Exercise 1: Prove that

$$\int_0^\infty \sin(x^2)dx = \int_0^\infty \cos(x^2)dx = \frac{\sqrt{2\pi}}{4}.$$

These are the **Fresnel integrals**. Here, $\int_0^\infty$ is interpreted as $\lim_{R\to\infty}\int_0^R$.

Solution. Let $f(z) = e^{iz^2}$. We integrate $f(z)$ around a circular sector of radius R running from $\theta = 0$ to $\frac{\pi}{4}$. Along the x axis the integral is $\int_0^R e^{ix^2}dx$. Along the curved part we have $z = Re^{i\theta}$ and the integral is

$$\int_0^{\pi/4} e^{iR^2e^{2i\theta}} iRe^{i\theta}d\theta = iR\int_0^{\pi/4} e^{-R^2\sin(2\theta)}e^{i(\theta+iR^2\cos(2\theta))}d\theta.$$

Finally, along the segment at angle $\frac{\pi}{4}$ we have $z = re^{i\pi/4}$ and the integral is $\int_R^0 e^{-r^2}e^{i\pi/4}dr$. The total integral is zero since f is analytic everywhere. As $R \to \infty$, the integral over the third piece approaches

$$-e^{i\pi/4}\int_0^\infty e^{-x^2}dx = -e^{i\pi/4}\frac{\sqrt{\pi}}{2} = -\frac{\sqrt{2\pi}}{4} - \frac{\sqrt{2\pi}}{4}i.$$

To estimate the integral over the curved piece, we used the fact that $\sin(2\phi) \geq \frac{4\phi}{\pi}$ for $0 \leq \phi \leq \frac{\pi}{4}$; this follows from the concavity of $\sin(2\phi)$. Using this,

$$\begin{aligned}\left|iR\int_0^{\pi/4} e^{-R^2\sin(2\theta)}e^{i(\theta+iR^2\cos(2\theta))}d\theta\right| &\leq R\int_0^{\pi/4}\left|e^{-R^2\sin(2\theta)}e^{i(\theta+iR^2\cos(2\theta))}\right|d\theta\\ &= R\int_0^{\pi/4} e^{-R^2\sin(2\theta)}d\theta\\ &\leq R\int_0^{\pi/4} e^{-4R^2\theta/\pi}d\theta\\ &= -\frac{\pi}{4R}e^{-4R^2\theta/\pi}\Big|_0^{\pi/4}\\ &= \frac{\pi(1-e^{-R^2})}{4R}.\end{aligned}$$

As $R \to \infty$, this approaches zero and we are left with

$$\int_0^\infty e^{ix^2}dx - \frac{\sqrt{2\pi}}{4} - \frac{\sqrt{2\pi}}{4}i = 0.$$

Taking real and imaginary parts, we have

$$\int_0^\infty \cos(x^2)dx = \int_0^\infty \sin(x^2)dx = \frac{\sqrt{2\pi}}{4}.$$

$\square$

Exercise 2: Show that $\int_0^\infty \frac{\sin x}{x}dx = \frac{\pi}{2}$.

Solution. We integrate $f(z) = \frac{e^{iz}}{z}$ around an indented semicircular contour bounded by circles of radius ϵ and R in the upper half plane. The integrals along the two portions of the real axis add up to

$$\int_{-R}^{\epsilon} \frac{\cos(x) + i\sin(x)}{x} dx + \int_{\epsilon}^{R} \frac{\cos(x) + i\sin(x)}{x} dx = 2i \int_{\epsilon}^{R} \frac{\sin(x)}{x} dx$$

because cosine is even and sine is odd. The integral around the arc of radius R tends to zero as $R \to \infty$, by the Jordan lemma; since this lemma isn't mentioned in the book, here's a proof for this specific case: On this arc, $z = Re^{i\theta}$ so the integral is

$$\int_0^{\pi} \frac{e^{iRe^{i\theta}}}{Re^{i\theta}} iRe^{i\theta} d\theta = i \int_0^{\pi} e^{-R\sin(\theta)} e^{iR\cos(\theta)} d\theta.$$

The absolute value of this integral is at most

$$\int_0^{\pi} e^{-R\sin(\theta)} d\theta = 2 \int_0^{\pi/2} e^{-R\sin(\theta)} d\theta$$

by symmetry. Now $\sin(\theta) \geq \frac{2\theta}{\pi}$ for $0 \leq \theta \leq \frac{\pi}{2}$ by the concavity of the sine function, so this is at most

$$2 \int_0^{\pi/2} e^{-2R\theta/pi} d\theta = -\frac{\pi e^{-2R\theta/pi}}{R} \Big|_0^{\pi/2} = \frac{\pi(1 - e^{-R})}{R}$$

which tends to 0 as $R \to \infty$.

Finally, the integral over the inner semicircle tends to $-\pi i$; this is an immediate consequence of the fractional residue theorem, but since that doesn't seem to be mentioned in this book either (gosh!), we can also see it from the fact that $\frac{e^{iz}}{z} = \frac{1}{z} + O(1)$ as $z \to 0$, and since the length of the semicircle is tending to zero, the integral over it approaches the integral of $\frac{1}{z}$ over it, which is

$$\int_{\pi}^{0} \frac{1}{\epsilon e^{i\theta}} i\epsilon e^{i\theta} d\theta = -\int_0^{\pi} 1 d\theta = -\pi i.$$

Putting the pieces together and letting $R \to \infty$ and $\epsilon \to 0$, we have

$$2i \int_0^{\infty} \frac{\sin(x)}{x} dx - \pi i = 0 \Rightarrow \int_0^{\infty} \frac{\sin(x)}{x} dx = \frac{\pi}{2}.$$

□

Exercise 3: Evaluate the integrals

$$\int_0^{\infty} e^{-ax} \cos bx \, dx \text{ and } \int_0^{\infty} e^{-ax} \sin bx \, dx, \quad a > 0$$

by integrating e^{-Ax}, $A = \sqrt{A^2 + B^2}$, over an appropriate sector with angle ω, with $\cos\omega = \frac{a}{A}$.

Solution. As indicated, we integrate $f(z) = e^{-Az}$ around a circular sector of radius R with $0 \leq \theta \leq \omega$, where $\omega = \cos^{-1}(\frac{a}{A})$ is strictly between 0 and

$\frac{\pi}{2}$. (Here we assume $b \neq 0$ since otherwise the integrals are trivially equal to $\frac{1}{a}$ and 0 respectively). The integral along the x axis is

$$\int_0^R e^{-Ax} dx \to \int_0^\infty e^{-Ax} dx = \frac{1}{A}$$

as $R \to \infty$. To estimate the integral over the curved part we use the fact that $\cos(\theta) \geq 1 - \frac{2\theta}{\pi}$ for $0 \leq \theta \leq \frac{\pi}{2}$, which follows from the concavity of the cosine in the first quadrant. Then we have

$$\begin{aligned}
\left| \int_0^\omega e^{-ARe^{i\theta}} Re^{i\theta} d\theta \right| &\leq \int_0^\omega \left| e^{-ARe^{i\theta}} Re^{i\theta} \right| d\theta \\
&= R \int_0^\omega e^{-AR\cos(\theta)} d\theta \\
&\leq R \int_0^\omega e^{-AR} e^{2AR\theta/\pi} d\theta \\
&= Re^{-AR} \frac{\pi}{2AR} e^{2AR\theta/\pi} \Big|_0^\omega \\
&= \frac{\pi}{2A} \left(e^{-AR(1-\frac{2\omega}{\pi})} - e^{-AR} \right).
\end{aligned}$$

Since $1 - \frac{2\omega}{\pi}$ is a positive constant, this tends to 0 as $R \to \infty$. Finally, on the segment with $\theta = \omega$, $z = re^{i\omega} = r\frac{a+bi}{A}$, so the integral is

$$\int_R^0 e^{-Ar(a+ib)/A} \frac{a+ib}{A} dr = \frac{a+ib}{A} \int_R^0 e^{-ar} e^{-ibr} dr.$$

Putting the pieces together and letting $R \to \infty$, we have

$$\frac{a+ib}{A} \int_\infty^0 e^{-ax} e^{-ibx} dx + \frac{1}{A} = 0 \Rightarrow \int_0^\infty e^{-ax} e^{ibx} dx = \frac{1}{a+ib} = \frac{a-ib}{a^2+b^2}.$$

Comparing the real and imaginary parts, we have

$$\int_0^\infty e^{-ax} \cos(bx) dx = \frac{a}{a^2+b^2} \text{ and } \int_0^\infty e^{-ax} \sin(bx) dx = \frac{b}{a^2+b^2}.$$

□

Exercise 4: Prove that for all $\xi \in \mathbb{C}$ we have $e^{-\pi\xi^2} = \int_{-\infty}^\infty e^{-\pi x^2} e^{2\pi i x\xi} dx$.

Solution. Let $\xi = a + bi$ with $a, b \in \mathbb{R}$. Then

$$\begin{aligned}
\int_{-\infty}^{\infty} e^{-\pi x^2} e^{-2\pi i x \xi} dx \quad \int_{-\infty}^{\infty} & e^{-\pi x^2} e^{-2\pi i x(a+bi)} dx \\
&= \int_{-\infty}^{\infty} e^{-\pi(x^2-2bx)} e^{-2\pi i a x} dx \\
&= e^{\pi b^2} \int_{-\infty}^{\infty} e^{-\pi(x-b)^2} e^{-2\pi i a x} dx \\
&= e^{\pi b^2} e^{-2\pi i a b} \int_{-\infty}^{\infty} e^{-\pi(x-b)^2} e^{-2\pi i a(x-b)} dx \\
&= e^{\pi b^2} e^{-2\pi i a b} \int_{-\infty}^{\infty} e^{-\pi u^2} e^{-2\pi i a u} du \\
&= e^{\pi b^2} e^{-2\pi i a b} e^{-\pi a^2} \\
&= e^{-\pi(a+bi)^2} \\
&= e^{-\pi \xi^2}.
\end{aligned}$$

□

Exercise 5: Suppose f is continuously *complex* differentiable on Ω, and $T \subset \Omega$ is a triangle whose interior is also contained in Ω. Apply Green's theorem to show that

$$\int_T f(z) dz = 0.$$

This provides a proof of Goursat's theorem under the additional assumption that f' is continuous.

Solution. Write $f(z)$ as $f(x, y) = u(x, y) + iv(x, y)$ where u, v are real-valued and $z = x + iy$. Then $dz = x + idy$ so

$$\begin{aligned}
\oint_T f(z) dz &= \oint_T (u(x, y) + iv(x, y))(dx + idy) \\
&= \oint_T u\,dx - v\,dy + i \oint_T v\,dx + u\,dy \\
&= \iint \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) dxdy + i \iint \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right) dxdy \\
&= 0
\end{aligned}$$

by the Cauchy-Riemann equations. (The double integrals are, of course, taken over the interior of T.) □

Exercise 6: Let Ω be an open subset of $\mathbb{C}$ and let $T \subset \Omega$ be a triangle whose interior is also contained in Ω. Suppose that f is a function holomorphic in Ω except possibly at a point w inside T. Prove that if f is bounded near w, then

$$\int_T f(z) dz = 0.$$

Solution. Let γ_ϵ be a circle of radius ϵ centered at w, where ϵ is sufficiently small that γ_ϵ lies within the interior of T. Since f is holomorphic in the region R between T and γ_ϵ,

$$\int_{\partial R} f(z)dz = \int_T f(z)dz - \int_{\gamma_\epsilon} f(z)dz = 0.$$

Thus, $\int_T f(z)dz = \int_{\gamma_\epsilon} f(z)dz$. But f is bounded near w and the length of γ_ϵ goes to 0 as $\epsilon \to 0$, so $\int_{\gamma_\epsilon} f(z)dz \to 0$ and therefore $\int_T f(z)dz = 0$. (Note: If we're not allowed to use Cauchy's theorem for a region bounded by two curves, one can use a "keyhole contour" instead; the result is the same.) □

Exercise 7: Suppose $f : \mathbb{D} \to \mathbb{C}$ is holomorphic. Show that the diameter $d = \sup_{x,w\in\mathbb{D}} |f(z) - f(w)|$ of the image of f satisfies

$$2|f'(0)| \leq d.$$

Moreover, it can be shown that equality holds precisely when f is linear, $f(z) = a_0 + a_1 z$.

Solution. By the Cauchy derivative formula,

$$f'(0) = \frac{1}{2\pi i} \oint_{C_r} \frac{f(\zeta)}{\zeta^2} d\zeta$$

where C_r is the circle of radius r centered at 0, $0 < r < 1$. Substituting $-\zeta$ for ζ and adding the two equations yields

$$2f'(0) = \frac{1}{2\pi i} \oint_{C_r} \frac{f(\zeta) - f(-\zeta)}{\zeta^2} d\zeta.$$

Then

$$|2f'(0)| \leq \frac{1}{2\pi} \oint \frac{|f(\zeta) - f(-\zeta)|}{r^2} d\zeta \leq \frac{M_r}{r} \leq \frac{d}{r},$$

where

$$M_r = \sup_{|\zeta|=r} |f(\zeta) - f(-\zeta)|.$$

Letting $r \to 1$ yields the desired result. □

Exercise 8: If f is a holomorphic function on the strip $-1 < y < 1$, $x \in \mathbb{R}$ with

$$|f(z)| \leq A(1 + |z|)^\eta, \quad \eta \text{ a fixed real number}$$

for all z in that strip, show that for each integer $n \geq 0$ there exists $A_n \geq 0$ so that

$$|f^{(n)}(x)| \leq A_n(1 + |x|)^\eta, \quad \text{for all } x \in \mathbb{R}.$$

Solution. For any x, consider a circle C centered at x of radius $\frac{1}{2}$. Applying the Cauchy estimates to this circle,

$$|f^{(n)}(x)| \leq \frac{n! \|f\|_C}{(1/2)^n}$$

where $\|f\|_C = \sup_{z\in C} |f(z)|$. Now for $z \in C$, $1 + |z| \leq 1 + |x| + |z - x| = \frac{3}{2} + |x| < 2(1 + |x|)$, so

$$|f(z)| \leq A(1 + |z|)^\eta \leq A2^\eta(1 + |x|)^\eta.$$

Hence, $\|f\|_C \leq A2^\eta(1+|x|)^\eta$, so

$$|f^{(n)}(x)| \leq n!2^n A2^\eta(1+|x|)^\eta = A_n(1+|x|)^\eta$$

with $A_n = n!2^n A2^\eta$. □

Exercise 9: Let Ω be a bounded open subset of $\mathbb{C}$, and $\phi : \Omega \to \Omega$ a holomorphic function. Prove that if there exists a point $z_0 \in \Omega$ such that

$$\phi(z_0) = z_0 \text{ and } \phi'(z_0) = 1$$

then ϕ is linear.

Solution. Let $f(z) = \phi(z+z_0) - z_0$ for $z \in \Omega - z_0$. Then $f(z) \in \Omega - z_0$ and f is linear iff ϕ is. Hence, we may assume WLOG that $z_0 = 0$. Expanding in a power series around 0, we have $\phi(z) = z + a_2 z^2 + \ldots$. Suppose a_n is the first nonzero coefficient with $n > 1$. Then $\phi(z) = z + a_n z^n + O(z^{n+1})$. By induction this implies that $\phi^k(z) = \phi \circ \cdots \circ \phi(z) = z + k a_n z^n + O(z^{n+1})$; the base case $k = 1$ has been established, and if it is true for k it follows that

$$\phi^{k+1}(z) = (z+ka_n z^n+O(z^{n+1}))+a_n\left(z+ka_n z^n+O(z^{n+1})\right)^n+O\left(\left(z+ka_n z^n+O(z^{n+1})\right)^{n+1}\right) = z+(k+1)a_n z^n$$

Now let $r > 0$ such that $z \in \Omega$ for $|z| \leq r$. By the Cauchy estimates,

$$\left|\left(\phi^k\right)^{(n)}(0)\right| \leq \frac{n!\|\phi^k\|_r}{r^n}$$

where $\|\phi^k\|_r = \sup_{|z|=r} |\phi^k(z)|$. But $\phi^k(z) \in \Omega$ which is bounded, so $\|\phi^k\|_r \leq M$ for some constant M independent of n and k. Now $(\phi^k)^{(n)} = kn!a_n$, so we have

$$kn!a_n \leq \frac{Mn!}{r^n} \Rightarrow a_n \leq \frac{M}{kr^n}$$

for all k. Letting $k \to \infty$, we have $a_n = 0$. Thus, there can be no nonzero terms of order $n > 1$ in the power series expansion of ϕ, so ϕ is linear. □

Exercise 11: Let f be a holomorphic function on the disc D_{R_0} centered at the origin and of radius R_0.

(a) Prove that whenever $0 < R < R_0$ and $|z| < R$, then

$$f(z) = \frac{1}{2\pi}\int_0^{2\pi} f(Re^{i\phi})\mathrm{Re}\left(\frac{Re^{i\phi}+z}{Re^{i\phi}-z}\right) d\phi.$$

(b) Show that

$$\mathrm{Re}\left(\frac{Re^{i\gamma}+r}{Re^{i\gamma}-r}\right) = \frac{R^2-r^2}{R^2-2Rr\cos\gamma+r^2}.$$

(Hint.)

Solution.

(a) Starting with the RHS,

$$\begin{aligned}
\frac{1}{2\pi}\int_0^{2\pi} f(Re^{i\phi})\mathrm{Re}\left(\frac{Re^{i\phi}+z}{Re^{i\phi}-z}\right)d\phi &= \frac{1}{4\pi}\int_0^{2\pi} f(Re^{i\phi})\left(\frac{Re^{i\phi}+z}{Re^{i\phi}-z}+\frac{Re^{-i\phi}+\bar{z}}{Re^{-i\phi}-\bar{z}}\right)d\phi \\
&= \frac{1}{4\pi}\int_0^{2\pi} f(Re^{i\phi})\left(2\frac{Re^{i\phi}}{Re^{i\phi}-z}-1+1-\frac{2\bar{z}}{\bar{z}-Re^{-i\phi}}\right)d\phi \\
&= \frac{1}{2\pi i}\int_0^{2\pi} f(Re^{i\phi})\frac{iRe^{i\phi}d\phi}{Re^{i\phi}-z}+\frac{1}{2\pi}\int_0^{2\pi} f(Re^{i\phi})\frac{\bar{z}}{\bar{z}-Re^{-i\phi}}d\phi \\
&= \frac{1}{2\pi i}\int_0^{2\pi} f(Re^{i\phi})\frac{iRe^{i\phi}d\phi}{Re^{i\phi}-z}+\frac{1}{2\pi}\int_0^{2\pi i} f(Re^{i\phi})\frac{iRe^{i\phi}}{Re^{i\phi}-R^2/\bar{z}}d\phi.
\end{aligned}$$

The first integral is equal to $f(z)$ by the Cauchy integral formula, and the latter is equal to zero since $\frac{f(\zeta)}{\zeta-R^2/w}$ is analytic on and inside the circle of radius R.

