

第五章 定积分

习题五

5.1

1. 写出下列定积分的定义式.

$$(1) \int_a^b \frac{dx}{1+x^2}$$

$$= \lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{1}{1+\xi_k^2} \Delta x_k$$

$$(2) \int_a^b \sin x dx$$

$$= \lim_{n \rightarrow \infty} \sum_{k=1}^n \sin \xi_k \Delta x_k$$

其中 $\lambda = \max_{k=1,2,\dots,n} \Delta x_k$

2. 设 $\int_1^3 f(x) dx = 18, \int_1^3 f(x) dx = 4, \int_1^3 g(x) dx = 3, \int_1^3 h(x) dx = 4$

$$(1) \int_1^3 f(x) dx = 18$$

$$= \int_1^3 f(x) dx - \int_1^3 f(x) dx$$

$$= 4 - 6 = -2$$

$$(2) \int_{-1}^1 \frac{1}{2} (f(x) + 3g(x)) dx$$

$$= \frac{1}{2} \int_{-1}^1 f(x) dx + \frac{3}{2} \int_{-1}^1 g(x) dx$$

$$= \frac{1}{2} \times 4 + \frac{3}{2} \times 3 = 5$$

3. 说明下列各对积分中哪一个的值较大.

(1) $\int_1^2 x^2 dx$ 还是 $\int_1^2 x^3 dx$

在 $[0, 1]$ 上 $x^2 > x^3$, 且 $x^2 \neq x^3$, 所以

$$\int_1^2 x^2 dx > \int_1^2 x^3 dx$$

(2) $\int_0^1 x dx$ 还是 $\int_0^1 \ln(1+x) dx$

在 $(0, 1]$ 上 $\ln(1+x) \leq x$, 且 $\ln(1+x) \neq x$, 所以

$$\int_0^1 \ln(1+x) dx < \int_0^1 x dx$$

(3) $\int_0^{\pi} \sin x dx$ 还是 $\int_0^{\pi} \sin x dx$

在 $[\pi/2, \pi]$ 上 $\sin x < 0$, 且 $\sin x \neq 0$, 所以

$$\int_0^{\pi} \sin x dx < 0$$

$$\int_0^{\pi} \sin x dx = \int_0^{\pi/2} \sin x dx + \int_{\pi/2}^{\pi} \sin x dx$$

$$< \int_0^{\pi/2} \sin x dx + 0 = \int_0^{\pi/2} \sin x dx$$

4. 设 $f(x)$ 连续, 且极限 $\lim_{x \rightarrow \infty} f(x)$ 存在, 试证:

$$\lim_{x \rightarrow \infty} \int_x^{x+a} f(x) dx = 0.$$

$$\begin{aligned} \lim_{x \rightarrow \infty} \int_x^{x+a} f(x) dx &= \lim_{x \rightarrow \infty} \frac{f(\xi)}{1} [(x+a) - x] \\ &= a \lim_{x \rightarrow \infty} \frac{f(\xi)}{1} = a \times 0 = 0 \end{aligned}$$

其中 ξ 介于 x 和 $x+a$ 之间, 当 $x \rightarrow \infty$ 时, $\xi \rightarrow \infty$, $\lim_{x \rightarrow \infty} \frac{f(\xi)}{1} = 0$

5. 设 $f(x)$ 在闭区间 $[0,1]$ 上可导, 且满足条件 $f(1) = 2 \int_0^1 x f(x) dx$, 试证: 存在 $\xi \in (0,1)$, 使得 $f(\xi) + \xi f'(\xi) = 0$.

由变数多中值定理, 存在 $\eta \in [0, \frac{1}{2}]$, 使得 $f(\eta) = 2 \int_0^{\frac{1}{2}} x f(x) dx = 2 \int_0^{\frac{1}{2}} f(\eta) (\frac{1}{2} - 0) = \eta f(\eta)$
 令 $F(x) = x f(x)$, 则 $F(x)$ 在 $[\eta, \frac{1}{2}]$ 上连续, 在 $(\eta, \frac{1}{2})$ 内可导, 且 $F(\eta) = \eta f(\eta) = f(\eta)$, $F(\frac{1}{2}) = \frac{1}{2} f(\frac{1}{2})$, 由罗尔定理, 存在 $\xi \in (\eta, \frac{1}{2}) \subset (0,1)$, 使得 $F(\xi) = f(\xi) + \xi f'(\xi) = 0$

