y-2y-C\end{aligned}$$

$$\begin{aligned}f(z)=u+\mathrm{i}v&=(x^3-3xy^2)+\mathrm{i}(-y^3+3x^2y)-2(x+\mathrm{i}y)+C(1-\mathrm{i})\\&=z^3-2z+C(1-\mathrm{i})\end{aligned}$$

四、解:1. $\int_{\Gamma}(x-y+\mathrm{i}x^2)\mathrm{d}z\overset{z=(1+\mathrm{i})t}{=\!=\!=}\int_0^1[(t-t)+\mathrm{i}t^2](1+\mathrm{i})\mathrm{d}t$

$$=\frac{\mathrm{i}}{3}(1+\mathrm{i})=\frac{1}{3}(-1+\mathrm{i})$$

2. $\frac{\mathrm{e}^z}{z-1}=-\left(1+z+\frac{z^2}{2!}+\cdots+\frac{z^n}{n!}+\cdots\right)(1+z+\cdots+z^n+\cdots)$

$$=-\sum_{n=0}^{\infty}\left(1+1+\frac{1}{2!}+\cdots+\frac{1}{n!}\right)z^n$$

$$\oint_C\frac{f(z)}{z^{n+1}}\mathrm{d}z=\frac{2\pi\mathrm{i}}{n!}f^{(n)}(0)=2\pi\mathrm{i}\frac{f^{(n)}(0)}{n!}$$

$$=-2\pi\mathrm{i}\left(1+1+\cdots+\frac{1}{n!}\right)$$

3. $\int_{-\infty}^{\infty}\frac{\cos x}{x^4+4x+5}\mathrm{d}x=\mathrm{Re}\int_{-\infty}^{\infty}\frac{\mathrm{e}^{\mathrm{i}x}}{x^2+4x+5}\mathrm{d}x$

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$$=\mathrm{Re}\left\{2\pi\mathrm{i}\,\mathrm{Res}\left[\frac{\mathrm{e}^{\mathrm{i}z}}{z^2+4z+5},-2+\mathrm{i}\right]\right\}$$

$$=\mathrm{Re}\left[2\pi\mathrm{i}\,\frac{\mathrm{e}^{\mathrm{i}(-2+\mathrm{i})}}{2\mathrm{i}}\right]$$

$$=\mathrm{Re}(\pi\mathrm{e}^{-1}\mathrm{e}^{-2\mathrm{i}})=\pi\mathrm{e}^{-1}\cos 2$$

4. $$f(z)=\frac{z-1}{z(z+2)}=-\frac{1}{2}\frac{1}{z}+\frac{3}{2}\frac{1}{z+2}$$

$$=-\frac{1}{2}\frac{1}{z+2\mathrm{i}}\frac{1}{1-\dfrac{2\mathrm{i}}{z+2\mathrm{i}}}+\frac{3}{2}\frac{1}{2-2\mathrm{i}}\frac{1}{1+\left(\dfrac{z+2\mathrm{i}}{2-2\mathrm{i}}\right)}$$

$$=-\frac{1}{2}\frac{1}{z+2\mathrm{i}}\sum_{n=0}^{\infty}\left(\frac{2\mathrm{i}}{z+2\mathrm{i}}\right)^n+\frac{3}{4}\frac{1}{1-\mathrm{i}}\sum_{n=0}^{\infty}(-1)^n\frac{(z+2\mathrm{i})^n}{2^n(1-\mathrm{i})^n}$$

$$=-\frac{1}{2}\sum_{n=1}^{\infty}(2\mathrm{i})^{n-1}\frac{1}{(z+2\mathrm{i})^n}$$

$$+\frac{3}{8}(1+\mathrm{i})\sum_{n=0}^{\infty}\frac{(-1)^n}{2^n(1-\mathrm{i})^n}(z+2\mathrm{i})^n,\ 2<|z+2\mathrm{i}|<2\sqrt{2}$$

5. $$\oint_{|z|=\frac{3}{2}}\frac{\mathrm{e}^z}{(z^2+1)(z^2+4)}\mathrm{d}z$$

$$=2\pi\mathrm{i}\left\{\mathrm{Res}\left[\frac{\mathrm{e}^z}{(z^2+1)(z^2+4)},\mathrm{i}\right]+\mathrm{Res}\left[\frac{\mathrm{e}^z}{(z^2+1)(z^2+4)},-\mathrm{i}\right]\right\}$$

$$=2\pi\mathrm{i}\left(\frac{\mathrm{e}^{\mathrm{i}}}{2\mathrm{i}\cdot 3}+\frac{\mathrm{e}^{-\mathrm{i}}}{-2\mathrm{i}\cdot 3}\right)=\frac{2}{3}\pi\mathrm{i}\,\frac{\mathrm{e}^{\mathrm{i}}-\mathrm{e}^{-\mathrm{i}}}{2\mathrm{i}}=\frac{2}{3}\pi\mathrm{i}\sin 1$$