(b) By straightforward calculations,

$$\begin{aligned}
\frac{Re^{i\gamma}+r}{Re^{i\gamma}-r} &= \frac{R\cos(\gamma)+r+iR\sin(\gamma)}{R\cos(\gamma)-r+iR\sin(\gamma)} \\
&= \frac{(R\cos(\gamma)+r+iR\sin(\gamma))(R\cos(\gamma)-r-iR\sin(\gamma))}{(R\cos(\gamma)-r)^2+R^2\sin^2(\gamma)} \\
&= \frac{R^2\cos^2(\gamma)-r^2+R^2\sin^2(\gamma)+i(\text{stuff})}{R^2-2Rr\cos(\gamma)+r^2} \\
&= \frac{R^2-r^2}{R^2-2Rr\cos(\gamma)+r^2}+(\text{imaginary stuff}).
\end{aligned}$$

□

Exercise 12: Let u be a real-valued function defined on the unit disc $\mathbb{D}$. Suppose that u is twice continuously differentiable and harmonic, that is,

$$\Delta u(x,y)=0$$

for all $x, y \in \mathbb{D}$.

(a) Prove that there exists a holomorphic function f on the unit disc such that

$$\mathrm{Re}(f)=u.$$

Also show that the imaginary part of f is uniquely defined up to an additive (real) constant.

(b) Deduce from this result, and from Exercise 11, the Poisson integral representation formula from the Cauchy integral formula: If u is harmonic in the unit disc and continuous on its closure, then if $z=re^{i\theta}$ one has

$$u(z)=\frac{1}{2\pi}\int_0^{2\pi} P_r(\theta-\phi)u(\phi)d\phi$$

where $P_r(\gamma)$ is the Poisson kernel for the unit disc given by

$$P_r(\gamma)=\frac{1-r^2}{1-2r\cos\gamma+r^2}.$$

Solution.

(a) If we are allowed to import a little manifold theory, we can observe that the 1-form $-u_y dx + u_x dy$ is closed, because its differential is

$$(-u_{xy}dx - u_{yy}dy) \wedge dx + (u_{xx}dx + u_{xy}dy) \wedge dy = (u_{yy} + u_{xx})dx \wedge dy = 0.$$

Since the unit disc is simply connected, every closed form is exact. Hence there exists a C^2 function v such that

$$dv = -u_y dx + u_x dy.$$

But by definition, $dv = v_x dx + v_y dy$. Hence $v_x = -u_y$ and $v_y = u_x$, so the function $f = u + iv$ satisfies the Cauchy-Riemann equations and is therefore holomorphic.
However, it is not necessary to import external knowledge about manifold theory here. By the equality of mixed partials,

$$\frac{\partial}{\partial \bar{z}} \frac{\partial}{\partial z} = \frac{1}{4} \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)$$

so if we let $g = 2\frac{\partial u}{\partial z}$, then $\frac{\partial g}{\partial \bar{z}} = 2\frac{\partial}{\partial \bar{z}}\frac{\partial}{\partial z} u = 0$. Hence g is holomorphic. By Theorem 2.1, $\exists F$ with $F' = g$. Then $\frac{\partial \mathrm{Re}(F)}{\partial z} = \frac{1}{2}\frac{\partial F}{\partial z} = \frac{1}{2} g = \frac{\partial u}{\partial z}$ by Proposition 2.3 on page 12, so $\mathrm{Re}(F)$ differs from u by a constant u_0. Then if $f(z) = F(z) - u_0$, f is holomorphic and $\mathrm{Re}(f) = u$.

(b) By Exercise 11,

$$u(z) + iv(z) = \frac{1}{2\pi} \int_0^{2\pi} \left(u(e^{i\phi}) + iv(e^{i\phi})\right) \mathrm{Re}\left(\frac{e^{i\phi} + z}{e^{i\phi} - z} \right) d\phi$$

so

$$\begin{aligned} u(re^{i\theta}) &= \frac{1}{2\pi} \int_0^{2\pi} u(e^{i\phi}) \mathrm{Re}\left(\frac{e^{i\phi} + re^{i\theta}}{e^{i\phi} - re^{i\theta}} \right) d\phi \\ &= \frac{1}{2\pi} \int_0^{2\pi} u(e^{i\phi}) \mathrm{Re}\left(\frac{e^{i(\phi-\theta)} + r}{e^{i(\phi-\theta)} - r} \right) d\phi \\ &= \frac{1}{2\pi} \int_0^{2\pi} u(e^{i\phi}) \frac{1 - r^2}{1 - 2r\cos(\theta - \phi) + r^2} \\ &= \frac{1}{2\pi} \int_0^{2\pi} P_r(\theta - \phi) u(e^{i\phi}) d\phi. \end{aligned}$$

$\square$

Exercise 13: Suppose f is an analytic function defined everywhere in $\mathbb{C}$ and such that for each $z_0 \in \mathbb{C}$ at least one coefficient in the expansion

$$f(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n$$

is equal to 0. Prove that f is a polynomial.

Solution. First, we prove the following lemma:

Lemma 1. *Let $S \subset \mathbb{C}$ be a subset of the plane with no accumulation points. Then S is at most countable.*

Proof. For each $x \in S$, since x is not an accumulation point of S, $\exists r_x > 0$ such that $B_{r_x} \cap S = \{x\}$. Then $\{B_{r_x/2}(x) : x \in S\}$ is a disjoint family of open sets; since each contains a distinct rational point, it is at most countable. But this set is bijective with S, so S is at most countable. $\square$

Now suppose that f is not a polynomial. Then none of its derivatives can be identically zero, because if $f^{(n)}$ were identically zero, then $f^{(k)}$ would be zero for $k \geq n$ and f would be a polynomial of degree $\leq n-1$. Since the derivatives of f are entire functions that are not everywhere zero, the set of zeros of $f^{(n)}$ has no accumulation points, so it is at most countable by the lemma. The set of zeros of *any* derivative of f must then be countable since it is a countable union of countable sets. But by hypothesis, every point $z \in \mathbb{C}$ is a zero of some derivative of f, since if $f(z) = \sum c_n(z-z_0)^n$ and $c_k = 0$, then $\frac{d^k}{dz^k} f(z)\Big|_{z_0} = 0$. Since $\mathbb{C}$ is uncountable, this is a contradiction, so f must be a polynomial. $\square$

Exercise 14: Suppose that f is holomorphic in an open set containing the closed unit disc, except for a pole at z_0 on the unit circle. Show that if

$$\sum_{n=0}^{\infty} a_n z^n$$

denotes the power series expansion of f in the open unit disc, then

$$\lim_{n\to\infty} \frac{a_n}{a_{n+1}} = z_0.$$

Solution. By replacing z with z/z_0, we may assume WLOG that $z_0 = 1$. Now let Ω be an open set containing $\bar{\mathbb{D}}$ such that f is holomorphic on Ω except for a pole at 1. Then

$$g(z) = f(z) - \sum_{j=1}^{N} \frac{a_{-j}}{(z-1)^j}$$

is holomorphic on Ω for some N and $a_{-1}, \ldots, a_{-N}$, where N is the order of the pole at 1. Next, we note that Ω must contain some disk of radius $1+\delta$ with $\delta > 0$: the set $\{z : |z| \leq 2\} \setminus \Omega$ is compact, so its image under the map $z \mapsto |z|$ is also compact and hence attains a lower bound, which must be strictly greater than 1 since the unit circle is contained in Ω. Now since g converges on the disk $|z| < 1+\delta$, we can expand it in a power series $\sum_{n=0}^{\infty} b_n z^n$ on this disk, and we must have $b_n \to 0$. (This follows from the fact that $\limsup \frac{b_{n+1}}{b_n} < 1$ when the radius of convergence is greater than 1.) Now for $|z| < 1$, we have

$$\sum_{n=0}^{\infty} a_n z^n = \sum_{j=1}^{N} \frac{a_{-j}}{(z-1)^j} + \sum_{n=0}^{\infty} b_n z^n.$$

Using the fact that

$$\frac{1}{(z-1)^j} = \frac{(-1)^j}{(j-1)!} \frac{d^{j-1}}{dz^{j-1}} \frac{1}{z-1} = \frac{(-1)^j}{(j-1)!} \sum_{s=0}^{\infty} \frac{(s+j-1)!}{s!} z^s \quad \text{for } |z| < 1,$$

we can write

$$\sum_{n=0}^{\infty} a_n z^n = \sum_{s=0}^{\infty}\left(\sum_{j=1}^{N}\frac{(-1)^j a_{-j}}{(j-1)!}\frac{(s+j-1)!}{s!}\right)z^s + \sum_{n=0}^{\infty} b_n z^n \Rightarrow a_n = P(n) + b_n$$

where $P(n)$ is a polynomial in n of degree at most $N-1$. Here the rearrangements of the series are justified by the fact that all these series converge uniformly on compact subsets of $\mathbb{D}$. Since $b_n \to 0$, $\frac{a_n}{a_{n+1}} \to \lim \frac{P(n)}{P(n+1)} = 1$. (Every polynomial P has the property that $\frac{P(n)}{P(n+1)} \to 1$ since if the leading coefficient is $c_k n^k$, $\frac{P(n)}{P(n+1)} \approx \frac{cn^k}{c(n+1)^k} = (1 - \frac{1}{n+1})^k \to 1$.) $\square$

Exercise 15: Suppose f is a non-vanishing continuous function on $\bar{\mathbb{D}}$ that is holomorphic in $\mathbb{D}$. Prove that if

$$|f(z)| = 1 \text{ whenever } |z| = 1,$$

then f is constant.

Solution. Define

$$F(z) = \begin{cases} f(z) & |z| \le 1 \\ \frac{1}{\overline{f(\frac{1}{\bar{z}})}} & \text{else.} \end{cases}$$

Then F is obviously continuous for $|z| < 1$ and $|z| > 1$; for $|z| = 1$ we clearly have continuity from the inside, and if $w \to z$ with $|w| > 1$, then $\frac{1}{\bar{w}} \to \frac{1}{\bar{z}} = z$ and $F(w) = \frac{1}{\overline{f(\frac{1}{\bar{w}})}} \to \frac{1}{\overline{f(z)}} = f(z) = F(z)$. Hence F is continuous everywhere. It is known to be holomorphic for $|z| < 1$. For $|z| > 1$ we can compute $\frac{\partial f}{\partial \bar{z}} = 0$; alternatively, if Γ is any contour lying in the region $|z| > 1$, let Γ' be the image of Γ under the map $w = \frac{1}{z}$. Then Γ' is a contour lying in the region $|w| < 1$ and excluding the origin from its interior (since the point at infinity does not lie within Γ), so

$$\oint_{\Gamma} F(z)dz = \oint_{\Gamma'} \frac{1}{\overline{f(\bar{w})}}\frac{-dw}{w^2} = 0$$

since $\frac{1}{w^2\overline{f(\bar{w})}}$ is analytic on and inside Γ'. To show F is analytic at points on the unit circle we follow the same procedure as with the Schwarz reflection principle, by subdividing a triangle which crosses the circle into triangles which either have a vertex on the circle or an edge lying "along" the circle (i.e. a chord of the circle). In the former case we may move the vertex by ϵ to conclude that the integral around the triangle is zero. In the case where a side of the triangle is a chord of the circle, we subdivide into smaller triangles (take the midpoint of the circular arc spanned by the chord) until the chord lies within ϵ of the circle and apply the same argument. The result is that F is entire. But F is bounded since $f(\bar{\mathbb{D}})$ is a compact set which excludes 0 and hence excludes a neighborhood of zero, so $\frac{1}{f}$ is bounded on $\mathbb{D}$. Since F is a bounded entire function, it is constant, so f is constant. $\square$

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Problem 3: Morera's theorem states that if f is continuous on $\mathbb{C}$, and $\int_T f(z)dz = 0$ for all triangles T, then f is holomorphic in $\mathbb{C}$. Naturally, we may ask if the conclusion still holds if we replace triangles by other sets.

(a) Suppose that f is continuous on $\mathbb{C}$, and

$$\int_C f(z)dz = 0$$

or every circle C. Prove that f is holomorphic.

(b) More generally, let Γ be any toy contour, and $\mathcal{F}$ the collection of all translates and dilates of Γ. Show that if f is continuous on $\mathbb{C}$, and

$$\int_\gamma f(z)dz = 0 \text{ for all } \gamma \in \mathcal{F}$$

then f is holomorphic. In particular, Morera's theorem holds under the weaker assumption that $\int_T f(z)dz = 0$ for all equilateral triangles.

Chapter 3.8, Page 103

Exercise 1: Using Euler's formula

$$\sin \pi z = \frac{e^{i\pi z} - e^{-i\pi z}}{2i},$$

show that the complex zeros of $\sin \pi z$ are exactly at the integers, and that they are each of order 1.

Calculate the residue of $1/\sin \pi z$ at $z = n \in \mathbb{Z}$.

Solution. Let $z = a + bi$ and suppose $\sin(\pi z) = 0$. Then

$$0 = 2i\sin(\pi(a+bi)) = (e^{-b\pi} - e^{b\pi})\cos(\pi a) + i(e^{b\pi} + e^{-b\pi})\sin(\pi a)$$

where we have used the formula $e^{i\theta} = \cos(\theta) + i\sin(\theta)$ for real θ. Since $e^{b\pi}$ and $e^{-b\pi}$ are both strictly positive, the imaginary part can only be zero if $\sin(\pi a) = 0$, which happens iff $a \in \mathbb{Z}$. For the real part to be zero, either $\cos(\pi a) = 0$ or $e^{-b\pi} = e^{b\pi}$. But $\cos(\pi a) = \pm 1$ when $a \in \mathbb{Z}$, so $e^{-b\pi} = e^{b\pi} \Rightarrow b = 0$. Hence $z = a + bi \in \mathbb{Z}$. To find the order of the zero, we note that

$$2i\frac{d}{dz}\Big|_{z=n} \sin(\pi z) = i\pi(e^{in\pi} + e^{-in\pi}) = 2i\pi e^{in\pi} \neq 0$$

so the zero is of order 1. Finally, the residue at n of $\frac{1}{\sin(\pi z)}$ is

$$\lim_{z\to n}(z-n)\frac{1}{\sin(\pi z)} = \frac{1}{\pi\cos(\pi n)} = \frac{(-1)^n}{\pi}$$

by L'Hôpital's Rule. $\square$

Exercise 2: Evaluate the integral

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^4}.$$

Where are the poles of $\frac{1}{1+z^4}$?

Solution. Since $1+z^4 = (z-\omega)(z-\omega^3)(z-\omega^5)(z-\omega^7)$ where $\omega = \frac{1+i}{\sqrt{2}}$ is a primitive 8th root of unity, the poles of $\frac{1}{1+z^4}$ are at $\omega, \omega^3, \omega^5$, and ω^7, of which ω and ω^3 are in the upper half plane. The residues may be calculated in various ways; the easiest is to note that if $\frac{1}{f}$ has a simple pole at z_0, the residue is $\frac{1}{f'(z_0)}$. This follows from the observation that if $f(z) = (z-z_0)h(z)$, then $f'(z_0) = h(z_0)$. In this case, the residues of interest are

$$\text{Res}(f;\omega) = \frac{1}{4\omega^3} = \frac{\omega^5}{4}$$

and

$$\text{Res}(f;\omega^3) = \frac{1}{4\omega} = \frac{\omega^7}{4}$$

which add up to

$$\frac{\omega^5+\omega^7}{4} = -\frac{\sqrt{2}}{4}i.$$

Now if we integrate $f(z) = \frac{1}{1+z^4}$ along a semicircular contour C_R of radius R, the Residue Theorem guarantees that

$$\oint_{C_R} f(z)dz = 2\pi i\left(-\frac{\sqrt{2}}{4}i\right) = \frac{\pi}{\sqrt{2}}.$$

Since $|1+z^4| \geq |z|^4 - 1$, the integral over the curved part is at most $\frac{\pi R}{R^4-1}$, which tends to zero as $R \to \infty$. Hence

$$\int_{-\pi}^{\pi} \frac{dx}{1+x^4} = \frac{\pi}{\sqrt{2}}.$$

□

Exercise 3: Show that

$$\int_{-\infty}^{\infty} \frac{\cos x}{x^2+a^2}dx = \pi\frac{e^{-a}}{a}, \quad \text{for } a > 0.$$

Solution. Let $f(z) = \frac{e^{iz}}{z^2+a^2}$. We integrate f around a semicircular contour of radius R in the upper half plane. Along the curved part of the semicircle, we note that $|e^{iz}| \leq 1$ for z in the UHP, so $|f(z)| \leq \frac{1}{|z^2+a^2|} \leq \frac{1}{R^2-a^2}$. Hence the integral over the curved part is at most $\frac{\pi R}{R^2-a^2} \to 0$ as $R \to \infty$. Hence

$$\int_{-\infty}^{\infty} f(z)dz = 2\pi i \sum_{z\in UHP} Res(f;z).$$

The residues of f are $z = \pm ai$, so the only pole in the UHP is at ai with residue

$$\lim_{z\to ai}(z-ai)\frac{e^{iz}}{(z-ai)(z+ai)} = \frac{e^{-a}}{2ai}.$$

Hence $\int_{-\infty}^{\infty} f(z)dz = \pi\frac{e^{-a}}{a}$. Taking the real parts of both sides,

$$\int_{-\infty}^{\infty} \frac{\cos x}{x^2+a^2}dx = \pi\frac{e^{-a}}{a}.$$

□

Exercise 4: Show that

$$\int_{-\infty}^{\infty} \frac{x \sin x}{x^2 + a^2} dx = \pi e^{-a}, \quad \text{for all } a > 0.$$

Solution. I'm getting rather fed up with the fact that this book doesn't even mention the Jordan lemma, so here it is, taken from page 216 of Gamelin (in Gamelin's lingo this is actually a corollary to Jordan's lemma):

Lemma 2 (Jordan's Lemma)**.** *If Γ_R is the semicircular contour $z(\theta) = Re^{i\theta}, 0 \leq \theta \leq \pi$, in the upper half-plane, and $P(z)$ and $Q(z)$ are polynomials with* $\deg Q(z) \geq \deg P(z) + 1$*, then*

$$\lim_{R \to \infty} \int_{\Gamma_R} \frac{P(z) e^{iz}}{Q(z)} dz = 0.$$

Proof. By concavity, $\sin \theta \geq \frac{2\theta}{\pi}$ for $0 \leq \theta \leq \frac{\pi}{2}$. Hence

$$|e^{iRe^{i\theta}}| = e^{-R \sin \theta} \leq e^{-2\theta/\pi}$$

for $0 \leq \theta \leq \frac{\pi}{2}$, and

$$\int_0^{\pi} e^{-R \sin \theta} d\theta = 2 \int_0^{\pi/2} e^{-R \sin \theta} d\theta \leq 2 \int_0^{\pi/2} e^{-2R\theta/\pi} d\theta = \frac{\pi}{R}(1 - e^{-R}) < \frac{\pi}{R}.$$

Since $|dz| = R$, this implies $\int_{\Gamma_R} |e^{iz}||dz| < \pi$. Since $\frac{|P(z)|}{|Q(z)|} = O(\frac{1}{R})$,

$$\int_{\Gamma_R} \frac{P(z) e^{iz}}{Q(z)} \leq O(\frac{1}{R}) \left| \int_{\Gamma_R} e^{iz} dz \right| \leq \pi O(\frac{1}{R}) \to 0.$$

□

Now back to our problem. Let $f(z) = \frac{z e^{iz}}{z^2 + a^2}$ and integrate f around a semicircular contour of radius R. By Jordan's lemma, the integral over the curved part tends to 0 as $R \to \infty$. Hence

$$\int_{-\infty}^{\infty} f(z) dz = 2\pi i \sum_{z \in UHP} Res(f; z).$$

The poles of f are at $z = \pm ai$; the residue at ai is

$$\lim_{z \to ai} (z - ai) \frac{e^{iz}}{(z - ai)(z + ai)} = aie^{-a} 2ai = \frac{e^{-a}}{2}.$$

Hence $\int_{-\infty}^{\infty} f(z) dz = \pi i e^{-a}$; taking imaginary parts,

$$\int_{-\infty}^{\infty} \frac{x \sin x}{x^2 + a^2} dx = \pi e^{-a}.$$

□

Exercise 5: Use contour integration to show that

$$\int_{-\infty}^{\infty} \frac{e^{-2\pi i x \xi}}{(1 + x^2)^2} dx = \frac{\pi}{2}(1 + 2\pi|\xi|) e^{-2\pi|\xi|}$$

for all ξ real.