得证

5.2

1. 计算下列积分:

$$\begin{aligned} (1) \frac{d}{dx} \int_0^x \sqrt{1+t^2} dt &= \frac{d}{dx} \left(- \int_0^x \sqrt{1+t^2} dt \right) \\ &= - \frac{d}{dx} \int_0^x \sqrt{1+t^2} dt \\ &= - \sqrt{1+x^2} \end{aligned}$$

$$\begin{aligned} (2) \frac{d}{dx} \int_0^{\cos x} \cos(x^2) dx &= \cos(\pi \cos^2 x) - (-\sin x) - \cos(\pi \sin^2 x) \cos x \\ &= -\sin x \cos(\pi \cos^2 x) - \cos x \cos(\pi \sin^2 x) \end{aligned}$$

2. 求由参数方程 $\begin{cases} x = \int_0^t \sin u du \\ y = \int_0^t u^2 \ln u du \end{cases}$ 所确定的函数对 x 的导数

$$\begin{aligned} \frac{dy}{dx} &= \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\left(\int_0^t u^2 \ln u du \right)'}{\left(\int_0^t \sin u du \right)'} \\ &= \frac{-\left(t^2 \ln t + \frac{1}{2} t^2 \right) \cdot (-\sin t)}{\cos t} = -t^2 \end{aligned}$$

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1. 求出 $\int_0^x e^t dt + \int_0^x \cos t dt = 0$ 所确定的隐函数对 x 的导数

$\frac{dy}{dx}$

两边关于 x 求导得

$e^x + \cos x = 0$

解得 $\frac{dy}{dx} = -e^x \cos x$

4. 设 $f(x)$ 连续, 且 $\int_0^x f(t) dt = x^2(1+x)$, 求 $f(x)$.

方程求导得

$f(x) = 2x + 3x^2$

5. 求下列极限.

(1) $\lim_{x \rightarrow 0} \frac{\int_0^x \cos t dt}{x}$

$$= \lim_{x \rightarrow 0} \frac{\cos x^2}{2x} = 1$$

6. 计算

(1) $\int_0^2 x^2 dx$

$$= \frac{1}{3} x^3 \Big|_0^2 = \frac{8}{3}$$

(2) $\lim_{x \rightarrow 0} \frac{\int_0^x \sqrt{\tan t} dt}{\sqrt{\sin x}}$

$$= \lim_{x \rightarrow 0} \frac{\frac{1}{2} \tan^{-1/2} x \cdot \cos x}{\frac{1}{2} \sin^{-1/2} x \cdot \cos x}$$

$$= \lim_{x \rightarrow 0} \frac{\sqrt{\tan x}}{\sqrt{\sin x}} = \lim_{x \rightarrow 0} \frac{\sqrt{\frac{\sin x}{\cos x}}}{\sqrt{\sin x}} = \lim_{x \rightarrow 0} \frac{1}{\sqrt{\cos x}} = 1$$

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(3) $\lim_{x \rightarrow a} \frac{\int_a^x x^2 f(t) dt}{x-a}$, 其中 $f(x)$ 连续.

$$= \lim_{x \rightarrow a} \frac{x^2 \int_a^x f(t) dt}{x-a}$$

$$= \lim_{x \rightarrow a} x^2 \lim_{x \rightarrow a} \frac{\int_a^x f(t) dt}{x-a} = a^2 \lim_{x \rightarrow a} \frac{\int_a^x f(t) dt}{x-a} = a^2 f(a)$$

$$= a^2 \lim_{x \rightarrow a} \frac{\int_a^x f(t) dt}{x-a} = a^2 f(a)$$

6. 计算下列定积分.

(1) $\int_0^3 x dx$

$$= \frac{1}{2} x^2 \Big|_0^3 = \frac{9}{2} - 0 = \frac{9}{2}$$

$$(2) \int_0^1 \cos x dx = \sin x \Big|_0^1 = 1 - 0 = 1$$

$$(3) \int_0^1 \frac{dx}{\sqrt{4-x^2}} = \arcsin \frac{x}{2} \Big|_0^1 = \arcsin \frac{1}{2} - 0 = \frac{\pi}{6}$$

$$(4) \int_0^1 \tan^{-1} 2x dx = \int_0^1 \tan^{-1} (2x-1+1) dx = (\tan^{-1} 2x - x) \Big|_0^1 = 1 - \frac{\pi}{2}$$

$$(5) \int_{-1}^2 \frac{dx}{x+x^2} = \int_{-1}^2 \left(\frac{1}{x} - \frac{x}{1+x^2} \right) dx = \int_{-1}^2 \frac{1}{x} dx - \int_{-1}^2 \frac{x}{1+x^2} dx = \ln|x| \Big|_{-1}^2 - \frac{1}{2} \ln|1+x^2| \Big|_{-1}^2 = \ln 2 - 0 - \frac{1}{2} (\ln 5 - \ln 2) = \frac{3}{2} \ln 2$$

$$(6) \int_x^{x+a} dx (a > 0)$$

当 $a \geq 1$ 时, $\int_0^1 x|x-a| dx = \int_0^1 x(a-x) dx = \int_0^1 (ax - x^2) dx = \left(\frac{ax^2}{2} - \frac{x^3}{3} \right) \Big|_0^1 = \frac{a}{2} - \frac{1}{3}$