五、解:当 $z_0\neq 0$ 时,由于 $f(z)$ 在点 z_0 处解析,根据 Taylor 展开定理,可得

$$f(z)=f(z_0)+f'(z_0)(z-z_0)+\frac{f''(z_0)}{2!}(z-z_0)^2+\cdots$$

$$=f'(z_0)(z-z_0)+\frac{f''(z_0)}{2!}(z-z_0)^2+\cdots$$

$$=(z-z_0)\left[f'(z_0)+\frac{f''(z_0)}{2!}(z-z_0)+\cdots\right]$$

$$=(z-z_0)\varphi(z)$$

这里 $\varphi(z)=f'(z_0)+\dfrac{f''(z_0)}{2!}(z-z_0)+\cdots$ 在 z_0 处解析且 $\varphi(z_0)=f'(z_0)\neq 0$. 于是

$$\frac{zf'(z)}{f(z)}=\frac{1}{z-z_0}\frac{zf'(z)}{\varphi(z)}=\frac{1}{z-z_0}g(z)$$

这里 $g(z)=\dfrac{zf'(z)}{\varphi(z)}$ 在 z_0 解析且 $g(z_0)\neq 0$,知 z_0 是 $\dfrac{zf'(z)}{f(z)}$ 的一阶极点,于是

$$\frac{1}{2\pi\mathrm{i}}\oint_{|z-z_0|=\delta}\frac{zf'(z)}{f(z)}\mathrm{d}z=\mathrm{Res}\left[\frac{zf'(z)}{f(z)},z_0\right]=\lim_{z\to z_0}\frac{zf'(z)}{\varphi(z)}$$

$$=\frac{z_0f'(z)}{\varphi(z_0)}=\frac{z_0f'(z_0)}{f'(z_0)}=z_0$$

$z_0=0$ 的情形可类似讨论.

六、解:由卷积定理,得到

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$$f*g(t)=\int_{-\infty}^{\infty}f(\tau)g(t-\tau)\mathrm{d}\tau=\int_{-\infty}^{\infty}g(\tau)f(t-\tau)\mathrm{d}\tau$$

显见当 $t<-1$ 时

$$f*g(t)=0$$

当 $-1\leqslant t\leqslant 1$ 时

$$f*g(t)=\int_{-1}^{t}1(t-\tau)^2\mathrm{d}\tau=-\frac{1}{3}(t-\tau)^3\Big|_{-1}^{t}=\frac{1}{3}(t+1)^3$$

当 $t>1$ 时

$$f*g(t)=\int_{-1}^{t}1(t-\tau)^2\mathrm{d}\tau=\int_{-1}^{1}(t-\tau)^2\mathrm{d}\tau=\frac{1}{3}[(t+1)^3-(t-1)^3]=\frac{2}{3}(3t^2+1)$$

即

$$f*g(t)=\begin{cases}0,t<-1\\ \dfrac{(t+1)^3}{3},-1\leqslant t\leqslant 1\\ \dfrac{2}{3}(3t^2+1),t>1\end{cases}$$

七、解：设 $Y(s)=\mathscr{L}[y(t)]$，对方程两边同时取 Laplace 变换并代入初始条件，得到

$$S^2Y(s)-S-SY(s)+1-6Y(s)=\frac{2}{s}$$

$$(s^2-s-6)Y(s)=\frac{2}{s}+s-1$$

$$Y(s)=\frac{1}{(s-3)(s+2)}\ \frac{s^2-s+2}{s}$$

$$y(t)=\mathscr{L}^{-1}[Y(s)]=\mathscr{L}^{-1}\left[\frac{s^2-s+2}{s(s+2)(s-3)}\right]$$

$$=\left\{\mathrm{Res}\left[\frac{(s^2-s+2)\mathrm{e}^{st}}{s(s+2)(s-3)},0\right]+\mathrm{Res}\left[\frac{(s^2-s+2)\mathrm{e}^{st}}{s(s+2)(s-3)},-2\right]\right.$$

$$\left.+\mathrm{Res}\left[\frac{(s^2-s+2)\mathrm{e}^{st}}{s(s+2)(s-3)},3\right]\right\}$$

$$=\left(\frac{2}{-6}+\frac{8}{(-2)(-5)}\mathrm{e}^{-2t}+\frac{8}{3\cdot 5}\mathrm{e}^{3t}\right)$$

$$=-\frac{1}{3}+\frac{4}{5}\mathrm{e}^{-2t}+\frac{8}{15}\mathrm{e}^{3t}$$

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