Solution. First, we note that if $\xi > 0$, substituting $u = -x$ reveals that

$$\int_{-\infty}^{\infty} \frac{e^{-2\pi i x\xi}}{(1+x^2)^2}dx = \int_{\infty}^{-\infty} \frac{e^{-2\pi i(-u)\xi}}{(1+u^2)^2}(-du) = \int_{-\infty}^{\infty} \frac{e^{-2\pi i u(-\xi)}}{(1+u^2)^2}du$$

so that we may assume WLOG that $\xi \le 0$. We integrate $f(z) = \frac{e^{-2\pi i z\xi}}{(1+z^2)^2}$ along a semicircular contour of radius R. Since $|e^{-2\pi i z\xi}| \le 1$ for $z \in UHP$ and $\xi \le 0$, the integral over the curved part is at most $\frac{2\pi R}{(R^2-1)^2} \to 0$. Hence

$$\int_{-\infty}^{\infty} f(z)dz = 2\pi i \sum_{z\in UHP} Res(f; z).$$

The only pole in the UHP is at $z = i$; because this is a double pole, the residue is

$$\lim_{z\to i}\frac{d}{dz}(z-i)^2 f(z) = \lim_{z\to i}\frac{(-2\pi i\xi(z+i)-2)e^{-2\pi i z\xi}}{(z+i)^3} = \frac{(2-4\pi\xi)e^{2\pi\xi}}{8i}.$$

Hence

$$\int_{-\infty}^{\infty} \frac{e^{-2\pi i x\xi}}{(1+x^2)^2}dx = 2\pi i\frac{(2-4\pi\xi)e^{2\pi\xi}}{8i} = \frac{\pi}{2}\left(1+2|\xi|\right)e^{-2\pi|\xi|}.$$

$\square$

Exercise 6: Show that

$$\int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^n+1} = \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots(2n)}\cdot\pi.$$

Solution. Let $f(z) = \frac{1}{(z^2+1)^{n+1}}$. Then the pole of z in the UHP is at $z = i$; because this is an $(n+1)$-fold pole, its residue is

$$\begin{aligned}
\frac{1}{n!}\lim_{z\to i}\frac{d^n}{dz^n}\frac{1}{(z+i)^{n+1}} &= \frac{1}{n!}\lim_{z\to i}(-n-1)(-n-2)\dots(-2n)\frac{1}{(z+i)^{2n+1}}\\
&= \frac{1}{n!}\lim_{z\to i}\frac{(-1)^n(2n)!}{n!}\frac{1}{z+i}^{2n+1}\\
&= \frac{(-1)^n(2n)!}{(n!)^2}\frac{1}{(2i)^{2n+1}}\\
&= \frac{(2n)!}{(2^n n!)^2}\frac{1}{2i}\\
&= \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots(2n)}\frac{1}{2i}.
\end{aligned}$$

Integrating f around a semicircular contour of radius R, the integral over the curved part is bounded by $\frac{\pi R}{(R^2-1)^{n+1}} \to 0$, so

$$\int_{-\infty}^{\infty} f(z)dz = 2\pi i Res(f; i) = \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots(2n)}\pi.$$

$\square$

Exercise 7: Prove that

$$\int_0^{2\pi} \frac{d\theta}{(a+\cos\theta)^2} = \frac{2\pi a}{(a^2-1)^{3/2}}, \quad \text{whenever } a > 1.$$

Solution. Letting $z = e^{i\theta}$ turns this into a contour integral around the unit circle C:

$$\begin{aligned}
\int_0^{2\pi} \frac{d\theta}{(a+\cos\theta)^2} &= \oint_C \frac{dz/iz}{\left(a+\left(\frac{z+1/z}{2}\right)\right)^2} \\
&= \oint_C \frac{4zdz}{i(z^2+2az+1)^2} \\
&= 2\pi i Res\left(\frac{4z}{i(z^2+2az+1)^2}; -a+\sqrt{a^2-1}\right) \\
&= 2\pi i \lim_{z\to -a+\sqrt{a^2-1}} \frac{d}{dz}\frac{4z}{i(z+a+\sqrt{a^2-1})^2} \\
&= 2\pi i \lim_{z\to -a+\sqrt{a^2-1}} \frac{4(a+\sqrt{a^2-1}-z)}{i(z+a+\sqrt{a^2-1})^3} \\
&= 2\pi i \frac{8a}{i(2\sqrt{a^2-1})^3} \\
&= 2\pi a(a^2-1)^{-3/2}.
\end{aligned}$$

□

Exercise 8: Prove that

$$\int_0^{2\pi} \frac{d\theta}{a+b\cos\theta} = \frac{2\pi}{\sqrt{a^2-b^2}}$$

if $a > |b|$ and $a, b \in \mathbb{R}$.

Solution. Again, we convert into a contour integral around the unit circle by substituting $z = e^{i\theta}$:

$$\begin{aligned}
\int_0^{2\pi} \frac{d\theta}{a+b\cos\theta} &= \oint_C \frac{dz/iz}{a+b\left(\frac{z+1/z}{2}\right)^2} \\
&= \oint_C \frac{2dz}{i(bz^2+2az+b)} \\
&= 4\pi Res\left(\frac{1}{bz^2+2az+b}; \frac{-a+\sqrt{a^2-b^2}}{b}\right) \\
&= 2\pi \frac{1}{2bz+2a}\Big|_{z=\frac{-a+\sqrt{a^2-b^2}}{b}} \\
&= 2\pi \frac{1}{2\sqrt{a^2-b^2}} \\
&= \frac{\pi}{\sqrt{a^2-b^2}}.
\end{aligned}$$

□

Exercise 9: Show that

$$\int_0^1 \log(\sin \pi x)dx = -\log 2.$$

Solution. Consider the function $f(z) = 1 - e^{2\pi iz} = -2ie^{\pi iz}\sin(\pi z)$. (This clever trick comes from Ahlfors, page 160.) We will integrate $\log f(z)$ along the rectangular contour bounded by 0, 1, iY, and $1 + iY$, with quarter-circle indentations of radius ϵ at 0 and 1. On this region, $e^{\pi iz}$ is in the upper half plane, and $\sin(\pi z)$ is in the right half plane. Hence, if we take the principal branch Arg of the argument function, $0 \le Arg(e^{\pi iz}) < \pi$ and $-\frac{\pi}{2} \le Arg(\sin(\pi z)) < \frac{\pi}{2}$. Then

$$\begin{aligned} -\pi &\le Arg(-2i) + Arg(e^{\pi iz}) + Arg(\sin(\pi z))) \\ \Rightarrow Arg(f(z)) &= Arg(-2i) + Arg(e^{\pi iz}) + Log(\sin(\pi z)) \\ \Rightarrow Log(f(z)) &= Log(-2i) + Log(e^{\pi iz}) + Log(\sin(\pi z)) \\ &= \log 2 - \frac{\pi}{2}i + \pi iz + Log(\sin(\pi z)) \end{aligned}$$

where Log is the principal branch of the logarithm function, corresponding to the principal branch Arg of the argument function.

Returning to our contour integral, we note that the two vertical pieces cancel out by the periodicity of the sine function. Moreover, the integral over the upper half of the rectangle approaches zero as $Y \to \infty$ since for $z = t + iY$ with $0 \le t \le 1$, $e^{2\pi iz} \to 0 \Rightarrow f(z) \to 1 \Rightarrow \log f(z) \to 0$ as $Y \to \infty$. What's more, the integrals over the quarter-circle indentations also approach zero: Since $f(z) \approx z$ for z near 0, $\log f(z) \approx \log z$ so the integral over the quarter-circle near zero is $O(\epsilon|\log \epsilon)|) \to 0$. Similar analysis holds near π. Since $Logf(z)$ is analytic on and inside our contour, we are left with

$$\int_0^1 Logf(z)dz = 0.$$

By our above comments concerning the branches of the logarithm, this implies

$$0 = \int_0^1 \left(\log 2 - \frac{\pi}{2}i + \pi iz + Log(\sin(\pi z))\right) dz = \log 2 + \int_0^1 \log\sin(\pi x)dx.$$

Hence

$$\int_0^1 \log\sin(\pi x)dx = -\log 2.$$

$\square$

Exercise 10: Show that if $a > 0$, then

$$\int_0^\infty \frac{\log x}{x^2 + a^2}dx = \frac{\pi}{2a}\log a.$$

Solution. We use the branch of the logarithm function with a branch cut along the negative imaginary axis, corresponding to a branch of the argument function such that $-\frac{\pi}{2} \le arg(z) < \frac{3\pi}{2}$. We integrate $f(z) = \frac{\log z}{z^2+a^2}$ around an indented semicircle of radius R and indentation radius ϵ. The integral around the outer curved part is $O(\frac{R\log R}{R^2}) \to 0$ and the integral around the indentation is $O(\epsilon|\log\epsilon|) \to 0$. The integral along the axis is

$$\begin{aligned}
&\int_{-R}^{-\epsilon} \frac{\log x}{x^2+a^2}dx + \int_{\epsilon}^{R} \frac{\log x}{x^2+a^2}dx \\
&= \int_{\epsilon}^{R} \left(\frac{\log x}{x^2+a^2} + \frac{\log(-x)}{x^2+a^2} \right) dx \\
&= 2\int_{\epsilon}^{R} \frac{\log x}{x^2+a^2}dx + i\pi \int_{\epsilon}^{R} \frac{1}{x^2+a^2}dx
\end{aligned}$$

Letting $R \to \infty$ and $\epsilon \to 0$ and using the fact that $\int_0^\infty \frac{dx}{x^2+a^2} = \frac{1}{a}\arctan(x/a)|_0^\infty = \frac{\pi}{2a}$,

$$\int_0^\infty \frac{\log x}{x^2+a^2} = \frac{1}{2}\left(2\pi i \sum_{z\in UHP} Res(f;z) - \frac{i\pi^2}{2a}\right).$$

The only pole in the UHP is at $z = ai$ with residue

$$\lim_{z\to ai} \frac{\log z}{z+ai} = \frac{\log(ai)}{2ai} = \frac{\pi}{4a} + \frac{\log a}{2ai}.$$

Hence

$$\int_0^\infty \frac{\log x}{x^2+a^2}dx = \frac{1}{2}\left(2\pi i\left(\frac{\pi}{4a} + \frac{\log a}{2ai}\right) - \frac{i\pi^2}{2a}\right) = \frac{\pi\log a}{2a}.$$

□

Exercise 11: Show that if $|a| < 1$, then

$$\int_0^{2\pi} \log|1 - ae^{i\theta}|d\theta = 0.$$

Then, prove that the above result remains true if we assume only that $|a| \leq 1$.

Solution. For $|a| < 1$, $1 - ae^{iz}$ is in the right half-plane for $z = \theta + yi$, $0 \leq \theta \leq 2\pi$, $y \geq 0$. Hence the principal branch of $\log(1-ae^{i\theta})$ is analytic on this region. We integrate $\log(1 - ae^{iz})$ around a rectangular contour bounded by 0, 2π, $2\pi + Yi$, and Yi. The integrals over the vertical portions cancel out, and the integral over the top tends to zero because $1 - ae^{i(\theta+Yi)} = 1 - ae^{-Y}e^{i\theta} \to 1$ uniformly so $\log(1 - ae^{iz}) \to 0$ uniformly in θ as $Y \to \infty$. Hence

$$0 = \int_0^{2\pi} \log(1 - ae^{i\theta})d\theta = \int_0^{2\pi} \log|1 - ae^{i\theta}|d\theta + i\int_0^{2\pi} \arg(1 - ae^{i\theta})d\theta.$$

But the integral of the argument is 0 by symmetry, since $\arg(1-ae^{i(\theta+\pi)}) = -\arg(1 - ae^{i\theta})$. Hence

$$\int_0^{2\pi} \log|1 - ae^{i\theta}|d\theta = 0$$

for $|a| < 1$. To get the same result for $|a| = 1$, I suppose some kind of continuity argument is needed, but I don't see it. □

Exercise 12: Suppose u is not an integer. Prove that

$$\sum_{n=-\infty}^{\infty} \frac{1}{(u+n)^2} = \frac{\pi^2}{(\sin \pi u)^2}$$

by integrating

$$f(z) = \frac{\pi \cot \pi z}{(u+z)^2}$$

over the circle $|z| = R_N = N + 1/2$ $(N \geq |u|)$, adding the residues of f inside the circle, and letting N tend to infinity.

Solution. Just to be contrary, I'll use a square rather than a circle. (Actually, the reason is that my solution is ripped off from Schaum's Outline of Complex Variables, page 188.) Specifically, let S_N be the boundary of the square lying inside the lines $|x| = N + \frac{1}{2}$ and $|y| = N + \frac{1}{2}$ in the complex plane $z = x + iy$. When $|y| \geq 1$ we have

$$\begin{aligned} |\cot \pi(x+iy)| &= \frac{|e^{\pi i x - \pi y} + e^{-\pi i x + \pi y}|}{|e^{\pi i x - \pi y} - e^{-\pi i x + \pi y}|} \\ &\leq \frac{|e^{\pi i x - \pi y}| + |e^{-\pi i x + \pi y}|}{||e^{\pi i x - \pi y}| - |e^{-\pi i x + \pi y}||} \\ &= \frac{1 + e^{-2\pi y}}{1 - e^{-2\pi |y|}} \\ &\leq \frac{1 + e^{-2\pi}}{1 - e^{-2\pi}} = C_1. \end{aligned}$$

Moreover, when $|y| \leq 1$ then $x = \pm(N + \frac{1}{2})$ and

$$|\cot(\pi(x+iy))| = |\cot \pi \left(N + \frac{1}{2} + iy\right)| = |\tanh(y)| \leq \tanh\left(\frac{\pi}{2}\right) = C_2.$$

Hence, $|\cot \pi z| \leq C$ on all such squares, where $C = \max(C_1, C_2)$ is a universal constant. Since $|\frac{1}{(z+u)^2}| = O(\frac{1}{N^2})$, and the length of the contour is $8N + 4$,

$$\lim_{N \to \infty} \oint_{S_N} \frac{\pi \cot(\pi z)}{(z+u)^2} dz = 0$$

by the ML estimate. The poles of f inside S_N are at $-n, \ldots, n$ with residues

$$\lim_{z \to n} \frac{(z-n)\pi \cos(\pi z)}{\sin(\pi z)(z+u)^2} = \frac{\pi \cos(\pi n)}{(n+u)^2} \lim_{z \to n} \frac{z-n}{\sin(\pi z)} = \frac{1}{(n+u)^2}$$

by L'Hôpital's Rule, as well as the pole at $-u$ with residue

$$\lim_{z \to -u} \frac{d}{dz}(z+u)^2 \frac{\pi \cot(\pi z)}{(z+u)^2} = \lim_{z \to -u} -\pi^2 \csc^2(\pi z) = -\frac{\pi^2}{\sin^2(\pi z)}.$$

By the Residue Theorem,

$$\oint_{S_R} f(z) dz = \sum_{n=-N}^{N} \frac{1}{(z+u)^2} - \frac{\pi^2}{\sin^2(\pi z)}.$$

As $N \to \infty$, the LHS approaches zero and we have

$$\sum_{n=-N}^{N} \frac{1}{(z+u)^2} = \frac{\pi^2}{\sin^2(\pi z)}.$$

$\square$

Exercise 13: Suppose $f(z)$ is holomorphic in a punctured disc $D_r(z_0) \setminus \{z_0\}$. Suppose also that

$$|f(z)| \leq A|z - z_0|^{-1+\epsilon}$$

for some $\epsilon > 0$, and all z near z_0. Show that the singularity of f at z_0 is removable.

Solution. Let $g(z) = (z - z_0)f(z)$. Then g is analytic in the punctured disc and $g(z) \to 0$ as $z \to z_0$; since g is bounded, the singularity at z_0 is removable, so we can take g to be analytic at z_0. Since g is a holomorphic function with $g(z_0) = 0$, $g(z) = (z - z_0)h(z)$ in some neighborhood of z_0, where h is holomorphic at z_0. Then $f(z) = h(z)$ in a deleted neighborhood of z_0, so by defining $f(z_0) = h(z_0)$ we can extend f to a holomorphic function at z_0. Thus the singularity is removable. $\square$

Exercise 14: Prove that all entire functions that are also injective take the form $f(z) = az + b$ with $a, b \in \mathbb{C}$ and $a \neq 0$.

Solution. Define $g : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ by $g(z) = f(\frac{1}{z})$. Then g is holomorphic on the punctured plane. It cannot have a removable singularity at 0, because this would imply that f is bounded as $|z| \to \infty$ and therefore constant (so certainly not injective!) by Liouville's theorem. Moreover, g cannot have an essential singularity at 0: Suppose it did. Let $z \in \mathbb{C} \setminus \{0\}$ and let $r < |z|$ and $r' < |z| - r$. By the Open Mapping Theorem, $g(B_r(z))$ contains a neighborhood of $g(z)$. But by the Casorati-Weierestrass theorem, $g(B_{r'}(0) \setminus \{0\})$ is dense and hence contains a point in said neighborhood of $g(z)$. This implies that g, and therefore f, is not injective. Thus, g must have a pole at 0, so

$$g(z) = \frac{a_m}{z^m} + \cdots + \frac{a_1}{z} + h(z)$$

where h is analytic at zero. But then

$$f(z) = a_m z^m + \cdots + a_1 z + h(\frac{1}{z})$$

cannot be analytic unless h is constant. So f is a polynomial. The only polynomials without multiple distinct roots are powers, so $f(z) = a(z-z_0)^m$ for some $a, z_0 \in \mathbb{C}$ and $m \in \mathbb{N}$. If $m > 1$ then $f(z_0+1) = f(z_0+e^{2\pi i/m})$ so f is not injective. Hence $m = 1$ and $f = a(z - z_0)$. Note that we cannot have $a = 0$ since then f is constant and obviously not injective. $\square$

Exercise 15: Use the Cauchy inequalities or the maximum modulus principle to solve the following problems:

(a) Prove that if f is an entire function that satisfies

$$\sup_{|z|=R} |f(z)| \le AR^k + B$$

for all $R > 0$, and for some integer $k \ge 0$ and some constant $A, B > 0$, then f is a polonymial of degree $\le k$.

(b) Show that if f is holomorphic in the unit disc, is bounded, and converges uniformly to zero in the sector $\theta < \arg z < \phi$ as $|z| \to 1$, then $f = 0$.

(c) Let $w_1, \dots, w_n$ be points on the unit circle in the complex plane. Prove that there exists a point z on the unit circle such that the product of the distances from z to the points w_j, $1 \le j \le n$, is exactly equal to 1.

(d) Show that if the real part of an entire function f is bounded, then f is a constant.

Solution.

(a) By the Cauchy inequalities,

$$|f^{(n)}(0)| \le \frac{n!(AR^k + B)}{R^n}.$$

For $n > k$, taking the limit as $R \to \infty$ implies $f^{(n)}(0) = 0$. Since f is entire and all derivatives higher than k vanish, it is a polynomial of degree at most k.

(b)

(c) Let $f(z) = (z - w_1) \dots (z - w_n)$. Then $|f(z)|$ is the product of the distances from z to the points $w_1, \dots, w_n$. Now $|f(0)| = 1$ because $|w_i| = 1$ for all i. Because f is analytic, the maximum modulus principle guarantees that $|f(z_0)| \ge 1$ for some z_0 with $|z_0| = 1$. Now consider the restriction of f to the unit circle. Then $|f(e^{i\theta})|$ is a continuous function of θ which takes on the values 0 (at each of the w_i) and a value at least equal to 1 (at z_0). By the intermediate value theorem, it is exactly equal to 1 somewhere on the unit circle.

(d) In fact, it is sufficient for the real part of f to be bounded from one side (WLOG above). Suppose $\mathrm{Re}(f) \le M$ everywhere. Let $g(z) = e^{f(z)}$. Then g is entire, and $|g| = e^{\mathrm{Re}(f)} \le e^M$. By Liouville's theorem, $g(z) = c$ for some constant c. Then $f(z) = \log c$ everywhere. Although there are multiple branches of the log function, they differ by $2\pi i$ and f is continuous, so f must be constant.

□

Exercise 16: Suppose f and g are holomorphic in a region containing the disc $|z| \le 1$. Suppose that f has a simple zero at $z = 0$ and vanishes nowhere else in $|z| \le 1$. Let

$$f_\epsilon(z) = f(z) + \epsilon g(z).$$

Show that if ϵ is sufficiently small, then

(a) $f_\epsilon(z)$ has a unique zero in $|z| \le 1$, and

(b) if z_ϵ is this zero, the mapping $\epsilon \mapsto z_\epsilon$ is continuous.

Solution.

(a) Since the closed unit disc is compact and $|g|$ is continuous, it attains a maximum M on it. Similarly, $|f|$ is continuous and nonzero on the unit circle, so it has a minimum value m. Then for $\epsilon < \frac{m}{M}$, $|f(z)| > |\epsilon g(z)|$ on the unit circle, so f and f_ϵ have the same number of zeros inside the circle by Rouché's theorem.

(b) Fix ϵ and z_ϵ. For $r > 0$, let N_r be the open neighborhood of z_ϵ of radius r. Then $|f_\epsilon| \geq \delta_r$ on $\mathbb{D} \setminus N_r$ for some $\delta_r > 0$, because $\mathbb{D} \setminus N_r$ is compact and $|f_\epsilon| > 0$ on $\mathbb{D} \setminus \{z_\epsilon\}$. Then for $|\epsilon' - \epsilon| < \frac{\delta_r}{M}$, $|f_{\epsilon'}(z) - f_\epsilon(z)| = |\epsilon' - \epsilon||g(z)| < \delta_r$ so $f_{\epsilon'}(z) \neq 0$ for $z \in \mathbb{D} \setminus N_r$. Hence $z_{\epsilon'} \in N_r$ provided $|\epsilon' - \epsilon| < \frac{\delta_r}{M}$.

□

Exercise 17: Let f be non-constant and holomorphic in an open set containing the closed unit disc.

(a) Show that if $|f(z)| = 1$ whenever $|z| = 1$, then the image of f contains the unit disc. (Hint.)

(b) If $|f(z)| \geq 1$ whenever $|z| = 1$ and there exists a point $z_0 \in \mathbb{D}$ such that $|f(z_0)| < 1$, then the image of f contains the unit disc.

Solution.

(a) By Rouché's theorem, $f(z)$ and $f(z) - w_0$ have the same number of zeros inside the unit circle provided $|w_0| < 1$. Hence, if f has a zero, its image includes the unit disc. If f is nonzero, then $\frac{1}{f}$ is holomorphic, so $\left|\frac{1}{f(z)}\right| \leq 1$ for $z \in \bar{D}$ by the maximum modulus principle. But then $|f(z)| \geq 1$ for $z \in \bar{D}$, which contradicts the open mapping theorem. (Pick any z with $|z| = 1$; then $f(\bar{D})$ contains a neighborhood of $f(z)$, which includes points w with $|w| < 1$ since $|f(z)| = 1$.)

(b) Let $w_0 = f(z_0)$ where $|z_0| < 1$ and $|w_0| < 1$. By Rouché's theorem again, $f(z)$ and $f(z) - w$ have the same number of zeros for all w with $|w| < 1$. Since there exists a w (namely w_0) for which $f(z) - w$ has a zero, it has a zero for all $w \in \mathbb{D}$. So the image of f contains $\mathbb{D}$.

□

Chapter 3.9, Page 108

Problem 3: If $f(z)$ is holomorphic in the deleted neighborhood $\{0 < |z - z_0| < r\}$ and has a pole of order k at z_0, then we can write

$$f(z) = \frac{a_{-k}}{(z - z_0)^k} + \cdots + \frac{a_{-1}}{(z - z_0)} + g(z)$$

where g is holomorphic in the disc $\{|z - z_0| < r\}$. There is a generalization of this expansion that holds even if z_0 is an essential singularity. This is a special case of the **Laurent series expansion**, which is valid even in a more general setting.

Let f be holomorphic in a region containing the annulus $\{z : r_1 \leq |z - z_0| \leq r_2\}$ where $0 < r_1 < r_2$. Then,

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

where the series converges absolutely in the interior of the annulus. To prove this, it suffices to write

$$f(z) = \frac{1}{2\pi i}\int_{C_{r_2}} \frac{f(\zeta)}{\zeta - z}d\zeta - \frac{1}{2\pi i}\int_{C_{r_1}} \frac{f(\zeta)}{\zeta - z}d\zeta$$

when $r_1 < |z - z_0| < r_2$, and argue as in the proof of Theorem 4.4, Chapter 2. Here C_{r_1} and C_{r_2} are the circles bounding the annulus.

Chapter 5.6, Page 153

Exercise 1: Give another proof of Jensen's formula in the unit disc using the functions (called Blaschke factors)

$$\psi_\alpha(z) = \frac{\alpha - z}{1 - \bar{\alpha}z}.$$

Solution. We use the same proof as before to establish Jensen's formula for functions with no zeros in the unit disc. Now suppose f is analytic on the unit disc and has zeros at $z_1, \ldots, z_N$, counted with multiplicity. Then the function

$$g(z) = \frac{f(z)}{\psi_1(z)\cdots\psi_N(z)}$$

is analytic on the unit disc and has no zeros, where we use the notation $\psi_k(z) = \psi_{z_k}(z)$. Hence

$$\log\left|\frac{f(0)}{\psi_1(0)\cdots\psi_N(0)}\right| = \frac{1}{2\pi}\int_0^{2\pi} \log\left|\frac{f(e^{i\theta})}{\psi_1(e^{i\theta})\cdots\psi_N(e^{i\theta})}\right| d\theta$$

$$\Rightarrow \log|f(0)| - \sum_{k=1}^N \log|\psi_k(0)| = \frac{1}{2\pi}\int_0^{2\pi} \log|f(e^{i\theta})|d\theta - \frac{1}{2\pi}\sum_{k=1}^N \int_0^{2\pi} \log|\psi_k(e^{i\theta})|d\theta.$$

Since $\psi_k(0) = z_k$ and

$$|\psi_k(e^{i\theta})| = \left|\frac{z_k - e^{i\theta}}{-e^{i\theta}(\overline{z_k - e^{i\theta}})}\right| = 1,$$

this becomes

$$\log|f(0)| = \sum_{k=1}^N \log|z_k| + \frac{1}{2\pi}\int_0^{2\pi} \log|f(e^{i\theta})|$$

as desired. □

Exercise 3: Show that if τ is fixed with $\mathrm{Im}(\tau) > 0$, then the Jacobi theta function

$$\Theta(z|\tau) = \sum_{n=-\infty}^{\infty} e^{\pi i n^2 \tau} e^{2\pi i n z}$$

is of order 2 as a function of z.

Solution. Let $\tau = s + it$ with $s, t \in \mathbb{R}$; by hypothesis, $t > 0$. Then

$$\begin{aligned}
|\Theta(z|\tau)| &\le \sum_{n=-\infty}^{\infty} |e^{\pi i n^2 \tau}||e^{2\pi i n z}| \\
&\le \sum_{n=-\infty}^{\infty} e^{-\pi n^2 t} e^{2\pi |n||z|} \\
&= 1 + 2\sum_{n=1}^{\infty} e^{-\pi n^2 t} e^{2\pi n|z|} \\
&= 1 + 2\sum_{n=0}^{\lfloor 4|z|/t \rfloor} e^{-\pi n^2 t + 2\pi n|z|} + 2\sum_{n=\lfloor 4|z|/t\rfloor+1}^{\infty} e^{-\pi n^2 t + 2\pi n|z|}.
\end{aligned}$$

For $n > 4|z|/t$, $-\pi n^2 t + 2\pi n|z| < -\pi n^2 t/2$, so the second sum is less than

$$\sum_{n=\lfloor 4|z|/t\rfloor+1}^{\infty} e^{-\pi n^2 t/2} \le \sum_{n=1}^{\infty} e^{-\pi n^2 t/2} = C'.$$

In the first sum, there are at most $4|z|/t$ terms, each of which is at most $e^{2\pi \cdot 4|z|/t \cdot |z|} = e^{8\pi |z|^2/t}$, so we have

$$|\Theta(z|\tau)| \le C_1 + C_2|z|e^{C_3|z|^2}$$

for constants C_1, C_2, C_3. This implies $|\Theta(z|\tau)| \le C_\epsilon e^{B_\epsilon |z|^{2+\epsilon}}$ for any $\epsilon > 0$, so Θ has order at most 2. $\square$

Exercise 4: Let $t > 0$ be given and fixed, and define $F(z)$ by

$$F(z) = \prod_{n=1}^{\infty}(1 - e^{-2\pi n t} e^{2\pi i z}).$$

Note that the product defines an entire function of z.

(a) Show that $|F(z)| \le Ae^{a|z|^2}$, hence F is of order 2.

(b) F vanishes exactly when $z = -int + m$ for $n \ge 1$ and n, m integers. Thus, if z_n is an enumoration of these zeros we have

$$\sum \frac{1}{|z_n|^2} = \infty \text{ but } \sum \frac{1}{|z_n|^{2+\epsilon}} < \infty.$$

Solution.

(a) Given z, let $c > \frac{1}{t}$ and let $N = \lfloor c|z| \rfloor$. Let

$$F_1(z) = \prod_{n=1}^{N}(1 - e^{-2\pi n t} e^{2\pi i z}), \quad F_2(z) = \prod_{n=N+1}^{N}(1 - e^{-2\pi n t} e^{2\pi i z}).$$

Since $e^{-2\pi(N+1)t}e^{2\pi|z|} \leq \frac{1}{2}$ by our choice of N (assuming $|z|$ sufficiently large), and since $|\log(1+y)| \leq 2|y|$ for $|y| \leq \frac{1}{2}$, we have

$$\begin{aligned}
|\log F_2(z)| &= \left|\sum_{n=N+1}^{\infty} \log(1 - e^{-2\pi nt}e^{2\pi iz})\right| \\
&\leq \sum_{n=N+1}^{\infty} |\log(1 - e^{-2\pi nt}e^{2\pi iz})| \\
&\leq \sum_{n=N+1}^{\infty} 2|e^{-2\pi nt}e^{2\pi iz}| \\
&\leq 2 \sum_{n=N+1}^{\infty} e^{-2\pi nt}e^{2\pi|z|} \\
&= 2e^{-2\pi(N+1)t+2\pi|z|} \sum_{k=0}^{\infty} e^{-2\pi kt} \\
&\leq \sum_{k=0}^{\infty} e^{-2\pi kt} \\
&= C
\end{aligned}$$

so $|F_2(z)|$ is bounded by a constant. Moreover, for any n,

$$|1 - e^{-2\pi nt}e^{2\pi iz}| \leq 1 + e^{2\pi|z|} \leq 2e^{2\pi|z|}.$$

Since F_1 is the product of N such terms, we have

$$|F_1(z)| \leq (2e^{2\pi|z|})^N = 2^N e^{2\pi N|z|} \leq 2^{c|z|}e^{2\pi c|z|^2} \leq e^{c'|z|^2}$$

for an appropriate choice of c'. Hence F_1 (and therefore F because $F = F_1F_2$ and F_2 is bounded) is of order at most 2.

(b) By Proposition 3.1 the product $F(z)$ is zero exactly when, for some integer $n \geq 1$,

$$e^{-2\pi(nt+iz)} = 1 \Rightarrow z = -int + m$$

for some integer m. Thus, if z_k is an enumeration of these zeros,

$$\sum \frac{1}{|z_k|^{2+\epsilon}} < \infty$$

by Theorem 2.1. However,

$$\begin{aligned}
\sum \frac{1}{|z_k|^2} &= \sum_{m=-\infty}^{\infty} \sum_{n=1}^{\infty} \frac{1}{m^2 + t^2 n^2} \\
&= \frac{\pi^2}{6t^2} + 2 \sum_{m,n=1}^{\infty} \frac{1}{m^2 + n^2 t^2} \\
&\geq 2 \sum_{m,n=1}^{\infty} \frac{1}{m^2 + n^2 t^2} \\
&\geq 2 \int_1^{\infty} \int_1^{\infty} \frac{1}{x^2 + t^2 y^2} dx dy \\
&= \frac{2}{t^2} \int_1^{\infty} \int_1^{\infty} \frac{1}{x^2 + u^2} dx du \\
&= \frac{2}{t^2} \int_0^{2\pi} \int_1^{\infty} \frac{1}{r^2} r dr d\theta \\
&= \frac{4\pi}{t^2} \int_1^{\infty} \frac{dr}{r} \\
&= \infty,
\end{aligned}$$

where the rearrangement of the double sum is allowed because all terms are nonnegative, and the sum-integral relation follows from the fact that $\frac{1}{x^2+y^2}$ is decreasing in both x and y.

□

Exercise 8: Prove that for every z the product below converges, and

$$\cos(z/2)\cos(z/4)\cos(z/8)\cdots = \prod_{k=1}^{\infty} \cos(z/2^k) = \frac{\sin z}{z}.$$

Solution. First, to show that the product converges, we note that $|1 - \cos(t)| \leq t^2$ for sufficiently small $|t|$; this follows from the Taylor series expansion for cosine. Thus, for large enough k we have $|1 - \cos(z/2^k)| \leq (z/2^k)^2 = z^2/4^k$, and since $\sum 1/4^k$ converges, Proposition 3.1 guarantees that $\prod \cos(z/2^k)$ converges as well. Now let $F(z) = \prod_{k=1}^{\infty} \cos(z/2^k)$. Using the trigonometric identity $\sin z = 2\cos(z/2)\sin(z/2)$ repeatedly, we have

$$\begin{aligned}
\sin z &= 2\cos(z/2)\sin(z/2) \\
&= 4\cos(z/2)\cos(z/4)\sin(z/4) \\
&= \dots \\
&= 2^N \sin(z/2^N) \prod_{k=1}^{N} \cos(z/2^N)
\end{aligned}$$

for each $N = 1, 2, \dots$. If we take the limit as $N \to \infty$, the product approaches $F(z)$, and since $\sin(t) \approx t$ for small $|t|$, we end up with

$$\sin z = zF(z) \Rightarrow F(z) = \frac{\sin z}{z}.$$

□

Exercise 9: Prove that if $|z| < 1$, then

$$(1+z)(1+z^2)(1+z^4)\cdots = \prod_{k=0}^{\infty}(1+z^{2^k}) = \frac{1}{1-z}.$$

Solution. We first note that the product is convergent; indeed, it converges uniformly on the compact subdisk $|z| \le R < 1$ by Proposition 3.2 since

$$\sum |z|^{2^k} \le \sum R^{2^k} < \sum R^k = \frac{1}{1-R} < \infty.$$

We next prove by induction that

(1) $$\prod_{k=0}^{N}(1+z^{2^k}) = \sum_{j=0}^{2^{N+1}-1} z^j.$$

The base case $N = 0$ is obvious. The inductive step is

$$\begin{aligned}
\prod_{k=0}^{N+1}(1+z^{2^k}) &= (1+z^{2^{N+1}})\prod_{k=0}^{N}(1+z^{2^k}) \\
&= (1+z^{2^{N+1}})\sum_{j=0}^{2^{N+1}-1} z^j \\
&= \sum_{j=0}^{2^{N+1}-1} z^j + z^{2^{N+1}}\sum_{j=0}^{2^{N+1}-1} z^j \\
&= \sum_{j=0}^{2^{N+1}-1} z^j + \sum_{j=2^{N+1}}^{2^{N+2}-1} z^j \\
&= \sum_{j=0}^{2^{N+2}-1} z^j.
\end{aligned}$$

Since a subsequence of a convergent sequence converges to the same limit,

$$\lim_{N\to\infty}\sum_{j=0}^{2^{N+2}-1} z^j = \lim_{M\to\infty}\sum_{j=0}^{M} z^j = \frac{1}{1-z}.$$

Thus, taking the limit of both sides of (1),

$$\prod_{k=0}^{\infty}(1+z^{2^k}) = \lim_{N\to\infty}\prod_{k=0}^{N}(1+z^{2^k}) = \lim_{N\to\infty}\sum_{j=0}^{2^{N+1}-1} z^j = \frac{1}{1-z}.$$

□

Exercise 11: Show that if f is an entire function of finite order that omits two values, then f is constant. This result remains true for any entire function and is known as Picard's little theorem.

Solution. Suppose f is never equal to a. Then $f(z)-a$ is an entire function which is nowhere zero; by Theorem 6.2 of Chapter 3, this implies $f(z)-a = e^{g(z)}$ for an entire function g. If f has finite order ρ, then $|g(z)| \le |z|^\rho$, which implies g is a polynomial. Every nonconstant polynomial takes on all complex values, so either g is constant (in which case f is as well) or it takes on every complex value, which implies e^g takes on every nonzero value and f takes on every value other than a.

(Note: In case it's not obvious that $|g(z)| \leq |z|^\rho$ implies g is a polynomial, write $g(z) = p(z) + z^k h(z)$ for a polynomial p, entire function h, and power $k > \rho$. Then h is entire and bounded, hence constant.) $\square$

Exercise 13: Show that the equation $e^z - z = 0$ has infinitely many solutions in $\mathbb{C}$.

Solution. Suppose the equation has finitely many solutions $a_1, \ldots, a_N$, where we allow the possibility $N = 0$. Since $e^z - z$ has order 1, Hadamard's theorem tells us that $e^z - z = p(z)e^{az+b}$ for some constants a, b, where

$$p(z) = \prod_{n=1}^{N} (1 - z/a_n).$$

Now for the equation $e^z - z = p(z)e^{az+b}$ to be true for large real values of z, we must have $a = 1$, and since e^b is a constant we can rewrite this as $e^z - z = p(z)e^z \Rightarrow p(z) = 1 - ze^{-z}$, a contradiction. Hence $e^z - z$ has infinitely many zeros.

Just for fun, let's also give a semi-constructive, real-variables proof. Let $z = x + iy$; then the equation $e^z = z$ becomes

$$e^{x+iy} = x + iy \Rightarrow e^x \cos y + ie^x \sin y = x + iy \Rightarrow e^x \cos y = x \text{ and } e^x \sin y = y.$$

We will narrow our search to solutions with $x, y > 0$. In this case the above equations are equivalent to the system of equations $x^2 + y^2 = e^{2x}$, $\frac{y}{x} = \tan y$. From this we see that $y = \sqrt{e^{2x} - x^2}$ and $\tan \sqrt{e^{2x} - x^2} = \sqrt{e^{2x}/x^2 - 1}$. This in turn implies

$$\sqrt{e^{2x} - x^2} = \arctan \sqrt{\frac{e^{2x}}{x^2} - 1} + k\pi \tag{2}$$

for some integer k. Now $\sqrt{e^{2x} - x^2}$ is equal to 1 at $x = 0$ and tends to infinity as $x \to \infty$. On the other hand, for $k \geq 0$ the right hand side of (2) is greater than 1 at $x = 0$, but is bounded. Hence, the intermediate value theorem guarantees at least one solution of (2) for each $k = 0, 1, 2, \ldots$. Given any such solution x, we can let $y = \sqrt{e^{2x} - x^2}$, and $x + iy$ will be a solution of $e^z = z$. $\square$

Exercise 14: Deduce from Hadamard's theorem that if F is entire and of growth order ρ that is non-integral, then F has infinitely many zeros.

Solution. If F is entire and has finitely many zeros, Hadamard's theorem implies that $F(z) = P_1(z)e^{P_2(z)}$ for some polynomials P_1 and P_2. But then F would have order $\deg P_2$, an integer, because $|P_1(z)| \leq C_\epsilon e^{|z|^\epsilon}$ for any $\epsilon > 0$. Thus, if the order of F is not an integer, F must have infinitely many zeros. $\square$

Exercise 15: Prove that every meromorphic function in $\mathbb{C}$ is the quotient of two entire functions. Also, if $\{a_n\}$ and $\{b_n\}$ are two disjoint sequences having no finite limit points, then there exists a meromorphic function in the whole complex plane that vanishes exactly at $\{a_n\}$ and has poles exactly at $\{b_n\}$.

Solution. Let f be a meromorphic function on $\mathbb{C}$. Let a_n be the poles of f counted with multiplicity. By the Weierstrass product theorem, there exists an entire function g with zeros exactly at a_n. Then the product fg is an entire function h, so $f = h/g$ where h and g are both entire. Now let a_n and b_n be two sequences with no finite limit points. Let F and G be entire functions with zeros precisely at the a_n and the b_n, respectively; such functions exist by the Weierstrass product theorem. Then the quotient F/G has zeros exactly at the a_n and poles exactly at the b_n. □

Exercise 16: Suppose that

$$Q_n(z) = \sum_{k=1}^{N_n} c_k^n z^k$$

are given polynomials for $n = 1, 2, \ldots$. Suppose also that we are given a sequence of complex numbers $\{a_n\}$ without limit points. Prove that there exists a meromorphic function $f(z)$ whose only poles are at $\{a_n\}$, and so that for each n, the difference

$$f(z) - Q_n\left(\frac{1}{z - a_n}\right)$$

is holomorphic near a_n. In other words, f has prescribed poles and principal parts at each of these poles. This result is due to Mittag-Leffler.

Solution. (Solution adapted from Gamelin, *Complex Analysis*, p. 348.) Let $K_m = \overline{B_m(0)} = \{z \in \mathbb{C} : |z| \leq m\}$. Let

$$f_m(z) = \sum_{a_k \in K_{m+1} \setminus K_m} Q_k\left(\frac{1}{z - a_k}\right).$$

This is a finite sum, so f_m is well-defined for $z \neq a_1, a_2, \ldots$. By the Runge approximation theorem (Theorem 5.7 of Chapter 2), there exist polynomials $g_m(z)$ such that $|f_m(z) - g_m(z)| \leq \frac{1}{2^m}$ on K_m. Let

$$f(z) = \sum_{m=1}^{\infty} \big(f_m(z) - g_m(z)\big).$$

This sum converges uniformly on compact subsets of $\mathbb{C}$, by the Weierstrass M-test. Hence f is well-defined on $\mathbb{C}$ and is meromorphic. Moreover, on K_m the tail $\sum_{j=m}^{\infty}(f_j(z) - g_j(z))$ is analytic, whereas $f_k(z) - g_k(z)$ for $k < m$ has poles precisely at those a_j in $K_{k+1} \setminus K_k$, with the prescribed principal parts; thus, the poles of f lying in K_m are precisely the a_k within K_m, with the correct principal parts, for each m. Hence f has the desired properties. □

Chapter 5.7, Page 156

Problem 1: Prove that if f is holomorphic in the unit disc, bounded, and not identically zero, and $z_1, z_2, \ldots$ are its zeros ($|z_k| < 1$), then

$$\sum_n (1 - |z_n|) < \infty.$$

Solution. By Jensen's formula, we have for each $R < 1$

$$\sum_{|z_k|<R} \log \left|\frac{R}{z_k}\right| = \int_0^{2\pi} \log|f(Re^{i\theta})| \frac{d\theta}{2\pi} - \log|f(0)|.$$

Because f is bounded, the right-hand side is bounded above by some constant M as R varies. Suppose now we fix R and let $R' > R$ be variable. We get

$$\sum_{|z_k|<R} \log \left|\frac{R'}{z_k}\right| \leq \sum_{|z_k|<R'} \log \left|\frac{R'}{z_k}\right| < M \tag{3}$$

since the first sum is a partial sum of the second and all terms are positive. Since the first sum in (3) is finite, we can let $R' \to 1$ and get

$$\sum_{|z_k|<R} \log \left|\frac{1}{z_k}\right| \leq M.$$

This is true for all $R < 1$, so letting $R \to 1$ we have

$$\sum_k \log \left|\frac{1}{z_k}\right| \leq M.$$

(If all the partial sums are at most M, the infinite sum is as well.) Now $1 - x \leq -\log x$ for all real $x > 0$, so

$$\sum_k 1 - |z_k| \leq \sum_k \log \left|\frac{1}{z_k}\right| \leq M < \infty.$$

$\square$

Chapter 6.3, Page 174

Exercise 1: Prove that

$$\Gamma(s) = \lim_{n\to\infty} \frac{n^s n!}{s(s+1)\cdots(s+n)}$$

whenever $s \neq 0, -1, -2, \ldots$.

Solution.

$$\begin{aligned}
\frac{1}{\Gamma(s)} &= se^{\gamma s}\prod_{n=1}^{\infty}\left(1+\frac{s}{n}\right)e^{-s/n}\\
&= s\left(\lim_{N\to\infty} e^{s(\sum_{n=1}^{N} 1/n-\log N)}\right)\left(\lim_{N\to\infty}\prod_{n=1}^{N}\left(1+\frac{s}{N}\right)e^{-s/n}\right)\\
&= \lim_{N\to\infty} sN^{-s}\prod_{n=1}^{N} e^{s/n}\prod_{n=1}^{N}\left(\frac{n+s}{n}\right)e^{-s/n}\\
&= \lim_{N\to\infty} sN^{-s}\prod_{n=1}^{N}\left(\frac{n+s}{n}\right)\\
&= \lim_{N\to\infty}\frac{s(s+1)\cdots(s+N)}{N^{S}N!}.
\end{aligned}$$

$\square$

Exercise 3: Show that Wallis's product formula can be written as

$$\sqrt{\frac{\pi}{2}} = \lim_{n\to\infty}\frac{2^{2n}(n!)^2}{(2n+1)!}(2n+1)^{1/2}.$$

As a result, prove the following identity:

$$\Gamma(s)\Gamma(s+1/2) = \sqrt{\pi}2^{1-2s}\Gamma(2s).$$

Solution. Wallis' product formula says

$$\frac{\pi}{2} = \lim_{n\to\infty}\prod_{k=1}^{n}\frac{(2k)^2}{(2k+1)(2k-1)}.$$

Now

$$\begin{aligned}
\prod_{k=1}^{n}(2k-1)(2k+1) &= \prod_{k=1}^{n}(2k-1)\prod_{k=1}^{n}(2k+1) \\
&= \left(\prod_{\substack{j=1 \\ j \text{ odd}}}^{2n-1} j\right)\left(\prod_{\substack{j=1 \\ j \text{ odd}}}^{2n+1} j\right) \\
&= (2n+1)\left(\prod_{\substack{j=1 \\ j \text{ odd}}}^{2n-1} j\right)^2 \\
&= (2n+1)\left(\frac{(2n)!}{\prod_{\substack{j=1 \\ j \text{ even}}}^{2n} j}\right)^2 \\
&= (2n+1)\left(\frac{(2n)!}{\prod_{k=1}^{n} 2k}\right)^2 \\
&= (2n+1)\left(\frac{(2n)!}{n!2^n}\right)^2
\end{aligned}$$

whereas

$$\prod_{k=1}^{n}(2k)^2 = 2^{2n}\prod_{k=1}^{n}k^2 = 2^{2n}(n!)^2.$$

Hence

$$\frac{\pi}{2} = \lim_{n\to\infty}\frac{2^{2n}(n!)^2}{(2n+1)\frac{((2n)!)^2}{(n!)^2 2^{2n}}} = \lim_{n\to\infty}\frac{2^{4n}(n!)^4}{(2n+1)((2n)!)^2} = \lim_{n\to\infty}\frac{2^{4n}(n!)^4(2n+1)}{((2n+1)!)^2}$$

and the result follows by taking square roots of both sides.

Using the result of Problem 1,

$$\begin{aligned}
\frac{\Gamma(2s)}{\Gamma(s)\Gamma(s+1/2)} &= \lim \frac{s(s+1)\cdots(s+n)}{n^s n!}\lim\frac{(s+1/2)\cdots(s+1/2+n)}{n^{s+1/2}n!}\lim\frac{(2n+1)^{2s}(2n+1)!}{(2s)(2s+1)\cdots(2s+2n+1)} \\
&= \lim \frac{s(s+1/2)(s+1)(s+3/2)\cdots(s+n+1/2)(2n+1)^{2s}(2n+1)!}{n^{2s+1/2}(n!)^2(2s)(2s+1)\cdots(2s+2n+1)} \\
&= \lim \frac{2s(2s+1)\cdots(2s+2n+1)}{2^{2n+2}}\frac{(2n+1)^{2s}(2n+1)!}{n^{2s+1/2}(n!)^2 2s(2s+1)\cdots(2s+2n+1)} \\
&= \lim\left(\frac{2n+1}{n}\right)^{2s}\sqrt{\frac{2n+1}{n}}\frac{(2n+1)!}{2^{2n+2}(n!)^2\sqrt{2n+1}} \\
&= 2^{2s}\sqrt{2}\frac{1}{4}\sqrt{\frac{\pi}{2}} \\
&= 2^{2s-1}\sqrt{\pi}.
\end{aligned}$$

□

Exercise 5: Use the fact that $\Gamma(s)\Gamma(1-s) = \pi/\sin \pi s$ to prove that

$$|\Gamma(1/2+it)| = \sqrt{\frac{2\pi}{e^{\pi t}+e^{-\pi t}}}, \text{ whenever } t \in \mathbb{R}.$$

Solution. Using the trigonometric identity $\sin(\theta + \pi/2) = \cos(\theta)$,

$$\Gamma(1/2+it)\Gamma(1/2-it) = \frac{\pi}{\sin \pi(1/2+it)} = \frac{\pi}{\cos(\pi i t)} = \frac{\pi}{\cosh \pi t} = \frac{2\pi}{e^{\pi t}+e^{-\pi t}}.$$

Using the fact that $\overline{\Gamma(z)} = \Gamma(\bar{z})$, which follows from the meromorphicity of Γ,

$$|\Gamma(1/2+it)|^2 = \Gamma(1/2+it)\Gamma(1/2-it) = \frac{2\pi}{e^{\pi t}+e^{-\pi t}}$$

and the result follows by taking square roots of both sides. □

Exercise 7: The **Beta function** is defined for $\mathrm{Re}(\alpha) > 0$ and $\mathrm{Re}(\beta) > 0$ by

$$B(\alpha,\beta) = \int_0^1 (1-t)^{\alpha-1} t^{\beta-1} dt.$$

(a) Prove that $B(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$.
(b) Show that $B(\alpha,\beta) = \int_0^\infty \frac{u^{\alpha-1}}{(1+u)^{\alpha+\beta}} du$.

Solution.
(a)

$$\Gamma(\beta)\Gamma(\alpha) = \left(\int_0^\infty x^{\beta-1}e^{-x}dx\right)\left(\int_0^\infty y^{\alpha-1}e^{-y}dy\right) = \int_0^\infty \int_0^\infty x^{\beta-1}y^{\alpha-1}e^{-(x+y)}dxdy.$$

Making the change of variables $u = x+y, v = \frac{x}{x+y}$, we have $x = uv$ and $y = u(1-v)$, so

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} v & u \\ 1-v & -u \end{vmatrix} = -u,$$

so

$$\begin{aligned}\Gamma(\alpha)\Gamma(\beta) &= \int_0^\infty \int_0^1 (uv)^{\beta-1}(u(1-v))^{\alpha-1}e^{-u}udvdu \\ &= \left(\int_0^\infty u^{\alpha+\beta-1}e^{-u}du\right)\left(\int_0^1 v^{\beta-1}(1-v)^{\alpha-1}dv\right) \\ &= \Gamma(\alpha+\beta)B(\alpha,\beta).\end{aligned}$$

(b) We make the change of variables $u = \frac{1}{t} - 1$, so $t = \frac{1}{u+1}$, $1-t = \frac{u}{u+1}$, and $dt = -\frac{du}{(u+1)^2}$. Then

$$\begin{aligned}B(\alpha,\beta) &= \int_0^1 (1-t)^{\alpha-1}t^{\beta-1}dt \\ &= \int_\infty^0 \left(\frac{u}{u+1}\right)^{\alpha-1}\left(\frac{1}{u+1}\right)^{\beta-1}\frac{-du}{(u+1)^2} \\ &= \int_0^\infty \frac{u^{\alpha-1}}{u^{\alpha+\beta}}du.\end{aligned}$$

□

Exercise 9: The hypergeometric series $F(\alpha, \beta, \gamma; z)$ was defined in Exercise 16 of Chapter 1. Show that

$$F(\alpha, \beta, \gamma; z) = \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)} \int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-zt)^{-\alpha} dt.$$

Solution.

$$\begin{aligned}
&\frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)} \int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-zt)^{-\alpha} dt \\
&= \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)} \int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1} \left(1 + \sum_{n=1}^{\infty} \frac{(-\alpha)(-\alpha-1)\cdots(-\alpha-n-1)}{n!}(-zt)^n\right) dt \\
&= \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)} \left(B(\beta, \gamma-\beta) + \sum_{n=1}^{\infty} \int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1} \frac{\alpha(\alpha+1)\cdots(\alpha+n-1)}{n!} z^n t^n dt\right) \\
&= 1 + \sum_{n=1}^{\infty} \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)} B(n+\beta, \gamma-\beta) \frac{\alpha(\alpha+1)\cdots(\alpha+n-1)}{n!} z^n \\
&= 1 + \sum_{n=1}^{\infty} \frac{\Gamma(\gamma)\Gamma(n+\beta)}{\Gamma(\beta)\Gamma(n+\gamma)} \frac{\alpha(\alpha+1)\cdots(\alpha+n-1)}{n!} z^n \\
&= 1 + \sum_{n=1}^{\infty} \frac{\Gamma(\gamma)\beta(\beta+1)\cdots(\beta+n-1)\Gamma(\beta)}{\Gamma(\beta)\gamma(\gamma+1)\cdots(\gamma+n-1)\Gamma(\gamma)} \frac{\alpha(\alpha+1)\cdots(\alpha+n-1)}{n!} z^n \\
&= 1 + \sum_{n=1}^{\infty} \frac{\alpha(\alpha+1)\cdots(\alpha+n-1)\beta(\beta+1)\cdots(\beta+n-1)}{n!\gamma(\gamma+1)\cdots(\gamma+n-1)} z^n \\
&= F(\alpha, \beta, \gamma; z)
\end{aligned}$$

where the sum and integral can be interchanged because all terms are non-negative; here we have used the identity $\Gamma(s+1) = s\Gamma(s)$ as well as the properties of the beta function derived in Exercise 7. Since $(1-w)^{-\alpha}$ is holomorphic for w in the plane slit along the ray $[1, \infty)$, Theorem 5.4 of Chapter 2 guarantees that the integral representation above is holomorphic for z in the same slit plane, yielding an analytic continuation of the hypergeometric function. □

Exercise 10: An integral of the form

$$F(z) = \int_0^{\infty} f(t)t^{z-1} dt$$

is called a **Mellin transform**, and we shall write $\mathcal{M}(f)(z) = F(z)$. For example, the gamma function is the Mellin transform of the function e^{-t}.

(a) Prove that

$$\mathcal{M}(\cos)(z) = \int_0^{\infty} \cos(t)t^{z-1} dt = \Gamma(z)\cos\left(\pi\frac{z}{2}\right) \text{ for } 0, \mathrm{Re}(z) < 1,$$

and

$$\mathcal{M}(\sin)(z) = \int_0^{\infty} \sin(t)t^{z-1} dt = \Gamma(z)\sin\left(\pi\frac{z}{2}\right) \text{ for } 0 < \mathrm{Re}(z) < 1.$$

(b) Show that the second of the above identities is valid in the larger strip $-1 < \mathrm{Re}(z) < 1$, and that as a consequence, one has

$$\int_0^\infty \frac{\sin x}{x} dx = \frac{\pi}{2} \text{ and } \int_0^\infty \frac{\sin x}{x^{3/2}} dx = \sqrt{2\pi}.$$

Solution.

(a) Let C_R be the contour in the first quadrant bounded by the quarter-circles of radius R and $1/R$ and the axes. Let $f(w) = e^{-w}w^{z-1}$. In evaluating this integral we will use the fact that $|w^{z-1}| \leq C||w|^{z-1}|$; this follows from writing $z - 1 = x + iy$ and $w = Re^{i\theta}$, from which $|w^{z-1}| = R^x e^{-\theta y} \leq CR^x = C||w|^{z-1}|$ where $C = e^{\pi/2|\mathrm{Im}(z)|}$. The integral of f around the quarter-circle of radius $1/R$ tends to 0 since $|f(w)| \leq e^{-\mathrm{Re}(w)}C|w|^{\mathrm{Re}(z)-1} \leq Ce^{1/R}R^{1-\alpha}$ on this segment, where $\alpha = \mathrm{Re}(z)$. The length of the segment is $\frac{\pi}{2R}$, so by the ML estimate the integral is at most $\frac{C\pi e^{1/R}}{2R^\alpha}$, which tends to 0 as $R \to \infty$. The integral over the outer quarter-circle also tends to zero by the Jordan lemma. Since Stein and Shakarchi, annoyingly enough, never mention the Jordan lemma, I'll prove it from scratch all over again, just like I did a half-dozen times last quarter in the section on contour integrals. On the outer quarter circle, $|f(w)| \leq CR^{\alpha-1}|e^{-Re^{i\theta}}| = CR^{\alpha-1}e^{-R\cos\theta}$. Now $-\cos\theta \geq \frac{2}{\pi}\theta - 1$ for $0 \leq \theta \leq \frac{\pi}{2}$, so

$$\begin{aligned}
\left| \int_0^{\pi/2} f(Re^{i\theta})Re^{i\theta}d\theta \right| &\leq \int_0^{\pi/2} |f(Re^{i\theta})Re^{i\theta}|d\theta \\
&\leq CR^{\alpha-1}\int_0^{\pi/2} e^{-R\cos\theta}Rd\theta \\
&\leq CR^\alpha \int_0^{\pi/2} e^{R(2\theta/\pi-1)}d\theta \\
&= CR^\alpha e^{-R}\frac{\pi e^{2R\theta/\pi}}{2R}\Big|_0^{\pi/2} \\
&= CR^\alpha \frac{\pi}{2R}\left(1 - e^{-R}\right)
\end{aligned}$$

which tends to 0 as $R \to \infty$ since $\alpha < 1$. Now f is analytic on and inside the contour, so we're left with

$$\begin{aligned}
0 &= \int_0^\infty e^{-t}t^{z-1}dt + \int_\infty^0 e^{-iu}(iu)^{z-1}idu \\
\Rightarrow \Gamma(z) &= \int_0^\infty (\cos u - i\sin u)u^{z-1}i^z du \\
\Rightarrow i^{-z}\Gamma(z) &= \int_0^\infty (\cos u - i\sin u)u^{z-1}du \\
\Rightarrow \left(\cos\left(\frac{\pi z}{2}\right) - i\sin\left(\frac{\pi z}{2}\right)\right)\Gamma(z) &= \mathcal{M}(\cos)(z) - i\mathcal{M}(\sin)(z).
\end{aligned}$$

For real z, we can compare real and imaginary parts to conclude that

$$\mathcal{M}(\cos)(z) = \Gamma(z)\cos\left(\frac{\pi z}{2}\right) \text{ and } \mathcal{M}(\sin)(z) = \Gamma(z)\sin\left(\frac{\pi z}{2}\right).$$

By analytic continuation, these relations both hold for z in the strip $0 < \operatorname{Re}(z) < 1$.

(b) The right-hand side of the above equation for $\mathcal{M}(\sin)$ is analytic for $-1 < \operatorname{Re}(z) < 1$ because the zero of sine cancels the pole of Γ at the origin. The left-hand side is also analytic on this strip by Theorem 5.4 of chapter 2; the integral converges near 0 because $\sin t \approx t$ and converges at ∞ because $\operatorname{Re}(z) < 1$. Hence, by analytic continuation,

$$\mathcal{M}(\sin)(z) = \Gamma(z)\sin\left(\frac{\pi z}{2}\right) \text{ for } -1 < \operatorname{Re}(z) < 1.$$

Letting $z = 0$ we get

$$\mathcal{M}(\sin)(0) = \int_0^\infty \frac{\sin t}{t} dt = \Gamma(0)\sin\frac{\pi \cdot 0}{2}.$$

In order to evaluate the right-hand side we rearrange the functional equation of Γ to read

$$\frac{\pi}{\Gamma(1-s)} = \Gamma(s)\sin(\pi s) = 2\Gamma(s)\sin(\pi s/2)\cos(\pi s/2) \Rightarrow \Gamma(s)\sin(\pi s/2) = \frac{\pi}{2\Gamma(1-s)\cos(\pi s/2)}.$$

This equals $\frac{\pi}{2}$ when $s = 0$, so $\int_0^\infty \frac{\sin t}{t} dt = \frac{\pi}{2}$.
Letting $z = -1/2$ in the Mellin transform,

$$\int_0^\infty \frac{\sin t}{t^{3/2}} dt = \Gamma\left(\frac{1}{2}\right)\sin\left(\frac{-\pi}{4}\right) = (-2\sqrt{\pi})\left(-\frac{1}{\sqrt{2}}\right) = \sqrt{2\pi}.$$

$\square$

Exercise 13: Prove that

$$\frac{d^2 \log\Gamma(s)}{ds^2} = \sum_{n=0}^\infty \frac{1}{(s+n)^2}$$

whenever s is a positive number. Show that if the left-hand side is interpreted as $(\Gamma'/\Gamma)'$, then the above formula also holds for all complex numbers s with $s \neq 0, -1, -2, \ldots$.

Solution. For positive s we can take the logarithm of the Hadamard factorization

$$\frac{1}{\Gamma(s)} = se^{\gamma s}\prod_{n=1}^\infty \left(1 + \frac{s}{n}\right)e^{-s/n}$$

to obtain

$$-\log\Gamma(s) = \log s + \gamma s + \sum_{n=1}^\infty \left(\log\left(1+\frac{s}{n}\right) - \frac{s}{n}\right).$$

Differentiating,

$$-\frac{d\log\Gamma(s)}{ds} = \frac{1}{s} + \gamma + \sum_{n=1}^\infty \left(\frac{1}{n+s} - \frac{1}{n}\right)$$

where the termwise differentiation is justified because the differentiated sum converges uniformly on compact intervals for s. Differentiating again,

$$-\frac{d^2\log\Gamma(s)}{ds^2} = -\frac{1}{s^2} + \sum_{n=1}^\infty \frac{-1}{(n+s)^2} = -\sum_{n=0}^\infty \frac{1}{(n+s)^2}.$$

The right-hand side defines an analytic function of s on the region $s \neq 0, -1, -2, \ldots$ because the sum converges uniformly on compact subsets of this region. Moreover, for positive s the second derivative on the left is equal to $(\Gamma'/\Gamma)'$, which is analytic on the same region since Γ is analytic and nonzero there. Hence the above relation holds throughout this region by analytic continuation. $\square$

Exercise 14: This exercise gives an asymptotic formula for $\log n!$. A more refined formula for $\Gamma(s)$ as $s \to \infty$ (Stirling's formula) is given in Appendix A.

(a) Show that

$$\frac{d}{dx}\int_x^{x+1} \log\Gamma(t)dt = \log x, \text{ for } x > 0,$$

and as a result

$$\int_x^{x+1} \log\Gamma(t)dt = x\log x - x + c.$$

(b) Show as a consequence that $\log\Gamma(n) \sim n\log n$ as $n \to \infty$. In fact, prove that $\log\Gamma(n) \sim n\log n + O(n)$ as $n \to \infty$.

Solution.

(a) By the Fundamental Theorem of Calculus,

$$\frac{d}{dx}\int_x^{x+1} \log\Gamma(t)dt = \log\Gamma(x+1) - \log\Gamma(x) = \log\frac{\Gamma(x+1)}{\Gamma(x)} = \log x.$$

Integrating both sides with respect to x,

$$\int_x^{x+1} \log\Gamma(t)dt = x\log x - x + c.$$

(b) Since $\Gamma(t)$ is increasing for $t \geq 2$,

$$\log\Gamma(n) \leq \int_n^{n+1} \log\Gamma(t)dt \leq \log\Gamma(n+1) = \log n + \log\Gamma(n)$$
$$\Rightarrow (n-1)\log n - n + c \leq \log\Gamma(n) \leq n\log n - n + c.$$

Thus, $\log\Gamma(n) = n\log n - n + o(\log n)$.

$\square$

Exercise 15: Prove that for $\mathrm{Re}(s) > 1$,

$$\zeta(s) = \frac{1}{\Gamma(s)}\int_0^\infty \frac{x^{s-1}}{e^x - 1}dx.$$

Solution. Using the geometric series $\frac{1}{e^x-1} = e^{-x}\frac{1}{1-e^{-x}} = \sum_{n=1}^{\infty} e^{-nx}$, we have

$$\begin{aligned}
\int_0^{\infty} \frac{x^{s-1}}{e^x-1}dx &= \int_0^{\infty} x^{s-1}\sum_{n=1}^{\infty} e^{-nx}dx \\
&= \sum_{n=1}^{\infty}\int_0^{\infty} x^{s-1}e^{-nx}dx \\
&= \sum_{n=1}^{\infty}\int_0^{\infty} \left(\frac{t}{n}\right)^{s-1} e^{-t}\frac{dt}{n} \\
&= \sum_{n=1}^{\infty}\frac{1}{n^s}\int_0^{\infty} t^{s-1}e^{-t}dt \\
&= \Gamma(s)\zeta(s).
\end{aligned}$$

Here the sum and integral can be interchanged because all terms are non-negative. □

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Exercise 1: Suppose that $\{a_n\}_{n=1}^{\infty}$ is a sequence of real numbers such that the partial sums

$$A_n = a_1 + \cdots + a_n$$

are bounded. Prove that the Dirichlet series

$$\sum_{n=1}^{\infty}\frac{a_n}{n^s}$$

converges for $\mathrm{Re}(s) > 0$ and defines a holomorphic function in this half-plane.

Solution. Using summation by parts,

$$\sum_{n=1}^{N}\frac{a_n}{n^s} = \frac{A_N}{N^s} - \sum_{n=1}^{N} A_n\left(\frac{1}{(n+1)^s} - \frac{1}{n^s}\right).$$

Taking the limit as $N \to \infty$, the first term on the right vanishes and

$$\sum_{n=1}^{\infty}\frac{a_n}{n^s} = -\sum_{n=1}^{\infty} A_n\left(\frac{1}{(n+1)^s} - \frac{1}{n^s}\right).$$

To prove that this converges, we note that by hypothesis $|A_n| \le M$ for some constant M. Now

$$\begin{aligned}
\left|\frac{1}{(n+1)^s} - \frac{1}{n^s}\right| &= \left|\int_n^{n+1}\frac{-1}{st^{s+1}}dt\right| \\
&\le |(n+1)-n|\max_{x\in(n,n+1)}\left|\frac{1}{st^{s+1}}\right| \\
&\le \frac{1}{\sigma n^{\sigma+1}}
\end{aligned}$$

where $\sigma = \mathrm{Re}(s)$. Thus, the tails of the series are dominated by

$$\sum_{n=N}^{\infty} \frac{M}{\sigma n^{\sigma+1}}$$

which converges, so the series converges. In addition, this shows that the convergence is uniform on closed half-planes $\sigma \geq \sigma_0 > 0$, so the series defines a holomorphic function on the right half-plane. $\square$

Exercise 2: The following links the multiplication of Dirichlet series with the divisibility properties of their coefficients.

(a) Show that if $\{a_m\}$ and $\{b_k\}$ are two bounded sequences of complex numbers, then

$$\left(\sum_{m=1}^{\infty} \frac{a_m}{m^s}\right)\left(\sum_{k=1}^{\infty} \frac{b_k}{k^s}\right) = \sum_{n=1}^{\infty} \frac{c_n}{n^s} \text{ where } c_n = \sum_{mk=n} a_m b_k.$$

The above series converges absolutely when $\mathrm{Re}(s) > 1$.

(b) Prove as a consequence that one has

$$(\zeta(s))^2 = \sum_{n=1}^{\infty} \frac{d(n)}{n^s} \text{ and } \zeta(s)\zeta(s-a) = \sum_{n=1}^{\infty} \frac{\sigma_a(n)}{n^s}$$

for $\mathrm{Re}(s) > 1$ and $\mathrm{Re}(s-a) > 1$, respectively. Here $d(n)$ equals the number of divisors of n, and $\sigma_a(n)$ is the sum of the ath powers of divisors of n. In particular, one has $\sigma_0(n) = d(n)$.

Solution.

(a) For $\sigma = \mathrm{Re}(s) > 1$, the two sums on the left both converge absolutely since

$$\sum_{m=1}^{\infty} \left|\frac{a_m}{m^s}\right| \leq \sum_{m=1}^{\infty} \frac{A}{m^\sigma} < \infty$$

where A is a bound for $|a_m|$. Thus, Fubini's theorem allows us to write the product as the double sum

$$\sum_{m,k} \frac{a_m}{m^s}\frac{b_k}{k^s} = \sum_{m,k} \frac{a_m b_k}{(mk)^s}$$

where the terms can be summed in any order. In particular, we can group them according to the product mk to obtain

$$\sum_{n=1}^{\infty} \sum_{\substack{m,k \\ mk=n}} \frac{a_m b_k}{n^s} = \sum_{n=1}^{\infty} \frac{c_n}{n^s}.$$

(b) If we let $a_m = b_k = 1$ for all k, m above, then $c_{n,k} = \sum_{mk=n} 1 = d(n)$ and we have

$$(\zeta(s))^2 = \left(\sum_{m=1}^{\infty} \frac{1}{m^s}\right)\left(\sum_{k=1}^{\infty} \frac{1}{k^s}\right) = \sum_{n=1}^{\infty} \frac{d(n)}{n^s}.$$

Assuming $\mathrm{Re}(a) > 0$ (which presumably was intended in the problem statement) and $\mathrm{Re}(s-a) > 1$, we can replace s with $s-a$ in part (a) and let $a_m = m^{-a}$, $b_k = 1$. The result is

$$\zeta(s)\zeta(s-a) = \left(\sum_{m=1}^{\infty} \frac{m^{-a}}{m^{s-a}}\right)\left(\sum_{k=1}^{\infty} \frac{1}{k^{s-a}}\right) = \sum_{n=1}^{\infty} \frac{c_n}{n^{s-a}} = \sum_{n=1}^{\infty} \frac{n^a c_n}{n^s}$$

where $c_n = \sum_{mk=n} m^{-a} = \frac{1}{n^a}\sum_{mk=n} k^a = \sigma_a(n)$.

□

Exercise 3: In line with the previous exercise, we consider the Dirichlet series for $1/\zeta$.

(a) Prove that for $\mathrm{Re}(s) > 1$,

$$\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s},$$

where $\mu(n)$ is the **Möbius function** defined by

$$\mu(n) = \begin{cases} 1 & \text{if } n = 1, \\ (-1)^k & \text{if } n = p_1 \cdots p_k, \text{ and the } p_j \text{ are distinct primes,} \\ 0 & \text{otherwise.} \end{cases}$$

Note that $\mu(nm) = \mu(n)\mu(m)$ whenever n and m are relatively prime.

(b) Show that

$$\sum_{k|n} \mu(k) = \begin{cases} 1 & \text{if } n = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Solution.

(a) Consider the finite product

$$\prod_{n=1}^{N}\left(1 - \frac{1}{p_n^s}\right).$$

Applying the distributive law, this is equal to

$$\sum_{n=1}^{\infty} \frac{\mu_N(n)}{n^s} \text{ where } \mu_N(n) = \begin{cases} 1 & n = 1 \\ (-1)^k & n = p_1 \cdots p_k \text{ and } 1, 2, \ldots, k \leq N \\ 0 & \text{else.} \end{cases}$$

Note that $\mu_N(n) = \mu(n)\chi_N(n)$ where $\chi_N(n) = 1$ if n has no prime factors larger than p_N, and 0 otherwise. Hence, this sum is a (rearranged) partial sum of $\sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}$. Because this latter sum is absolutely convergent for $\sigma > 1$, we can take the limit as $N \to \infty$ to obtain

$$\frac{1}{\zeta(s)} = \prod_{n=1}^{\infty}\left(1 - \frac{1}{p_n^s}\right) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}.$$

An alternate proof uses the fact that $\sum_{d|n} \mu(d) = 0$ for $n > 1$. One way to establish this is as follows: Let $p_1, \ldots, p_k$ be the distinct prime factors of n. For each $j = 1, \dot{,} k$, there will be $\binom{k}{j}$ squarefree divisors of

n which have j distinct prime factors, and $\mu(d) = (-1)^j$ if d is any of these divisors. Hence the sum of μ over all divisors is

$$\sum_{j=0}^{k} \binom{k}{j} (-1)^j = (1-1)^k = 0$$

by the binomial theorem. Given this, we can use the previous exercise to write

$$\left(\sum_{m=1}^{\infty} \frac{1}{m^s}\right)\left(\sum_{k=1}^{\infty} \frac{\mu(k)}{k^s}\right) = \sum_{n=1}^{\infty} \frac{c_n}{n^s}$$

where

$$c_n = \sum_{mk=n} \mu(k) = \begin{cases} 1 & n = 1 \\ 0 & \text{else.} \end{cases}$$

Thus,

$$\zeta(s) \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = 1 \Rightarrow \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = \frac{1}{\zeta(s)}.$$

(b) This was already proved in part (a) using the binomial theorem. Alternatively, if (a) was established using the Euler product, then multiplying the series for ζ and $1/\zeta$ using Exercise 2 yields part (b) as a result. Thus, one can use (b) (proved using the binomial theorem) and Exercise 2 to establish (a), or one can use (a) (proved using the Euler product) and Exercise 2 to establish (b).

□

Exercise 5: Consider the following function

$$\bar{\zeta}(s) = 1 - \frac{1}{2^s} + \frac{1}{3^s} - \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^s}.$$

(a) Prove that the series defining $\bar{\zeta}(s)$ converges for $\mathrm{Re}(s) > 0$ and defines a holomorphic function in that half-plane.
(b) Show that for $s > 1$ one has $\bar{\zeta}(s) = (1 - 2^{1-s})\zeta(s)$.
(c) Conclude, since $\bar{\zeta}$ is given as an alternating series, that ζ has no zeros on the segment $0 < \sigma < 1$. Extend this last assertion to $\sigma = 0$ by using the functional equation.

Solution.
(a) We rewrite the series as

$$\bar{\zeta}(s) = \sum_{n=1}^{\infty} \left(\frac{1}{(2n-1)^s} - \frac{1}{(2n)^s}\right).$$

Now

$$\begin{aligned}\left|\frac{1}{(2n-1)^s} - \frac{1}{(2n)^s}\right| &= \left|\int_{2n-1}^{2n} \frac{d}{dt} t^{-s} dt\right| \\ &\leq \int_{2n-1}^{2n} \left|(-s)t^{-s-1}\right| dt \\ &= |s| \int_{2n-1}^{2n} t^{-\sigma-1} dt \\ &\leq |s|(2n-1)^{-\sigma-1}.\end{aligned}$$

Since

$$\sum_{n=1}^{\infty} |s|(2n-1)^{-\sigma-1}$$

converges for $\sigma > 0$ and converges uniformly on closed sub-half-planes $\sigma \geq \sigma_0 > 0$, the series defining $\bar{\zeta}$ defines a holomorphic function on the right half-plane. (To complete the proof we should also note that the individual terms $\frac{(-1)^{n+1}}{n^s}$ tend to 0.)

(b) For $s > 1$,

$$2^{1-s}\zeta(s) = 2\sum_{n=1}^{\infty} 2^{-s} n^{-s} = 2 \sum_{n \text{ even}} \frac{1}{n^s}$$

and the absolute converge of the series for ζ on closed half-planes $\sigma \geq \sigma_0 > 1$ allows us to rearrange terms and obtain

$$(1 - 2^{1-s})\zeta(s) = \sum_n \frac{1}{n^s} - 2\sum_{n \text{ even}} \frac{1}{n^s} = \sum_n \frac{(-1)^{n+1}}{n^s} = \bar{\zeta}(s).$$

(c) I assume the problem statement intends to say $0 < s < 1$ (i.e. s real), since otherwise the zeros of ζ in the critical strip would create a contradiction. For real s the terms

$$\left(\frac{1}{(2n-1)^s} - \frac{1}{(2n)^s}\right)$$

are all strictly positive, so the sum cannot be zero. Taking the limit of both sides of the equation $\bar{\zeta}(s) = (1 - 2^{1-s})$ as $s \to 0$, the left-hand side becomes the alternating series

$$\bar{\zeta}(0) = 1 - \frac{1}{2} + \frac{1}{3} - \cdots = \log 2,$$

so the right-hand side cannot be zero.

□

Exercise 10: In the theory of primes, a better approximation to $\pi(x)$ (instead of $x/\log x$) turns out to be $\mathrm{Li}(x)$ defined by

$$\mathrm{Li}(x) = \int_2^x \frac{dt}{\log t}.$$

(a) Prove that

$$\mathrm{Li}(x) = \frac{x}{\log x} + O\left(\frac{x}{(\log x)^2}\right) \quad \text{as } x \to \infty,$$

and that as a consequence

$$\pi(x) \sim \mathrm{Li}(x) \quad \text{as } x \to \infty.$$

(b) Refine the previous analysis by showing that for every integer $N > 0$ one has the following asymptotic expansion

$$\mathrm{Li}(x) = \frac{x}{\log x} + \frac{x}{(\log x)^2} + 2\frac{x}{(\log x)^3} + \cdots + (N-1)!\frac{x}{(\log x)^N} + O\left(\frac{x}{(\log x)^{N+1}}\right)$$

Solution.

(a) Substituting $u = \frac{1}{\log t}$ and $v = t$ in the definition of $\mathrm{Li}(x)$ and integrating by parts,

$$\mathrm{Li}(x) = \int_2^x \frac{dt}{\log t} = \frac{t}{\log t}\bigg|_2^x + \int_2^x \frac{dt}{(\log t)^2} = \frac{x}{\log x} - \frac{2}{\log 2} + \int_2^x \frac{dt}{(\log t)^2}.$$

We will estimate the latter integral in two pieces, one from 2 to $\sqrt{x}$ and one from $\sqrt{x}$ to x. The graphs of the functions $f(t) = (\log t)^2$ and $g(t) = \frac{(\log\sqrt{x})^2}{\sqrt{x}-1}(t-1)$ both pass through the points $(1,0)$ and $(\sqrt{x}, (\log\sqrt{x})^2$; by concavity, $f \geq g$ for $1 \leq t \leq \sqrt{x}$. Taking reciprocals,

$$\frac{1}{(\log t)^2} \leq \frac{(\sqrt{x}-1)}{(\log\sqrt{x})^2}\frac{1}{t-1}$$

$$\Rightarrow \int_2^x \frac{1}{(\log t)^2}dt \leq \int_2^x \frac{(\sqrt{x}-1)}{(\log\sqrt{x})^2}\frac{1}{t-1}dt = \frac{(\sqrt{x}-1)\log(\sqrt{x}-1)}{(\log\sqrt{x})^2} \sim \frac{\sqrt{x}}{\log\sqrt{x}} = O\left(\frac{x}{(\log x)^2}\right).$$

For the integral from $\sqrt{x}$ to x it suffices to approximate the integrand by a constant:

$$\int_{\sqrt{x}}^x \frac{dt}{(\log t)^2} \leq (x - \sqrt{x})\frac{1}{(\log\sqrt{x})^2} \sim \frac{x}{\left(\frac{1}{2}\log x\right)^2} = O\left(\frac{x}{(\log x)^2}\right).$$

Putting the pieces together, we have

$$\int_2^x \frac{dt}{\log t} = \frac{x}{\log x} - \frac{2}{\log 2} + O\left(\frac{x}{(\log x)^2}\right) = \frac{x}{\log x} + O\left(\frac{x}{(\log x)^2}\right).$$

(b) Integrating by parts with the same substitution as above, we have more generally

$$\int_2^x \frac{dt}{(\log t)^2} = \frac{t}{(\log t)^k}\bigg|_2^x + k\int_2^x \frac{dt}{(\log t)^{k+1}}.$$

This plus an easy induction yields

$$\mathrm{Li}(x) = \sum_{k=1}^N (k-1)!\frac{t}{(\log t)^k}\bigg|_2^x + \int_2^x \frac{dt}{(\log t)^{N+1}}$$

for each $N = 1, 2, \ldots$. Evaluation at the lower terms yields a constant, so we can write this as

$$\mathrm{Li}(x) = C_N + \sum_{k=1}^N (k-1)!\frac{x}{(\log x)^k} + \int_2^x \frac{dt}{(\log t)^{N+1}}.$$

We can estimate this integral in the same manner as before. By concavity,

$$(\log t)^{N+1} \geq \frac{(\log\sqrt{x})^{N+1}}{\sqrt{x}-1}(t-1) \quad \text{for } 1 \leq t \leq \sqrt{x}$$

$$\Rightarrow \frac{1}{(\log t)^{N+1}} \leq \frac{\sqrt{x}-1}{(\log\sqrt{x})^{N+1}}\frac{1}{t-1} \quad \text{for } 1 \leq t \leq \sqrt{x}$$

$$\Rightarrow \int_2^{\sqrt{x}} \frac{dt}{(\log t)^{N+1}} \leq \frac{\sqrt{x}-1}{(\log\sqrt{x})^{N+1}} \int_2^{\sqrt{x}} \frac{dt}{t-1} = \frac{(\sqrt{x}-1)\log(\sqrt{x}-1)}{(\log\sqrt{x})^{N+1}} = O\left(\frac{x}{(\log x)^{N+1}}\right).$$

Also,

$$\int_{\sqrt{x}}^{x} \frac{dt}{(\log t)^{N+1}} \leq (x-\sqrt{x})\frac{1}{(\log\sqrt{x})^{N+1}} = O\left(\frac{x}{(\log x)^{N+1}}\right)$$

so that

$$\int_2^{x} \frac{dt}{(\log t)^{N+1}} = O\left(\frac{x}{(\log x)^{N+1}}\right) \Rightarrow \mathrm{Li}(x) = \sum_{k=1}^{N}(k-1)!\frac{x}{(\log x)^k} + O\left(\frac{x}{(\log x)^{N+1}}\right).$$

□

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Exercise 5: Prove that $f(z) = -\frac{1}{2}(z + 1/z)$ is a conformal map from the half-disc $\{z = x + iy : |z| < 1, y > 0\}$ to the upper half-plane.

Solution. Clearly f is holomorphic on the upper half-disc U. To show it is injective, consider that if $f(z) = f(z') = w$, then z and z' are both roots of the equation $t^2 + 2tw + 1 = 0$; the product of the roots of this equation is 1, so only one of the roots can have norm less than 1. So f is injective on U. It maps U into the upper half-plane $\mathcal{H}$ because if $z \in U$, then

$$\mathrm{Im}(-(z+1/z)) = -\mathrm{Im}(z) - \mathrm{Im}(1/z) = -\mathrm{Im}(z) - \frac{1}{|z}\mathrm{Im}(z) = \left(\frac{1}{|z|} - 1\right)\mathrm{Im}(z),$$

and since $|z| < 1$ this is positive. Finally, f is surjective from U onto $\mathcal{H}$ because for any $w \in \mathcal{H}$, the equation $w = -\frac{z+1/z}{2}$, equivalent to $z^2+2zw+1$, has two roots with product 1, so one is inside the disc and one outside. (They cannot both be on the disc because $z + 1/z$ is real for $|z| = 1$.) Let z_0 be the root inside the disc. Then

$$\mathrm{Im}(w) = \left(\frac{1}{|z_0|} - 1\right)\mathrm{Im}(z_0) > 0 \Rightarrow \mathrm{Im}(z_0) > 0$$

so z_0 is in U. □

Exercise 8: Find a harmonic function u in the open first quadrant that extends continuously up to the boundary except at the points 0 and 1, and that takes on the following boundary values: $u(x, y) = 1$ on the half-lines $\{y = 0, x > 1\}$ and $\{x = 0, y > 0\}$, and $u(x, y) = 0$ on the segment $\{0 < x < 1, y = 0\}$.

Solution. We follow the outline given in Figure 11. Define $F_1 : FQ \to UHD$, where FQ is the (open) first quadrant and UHD the upper half-disc, by $F_1(z) = \frac{z-1}{z+1}$. This is the inverse of the map in Example 3 on page 210. The boundary is mapped as follows: $\{y = 0, x > 1\}$ is mapped to the positive real axis, $\{0 < x < 1, y = 0\}$ to the negative real axis, and $\{x = 0, y > 0\}$ to the semicircular part of the boundary. Next, define $F_2 : UHD \to LHS$, where LHS is the left half-strip $\{x < 0, 0 < y < \pi\}$ by $F_2(z) = \log z$, using the principal branch. Here the positive real axis is mapped to the negative real axis, the negative real axis to the line $\{x < 0, y = i\pi\}$, and the semicircular part to the segment of the imaginary axis from 0 to $i\pi$. Next, we take the map $F_3 : LHS \to UHS$, where UHS is the upper half-strip $\{-\pi/2 < x < \pi/2, y > 0\}$, where $F_3(z) = \frac{z}{i} - \frac{\pi}{2}$. This takes the imaginary segment to the real segment, upper and lower boundaries of the strip to the left and right boundaries respectively. Next, define $F_4 : UHS \to U$, where U is the upper half-plane, by $F_4(z) = \frac{1+\sin(z)}{2}$, which is conformal on this domain by Example 8 on page 212. This takes the real segment to the segment between 0 and 1, and the left and right boundaries to the rays $\{x < 0, y = 0\}$ and $\{x > 1, y = 0\}$ respectively. Finally, let $F_5 : U \to U$ be defined by $F_5(z) = z - 1$. Now since $\frac{1}{\pi}\arg(z)$ is harmonic on U and equals 0 on the positive real axis and 1 on the negative real axis, the composition $z \mapsto \frac{1}{\pi}\arg(F_5(F_4(F_3(F_2(F_1(z))))))$ is harmonic on the first quadrant and has the desired boundary values. □

Exercise 9: Prove that the function u defined by

$$u(x, y) = \operatorname{Re}\left(\frac{i+z}{i-z}\right) \text{ and } u(0, 1) = 0$$

is harmonic in the unit disc and vanishes on its boundary. Note that u is not bounded in $\mathbb{D}$.

Solution. The real part of an analytic function is harmonic, and $\frac{i+z}{i-z}$ is analytic on the open unit disc, so u is harmonic in $\mathbb{D}$. Moreover, on the boundary points other than $(0, 1)$, write $z = \cos\theta + i\sin\theta$; then

$$\frac{i+z}{i-z} = \frac{\cos\theta + i(1+\sin\theta)}{-\cos\theta + i(1-\sin\theta)} \cdot \frac{-cos\theta + i(\sin\theta - 1)}{-cos\theta + i(\sin\theta - 1)} = \frac{-2i\cos\theta}{\cos^2\theta + (1-\sin\theta)^2}$$

is pure imaginary, so its real part is zero. □

Exercise 10: Let $F : \mathcal{H} \to \mathbb{C}$ be a holomorphic function that satisfies

$$|F(z)| \leq 1 \text{ and } F(i) = 0.$$

Prove that

$$|F(z)| \leq \left|\frac{z-i}{z+i}\right| \text{ for all } z \in \mathcal{H}.$$

Solution. Define $G : \mathbb{D} \to \mathbb{D}$ by

$$G(w) = F\left(i\frac{1-w}{1+w}\right).$$

Then G is holomorphic and $G(0) = F(i) = 0$. By the Schwarz lemma, $|G(w)| \le |w|$ for all $w \in \mathbb{D}$. Then for any $z \in \mathcal{H}$,

$$|F(z)| = \left| G\left(\frac{i-z}{i+z}\right)\right| \le \left|\frac{z-i}{z+i}\right|.$$

$\square$

Exercise 11: Show that if $f : D(0,R) \to \mathbb{C}$ is holomorphic, with $|f(z)| \le M$ for some $M > 0$, then

$$\left|\frac{f(z)-f(0)}{M^2 - \overline{f(0)}f(z)}\right| \le \frac{|z|}{MR}.$$

Solution. For $z \in \mathbb{D}$, let $g(z) = \frac{f(Rz)}{M}$. Since $Rz \in D(0,R)$, $|f(Rz)| \le M$ so $g(z) \in \mathbb{D}$. Thus $g : \mathbb{D} \to \mathbb{D}$ and is holomorphic. Let $\alpha = g(0) = \frac{f(0)}{M}$. Then $\psi_\alpha \circ g : \mathbb{D} \to \mathbb{D}$ satisfies $\psi_\alpha(g(0)) = 0$, where $\psi_\alpha(w) = \frac{\alpha - w}{1-\bar{\alpha}w}$. By the Schwarz lemma,

$$\begin{aligned}
|\psi_\alpha \circ g(\zeta)| &\le |\zeta| \qquad\qquad (\zeta \in \mathbb{D})\\
\Rightarrow \left|\frac{\alpha - g(\zeta)}{1-\bar{\alpha}g(\zeta)}\right| &\le |\zeta|\\
\Rightarrow \left|\frac{\frac{f(0)}{M} - \frac{f(R\zeta)}{M}}{1 - \frac{\overline{f(0)}}{M}\frac{f(R\zeta)}{M}}\right| &\le |\zeta|\\
\Rightarrow M\left|\frac{f(0)-f(R\zeta)}{M^2 - \overline{f(0)}f(R\zeta)}\right| &\le |\zeta|\\
\Rightarrow \left|\frac{f(0)-f(z)}{M^2 - \overline{f(0)}f(z)}\right| &\le \frac{|z|}{MR}
\end{aligned}$$

where $z = R\zeta \in D(0,R)$. $\square$

Exercise 13: The **pseudo-hyperbolic distance** between two points $z, w \in \mathbb{D}$ is defined by

$$\rho(z,w) = \left|\frac{z-w}{1-\bar{w}z}\right|.$$

(a) Prove that if $f : \mathbb{D} \to \mathbb{D}$ is holomorphic, then

$$\rho(f(z), f(w)) \le \rho(z,w) \text{ for all } z, w \in \mathbb{D}.$$

Moreover, prove that if f is an automorphism of $\mathbb{D}$ then f preserves the pseudo-hyperbolic distance

$$\rho(f(z), f(w)) = \rho(z,w) \text{ for all } z, w \in D.$$

(Hint.)

(b) Prove that

$$\frac{|f'(z)|}{1-|f(z)|^2} \le \frac{1}{1-|z|^2} \text{ for all } z \in \mathbb{D}.$$

Solution.

(a) Let

$$\psi_\alpha(z) = \frac{z-\alpha}{1-\bar{\alpha}z}.$$

Then it is easy to check that ψ_α is an automorphism of $\mathbb{D}$; in fact, its inverse can be explicitly computed as

$$\psi_\alpha^{-1}(z) = \frac{z+\alpha}{1+\bar{\alpha}z}.$$

Now let $g = \psi_{f(w)} \circ f \circ \psi_w^{-1}$. Then

$$g(0) = \psi_{f(w)}\left(f\left(\psi_w^{-1}(0)\right)\right) = \psi_{f(w)}\left(f\left(w\right)\right) = 0$$

and since g is the composition of three functions which map $\mathbb{D}$ into $\mathbb{D}$, so it also maps $\mathbb{D}$ into $\mathbb{D}$. By the Schwarz Lemma, $|g(y)| \leq |y|$ for all $y \in \mathbb{D}$. In particular, if $y = \psi_w(z)$ we have

$$\begin{aligned}
|g(y)| &\leq |y| \\
\left|\psi_{f(w)}(f(\psi_w^{-1}(y)))\right| &\leq |\psi_w(z)| \\
\left|\psi_{f(w)}(f(z))\right| &\leq \left|\frac{z-w}{1-\bar{w}z}\right| \\
\left|\frac{f(z)-f(w)}{1-\overline{f(w)}f(z)}\right| &\leq \left|\frac{z-w}{1-\bar{w}z}\right| \\
\rho(f(z), f(w)) &\leq \rho(z,w)
\end{aligned}$$

as desired. Moreover, if f is an automorphism of $\mathbb{D}$ then by Theorem 2.2,

$$f(z) = e^{i\theta}\frac{\alpha - z}{1-\bar{\alpha}z}$$

for some $\theta \in \mathbb{R}$ and $\alpha \in \mathbb{D}$. Then

$$\begin{aligned}
\rho(f(z), f(w)) &= \left|\frac{f(z)-f(w)}{1-\overline{f(w)}f(z)}\right| \\
&= \frac{\left|e^{i\theta}\left(\frac{\alpha-z}{1-\bar{\alpha}z} - \frac{\alpha-w}{1-\bar{\alpha}w}\right)\right|}{\left|1-\overline{e^{i\theta}\frac{\alpha-w}{1-\bar{\alpha}w}}e^{i\theta}\frac{\alpha-z}{1-\bar{\alpha}z}\right|} \\
&= \frac{\left|\frac{(z-w)(1-|\alpha|^2)}{(1-\bar{\alpha}z)(1-\bar{\alpha}w)}\right|}{\left|\frac{(1-|\alpha|^2)(1-\bar{w}z)}{(1-\alpha\bar{w})(1-\bar{\alpha}z)}\right|} \\
&= \frac{|z-w|}{|1-\bar{w}z|} \\
&= \rho(z,w)
\end{aligned}$$

so f preserves pseudo-hyperbolic distance.

(b) By a simple rearrangement,

$$\begin{aligned}
&\rho(f(w), f(z)) \le \rho(w, z) \\
\Rightarrow\ & \frac{|f(w) - f(z)|}{|1 - \overline{f(w)}f(z)} \le \frac{|w - z|}{1 - \bar{w}z} \\
\Rightarrow\ & \left|\frac{f(w) - f(z)}{w - z}\right| \le \frac{|1 - \overline{f(w)}f(z)|}{1 - \bar{w}z|}.
\end{aligned}$$

Taking the limit as $w \to z$, we have

$$|f'(z)| \le \frac{1 - |f(z)|^2}{1 - |z|^2} \Rightarrow \frac{|f'(z)|}{1 - |f(z)|^2} \le \frac{1}{1 - |z|^2}.$$

□

Exercise 14: Prove that all conformal mappings from the upper half-plane $\mathcal{H}$ to the unit disc $\mathbb{D}$ take the form

$$e^{i\theta}\frac{z - \beta}{z - \bar{\beta}}, \quad \theta \in \mathbb{R} \text{ and } \beta \in \mathcal{H}.$$

Solution. Let $g : \mathcal{H} \to \mathbb{D}$ be a conformal mapping. Let $\phi : \mathbb{D} \to \mathcal{H}$ be the conformal mapping defined by

$$z = \phi(w) = i\frac{1 - w}{1 + w}.$$

As shown in section 1.1, ϕ is a conformal mapping with inverse $w = \psi(z) = \frac{i-z}{i+z}$. Then $g \circ \phi : \mathbb{D} \to \mathbb{D}$ is a conformal automorphism of the disc, so by Theorem 2.2 there exist $\mu \in \mathbb{R}$ and $\alpha \in \mathbb{D}$ such that

$$\begin{aligned}
g(z) &= g\left(i\frac{1 - w}{1 + w}\right) \\
&= e^{i\mu}\frac{\alpha - w}{1 - \bar{\alpha}w} \\
&= e^{i\mu}\frac{\alpha - \frac{i-z}{i+z}}{1 - \bar{\alpha}\frac{i-z}{i+z}} \\
&= e^{i\mu}\frac{z(1 + \alpha) - i(1 - \alpha)}{z(1 + \alpha) - i(1 - \bar{\alpha})} \\
&= e^{i\mu}\frac{z - i\frac{1-\alpha}{1+\alpha}}{z\frac{1+\bar{\alpha}}{1+\alpha} + i\frac{1-\bar{\alpha}}{1+\alpha}} \\
&= e^{i\mu}\frac{z - \beta}{e^{i\gamma}(z - \bar{\beta})} \\
&= e^{i\theta}\frac{z - \beta}{z - \bar{\beta}}
\end{aligned}$$

where $\theta = \mu - \gamma \in \mathbb{R}$, $e^{i\gamma} = \frac{1+\bar{\alpha}}{1+\alpha}$ has unit length because conjugation preserves norm, and $\beta = i\frac{1-\alpha}{1+\alpha} \in \mathcal{H}$ because $\beta = \phi(\alpha)$ and $\alpha \in \mathbb{D}$. □

Exercise 15: Here are two properties enjoyed by automorpisms of the upper half-plane.

(a) Suppose Φ is an automorphism of $\mathcal{H}$ that fixes three distinct points on the real axis. Then Φ is the identity.

(b) Suppose (x_1, x_2, x_3) and (y_1, y_2, y_3) are two pairs of three distinct points on the real axis with

$$x_1 < x_2 < x_3 \text{ and } y_1 < y_2 < y_3.$$

Prove that there exists (a unique) automorphism Φ of $\mathcal{H}$ so that $\Phi(x_j) = y_j$, $j = 1, 2, 3$. The same conclusion holds if $y_3 < y_1 < y_2$ or $y_2 < y_3 < y_1$.

Solution.

- (a) By Theorem 2.4, there exist $a, b, c, d \in \mathbb{R}$ with $ad - bc = 1$ and

$$\Phi(z) = \frac{az + b}{cz + d}.$$

Suppose Φ fixes $x \in \mathbb{R}$. Then

$$x = \frac{ax + b}{cx + d} \Rightarrow cx^2 + (d - a)x - b = 0.$$

If $c \neq 0$ this equation has at most 2 distinct solutions; if $c = 0$ but $a \neq d$ it has only one. For it to have three or more both of these conditions must fail, so $c = 0$ and $a = d$; the equation then becomes $b = 0$, and the condition $ad - bc = 1$ then implies $a = d = \pm 1$, so $\Phi(z) = \frac{\pm z}{\pm 1} = z$.

(b) Let x_i and y_i be so chosen. The system of equations

$$\frac{ax_i + b}{cx_i + d} = y_i \Leftrightarrow ax_i + b = cx_i y_i + dy_i, \qquad i = 1, 2, 3$$

can be written as the vector equation

$$a\vec{x} - c\vec{xy} - d\vec{y} = b \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix},$$

where $\vec{xy} = \begin{pmatrix} x_1 y_1 \\ x_2 y_2 \\ x_3 y_3 \end{pmatrix}$. We want to show that this equation has a unique solution, up to multiplying a, b, c, d by a common factor. Consider three cases:

(i) $\vec{x}$, $\vec{y}$, and $\vec{xy}$ are linearly independent. In this case, $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ can be written as a unique linear combination of them, which yields our solution for a, b, c, d and hence our automorphism of $\mathcal{H}$.

(ii) $\vec{x}$ and $\vec{y}$ are linearly dependent, say $\vec{y} = \lambda \vec{x}$, where $\lambda \neq 0$. Then

$$\begin{vmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ x_1 y_1 & x_2 y_2 & x_3 y_3 \end{vmatrix} = \lambda \begin{vmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ x_1^2 & x_2^2 & x_3^2 \end{vmatrix}$$

is a Vandermonde determinant and hence nonzero for distinct x_1, x_2, x_3. This implies that $\operatorname{span}(\vec{x}, \vec{y}, \vec{xy})$ is 2-dimensional and that $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ does not lie in it, so that the only solution to our equation is $b = 0$ and a, b, c determined by the (projectively) unique dependence relation of $\vec{x}, \vec{y}, \vec{xy}$.

(iii) $\vec{x}$ and $\vec{y}$ are linearly dependent, and $\vec{xy}$ is a linear combination of them. I haven't figured out how to do this case; it's worth noting that this must be where the hypotheses $x_1 < x_2 < x_3$ and $y_1 < y_2 < y_3$ come into play, since so far I've only used that they're distinct without using their cyclic order.

□

Exercise 16: Let

$$f(z) = \frac{i-z}{i+z} \text{ and } f^{-1}(w) = i\frac{1-w}{1+w}.$$

(a) Given $\theta \in \mathbb{R}$, find real numbers a, b, c, d such that $ad - bc = 1$, and so that for any $z \in \mathcal{H}$

$$\frac{az+b}{cz+d} = f^{-1}\left(e^{i\theta} f(z)\right).$$

(b) Given $\alpha \in \mathbb{D}$ find real numbers a, b, c, d so that $ad - bc = 1$, and so that for any $z \in \mathcal{H}$

$$\frac{az+b}{cz+d} = f^{-1}(\psi_\alpha(f(z))),$$

with ψ_α defined in Section 2.1.

(c) Prove that if g is an automorphism of the unit disc, then there exist real numbers a, b, c, d such that $ad - bc = 1$ and so that for any $z \in \mathcal{H}$

$$\frac{az+b}{cz+d} = f^{-1} \circ g \circ f(z).$$

Solution.
(a)

$$\begin{aligned}
f^{-1}(e^{i\theta} f(z)) &= f^{-1}\left(e^{i\theta}\frac{i-z}{i+z}\right) \\
&= i\frac{1 - e^{i\theta}\frac{i-z}{i+z}}{1 + e^{i\theta}\frac{i-z}{i+z}} \\
&= i\frac{i + z - ie^{i\theta} + e^{i\theta}z}{i + z + ie^{i\theta} - e^{i\theta}z} \\
&= \frac{i(1+e^{i\theta})z + i(1-e^{i\theta})}{(1-e^{i\theta})z + i(1+e^{i\theta})} \\
&= \frac{2ie^{i\theta/2}\frac{e^{i\theta/2}+e^{-i\theta/2}}{2}z + 2e^{i\theta/2}\frac{e^{i\theta/2}-e^{-i\theta/2}}{2i}}{-2ie^{i\theta/2}\frac{e^{i\theta/2}-e^{-i\theta/2}}{2i}z + 2ie^{i\theta}\frac{e^{i\theta/2}+e^{-i\theta/2}}{2}} \\
&= \frac{i\cos(\theta/2)z + \sin(\theta/2)}{-i\sin(\theta/2)z + i\cos(\theta/2)}.
\end{aligned}$$

The determinant here is $ad - bc = -\cos^2(\theta/2) + i\sin^2(\theta/2) \neq 0$, so they can all be scaled by an appropriate amount to make the determinant equal 1.

(b)

$$
\begin{aligned}
f^{-1}(\psi_\alpha(f(z))) &= f^{-1}\left(\psi_\alpha\left(\frac{i-z}{i+z}\right)\right)\\
&= f^{-1}\left(\frac{\alpha - \frac{i-z}{i+z}}{1-\bar{\alpha}\frac{i-z}{i+z}}\right)\\
&= f^{-1}\left(\frac{\alpha i + \alpha z - i + z}{i+z-\bar{\alpha} i + \bar{\alpha} z}\right)\\
&= f^{-1}\left(\frac{(1+\alpha)z + i(\alpha-1)}{(1+\bar{\alpha})z + i(1-\bar{\alpha})}\right)\\
&= i\frac{1-\frac{(1+\alpha)z+i(\alpha-1)}{(1+\bar{\alpha})z+i(1-\bar{\alpha})}}{1+\frac{(1+\alpha)z+i(\alpha-1)}{(1+\bar{\alpha})z+i(1-\bar{\alpha})}}\\
&= i\frac{(1+\bar{\alpha})z + i(1-\bar{\alpha}) - (1+\alpha)z + i(1-\alpha)}{(1+\bar{\alpha})z + i(1-\bar{\alpha}) + (1+\alpha)z + i(\alpha-1)}\\
&= i\frac{(\bar{\alpha}-\alpha)z + i(2-\alpha-\bar{\alpha})}{(2+\alpha+\bar{\alpha})z + i(\alpha-\bar{\alpha})}\\
&= \frac{bz+(a-1)}{(a+1)z-b}
\end{aligned}
$$

where $\alpha = a + bi$. The determinant is $-b^2 - (a^2-1) = 1 - |\alpha|^2 \neq 0$, so it can be made 1 by an appropriate scaling.

(c) Let $R_\theta(z) = e^{i\theta}z$. Then $g = R_\theta \circ \psi_\alpha$ for some $\theta \in \mathbb{R}$ and $\alpha \in \mathbb{D}$, so

$$
\begin{aligned}
f^{-1}\circ g \circ f &= f^{-1}\circ R_\theta \circ \psi_\alpha \circ f\\
&= f^{-1}\circ R_\theta \circ f \circ f^{-1} \circ \psi_\alpha \circ f\\
&= (f^{-1}\circ R_\theta \circ f)\circ(f^{-1}\circ \psi_\alpha \circ f)
\end{aligned}
$$

is the composition of two Möbius transformations of determinant 1, by parts (a) and (b); this is another Möbius transformation of determinant 1, so we're done.

□

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Problem 2: Prove that a real-differentiable function $f : \Omega \to \mathbb{C}$ with $J_f(z_0) \neq 0$ is holomorphic with $f'(z_0) \neq 0$ iff f preserves angles at z_0.

Solution. If $\gamma : [0,1] \to \mathbb{C}$ and $\eta : [0,1] \to \mathbb{C}$ are two curves passing through z_0 with tangent vectors γ' and η', then the tangents to $f\circ\gamma$ and $f\circ\eta$ at z_0 are $J_f\gamma'$ and $J_f\eta'$ by the Chain Rule. For these always to have the same angle as γ' and η', it is necessary and sufficient that J_f be a nonzero multiple of a unitary matrix (this is a standard theorem from linear algebra), i.e. $J_f^T J_f = C\cdot Id$ where $C \neq 0$ and Id is the 2×2 identity matrix. Now

$$
J_f^T J_f = \begin{pmatrix} u_x & v_x \\ u_y & v_y \end{pmatrix}\begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} = \begin{pmatrix} u_x^2 + u_y^2 & u_x v_x + u_y v_y \\ u_x v_x + u_y v_y & v_x^2 + v_y^2 \end{pmatrix}
$$

so the condition for f to preserve angles is that $u_x^2 + u_y^2 = v_x^2 + v_y^2$ and $u_x v_x + u_y v_y = 0$. We will show that these are equivalent to the Cauchy-Riemann equations. Clearly if $u_x = v_y$ and $u_y = -v_x$ then $u_x^2 + u_y^2 = v_x^2 + v_y^2$ and $u_x v_x + u_y v_y = 0$. Conversely, if the latter two equations are true, and $f' \neq 0$, choose some nonzero component of u or v; WLOG $u_x \neq 0$. Then $v_x = -\frac{u_y}{u_x} v_y$ so

$$v_y^2 \left(1 + \frac{u_y^2}{u_x^2}\right) = v_x^2 + v_y^2 = u_x^2 + u_y^2 = u_x^2 \left(1 + \frac{u_y^2}{u_x^2}\right) \Leftrightarrow v_y = \pm u_x.$$

If $v_y = u_x$ then we immediately have $v_x = -u_y$ as well, so the Cauchy-Riemann equations are satisfied. The possibility $v_y = -u_x$ is impossible because it would imply that $\det(J_f) < 0$ which is impossible for a multiple of a unitary matrix (in an even number of dimensions). Hence, f preserving angles is equivalent to f being analytic for $J_f \neq 0$. $\square$

Problem 7: Applying ideas of Carathéodory, Koebe gave a proof of the Riemann mapping theorem by constructing (more explicitly) a sequence of functions that converges to the desired conformal map.

Starting with a Koebe domain, that is, a simply connected domain $\mathcal{K}_0 \subset \mathbb{D}$ that is not all of $\mathbb{D}$, and which contains the origin, the strategy is to find an injective function f_0 such that $f_0(\mathcal{K}_0) = \mathcal{K}_1$ is a Koebe domain "larger" than $\mathcal{K}_0$. Then, one iterates this process, finally obtaining functions $F_n = f_n \circ \cdots \circ f_0 : \mathcal{K}_0 \to \mathbb{D}$ such that $F_n(\mathcal{K}_0) = \mathcal{K}_{n+1}$ and $\lim F_n = F$ is a conformal map from $\mathcal{K}_0$ to $\mathbb{D}$.

The **inner radius** of a region $\mathcal{K} \subset \mathbb{D}$ that contains the origin is defined by $r_{\mathcal{K}} = \sup\{\rho \geq 0 : D(0, \rho) \subset \mathcal{K}\}$. Also, a holomorphic injection $f : \mathcal{K} \to \mathbb{D}$ is said to be an **expansion** if $f(0) = 0$ and $|f(z)| > |z|$ for all $z \in \mathcal{K} - \{0\}$.

(a) Prove that if f is an expansion, then $r_{f(\mathcal{K})} \geq r_{\mathcal{K}}$ and $|f'(0)| > 1$.

Suppose we begin with a Koebe domain $\mathcal{K}_0$ and a sequence of expansions $\{f_0, f_1, \ldots, f_n, \ldots\}$, so that $\mathcal{K}_{n+1} = f_n(\mathcal{K}_n)$ are also Koebe domains. We then define holomorphic maps $F_n : \mathcal{K}_0 \to \mathbb{D}$ by $F_n = f_n \circ \cdots \circ f_0$.

(b) Prove that for each n, the function F_n is an expansion. Moreover, $F_n'(0) = \prod_{k=0}^n f_k'(0)$, and conclude that $\lim_{n\to\infty} |f_n'(0)| = 1$.

(c) Show that if the sequence is osculating, that is, $r_{\mathcal{K}_n} \to 1$ as $n \to \infty$, then a subsequence of $\{F_n\}$ converges uniformly on compact subsets of $\mathcal{K}_0$ to a conformal map $F : \mathcal{K}_0 \to \mathbb{D}$.

To construct the desired osculating sequence we shall use the automorphisms $\psi_\alpha = (\alpha - z)/(1 - \bar{\alpha} z)$.

(d) Given a Koebe domain $\mathcal{K}$, choose a point $\alpha \in \mathbb{D}$ on the boundary of $\mathcal{K}$ such that $|\alpha| = r_{\mathcal{K}}$, and also choose $\beta \in \mathbb{D}$ such that $\beta^2 = \alpha$. Let S denote the square root of ψ_α on $\mathcal{K}$ such that $S(0) = 0$. Why is such a function well defined? Prove that the function $f : \mathcal{K} \to \mathbb{D}$ defined by $f(z) = \psi_\beta \circ S \circ \psi_\alpha$ is an expansion. Moreover, show that $|f'(0)| = (1 + r_{\mathcal{K}}) 2\sqrt{r_{\mathcal{K}}}$.

(e) Use part (d) to construct the desired sequence.

Solution.

(a) Since f is a holomorphic injection, it is a homeomorphism, so it maps $\mathcal{K}_0$ to a simply connected domain. Let $\rho < r_{\mathcal{K}}$ and consider the image

under f of $C(0,\rho)$. This curve is mapped to another curve all of whose points are at least ρ away from the origin; since the image under f of $\mathcal{K}_0$ is simply connected and includes the origin, it includes the interior of this curve. In particular, it contains $D(0,\rho)$. Thus, $D(0,\rho) \subset \mathcal{K} \Rightarrow D(0,\rho) \subset f(\mathcal{K})$. Taking suprema, $r_{f(\mathcal{K})} \geq r_K$.
Since f is holomorphic on $\mathcal{K}_0$ and $f(0) = 0$, we can write $f(z) = zg(z)$ on a neighborhood of 0, where g is holomorphic. Then $f'(0) = g(0)$. Now $g \neq 0$ since $|g(z)| = \frac{|f(z)|}{|z|} > 1$ for $z \neq 0$ and g is continuous. Thus, by the Minimum Modulus Principle applied to a small circle near the origin, since $|g| > 1$ on the circle, $|g(0)| > 1$. (The Minimum Modulus Principle is just the Maximum Modulus Principle applied to $\frac{1}{g}$, which is valid since $g \neq 0$ in the region under consideration.)

(b) Since each f_i fixes the origin, so does F_n. Moreover, it is easy to see by induction that $|F_n(z)| > |z|$ since $|F_n(z)| = |f_n(F_{n-1}(z))| > |F_{n-1}(z)| > |z|$ by the induction hypothesis. By the Chain Rule,

$$F_n'(0) = (f_n \circ F_{n-1})'(0) = f_n'(F_{n-1}(0))F_{n-1}'(0) = f_n'(0)F_{n-1}'(0)$$

so by another easy induction we have $F_n'(0) = \prod_{k=1}^n f_k'(0)$. However, if we let $\rho = r_{\mathcal{K}_0}/2$, then $D(0,\rho) \subset \mathcal{K}_n$ for all n, so we can define $G_n(z) : \mathbb{D} \to \mathbb{D}$ by $G_n(z) = F_n(\rho z)$. Then by the Schwarz Lemma,

$$\rho|F_n'(0)| = |G_n'(0)| < 1 \Rightarrow |F_n'(0)| < \frac{1}{\rho}.$$

Thus, the sequence $\{|F_n'(0)|\}$ is bounded above. This implies that $|f_k'(0)| \to 1$ as otherwise the product would be infinite.

(c) Since each F_n maps into $\mathbb{D}$, the sequence $\{F_n\}$ is uniformly bounded. By Montel's theorem, there is a subsequence F_{n_k} that converges uniformly on all compact subsets $K \subset \mathcal{K}_0$. The limit function $F : \mathcal{K}_0 \to \mathbb{D}$ must be holomorphic because the uniform limit of holomorphic functions is holomorphic (and $\mathcal{K}_0$ is the union of its compact subsets). Then F is injective by Proposition 3.5 (it cannot be constant because then it would be everywhere zero, and for $z \in \mathcal{K} \setminus \{0\}$, $|f_{n_k}(z)| > |z|$ for all $n \Rightarrow |F(z)| = \lim |f_{n_k}(z)| \geq |z| > 0$). Thus F is an injective holomorphic function, hence a homeomorphism, so $F(\mathcal{K}_0)$ is simply connected. I claim that $r_{F(\mathcal{K}_0)} \geq r_{F_N(\mathcal{K}_0)}$ for all N. To see this, let $\rho < r_{F_N(\mathcal{K}_0)}$. For $n_k \geq N$, if $w \in C(0,\rho)$, then $|F_{n_k}(w)| > \rho$. This implies $|F(w)| \geq \rho$. Hence F maps $C(0,\rho)$ to a smooth closed curve whose points are all at least ρ from 0; since the image of F is simply connected, it contains the image of this curve, so it contains $D(0,\rho)$. Taking suprema yields $r_{F(\mathcal{K}_0)} \geq r_{F_N(\mathcal{K}_0)}$. Since $r_{F_N(\mathcal{K}_0)} \to 1$ as $N \to \infty$, this implies $r_{F(\mathcal{K}_0)} \geq 1$. Hence $D(0,\rho) \subset F(\mathcal{K}_0)$ for all $\rho < 1$, so F is surjective. So F is a conformal map from $\mathcal{K}_0$ to $\mathbb{D}$.

$\square$