当 $0 < a < 1$ 时, $\int_0^1 x|x-a| dx = \int_0^a x(a-x) dx + \int_a^1 x(x-a) dx = \left(\frac{ax^2}{2} - \frac{x^3}{3} \right) \Big|_0^a + \left(\frac{x^2}{2} - \frac{ax}{2} \right) \Big|_a^1 = \frac{a^3}{2} - \frac{a}{2} + \frac{1}{2} - \frac{a}{2} = \frac{1}{2} - \frac{a^2}{2}$

$$(7) \int_{-1}^1 f(x) dx, \text{ 其中 } f(x) = \begin{cases} \frac{1}{2}x^2, & x > 1 \\ x+1, & x \leq 1 \end{cases}$$

$$= \int_0^1 (x+1) dx + \int_1^2 \frac{1}{2}x^2 dx = \left(\frac{x^2}{2} + x \right) \Big|_0^1 + \frac{1}{6}x^3 \Big|_1^2 = \left(\frac{1}{2} + 1 \right) + \frac{1}{6}(8-1) = \frac{8}{3}$$

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7. 设 $f(x) = \begin{cases} \frac{1}{2} \sin x, & 0 \leq x \leq \pi \\ 0, & x < 0 \text{ 或 } x > \pi \end{cases}$, 求 $F(x) = \int_0^x f(t) dt$ 在区
($b \neq 0$)

解: 在 $(-\infty, +\infty)$ 内的表达式:
 当 $x < 0$ 时, $F(x) = \int_0^x 0 dt = 0$
 当 $0 \leq x \leq \pi$ 时, $F(x) = \int_0^x \frac{1}{2} \sin t dt = -\frac{1}{2} \cos t \Big|_0^x = \frac{1 - \cos x}{2}$
 当 $x > \pi$ 时, $F(x) = \int_0^\pi \frac{1}{2} \sin t dt + \int_\pi^x 0 dt = \frac{1 - \cos \pi}{2} = 1$

8. 利用定积分的定义计算下列极限:
 (1) $\lim_{n \rightarrow \infty} \frac{1}{n} (\sqrt{1} + \sqrt{2} + \dots + \sqrt{n})$
 解: $\lim_{n \rightarrow \infty} \frac{1}{n} (\sqrt{1} + \sqrt{2} + \dots + \sqrt{n}) = \lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{\sqrt{k}}{n} = \int_0^1 \sqrt{x} dx = \frac{2}{3} x^{\frac{3}{2}} \Big|_0^1 = \frac{2}{3}$

(2) $\lim_{n \rightarrow \infty} \frac{1}{n} [\sin a + \sin(a + \frac{b}{n}) + \dots + \sin(a + \frac{(n-1)b}{n})]$
 ($b \neq 0$)
 解: $\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} \sin(a + kb) = \int_0^1 \sin(a + bx) dx = -\frac{1}{b} \cos(a + bx) \Big|_0^1 = \frac{1}{b} [\cos a - \cos(a+b)]$

9. 设 $f(x)$ 在闭区间 $[0, 1]$ 上连续, 且 $f(x) = 3x - \sqrt{1-x^2}$.
 $\int_0^1 f(x) dx$, 求 $f(x)$.

解: $A = \int_0^1 [3x - A\sqrt{1-x^2}] dx = \int_0^1 [3x^2 - 6Ax\sqrt{1-x^2} + A^2(1-x^2)] dx = [x^3 - 3Ax^2 + A^2x - \frac{2}{3}A^2(1-x^2)^{\frac{3}{2}}] \Big|_0^1 = 3 - 2A + \frac{2}{3}A^2$
 由 $2A^2 - 9A + 4 = 0$, 解得 $A = 3$ 或 $A = \frac{2}{3}$.
 或 $4A^2 - 9A - 2 = 0$ 解得 $A = \frac{3}{2}$ 或 $A = -\frac{2}{3}$.

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5.3

1. 计算下列定积分.

(1) $\int_1^4 \frac{x}{\sqrt{5-x}} dx$

令 $t = \sqrt{5-x}$, 则 $x = 5-t^2$, $dx = -2t dt$,
 于是 $\int_1^4 \frac{x}{\sqrt{5-x}} dx = \int_2^1 \frac{5-t^2}{t} \cdot (-2t) dt$
 $= \int_1^2 2(5-t^2) dt = 2 \left[5t - \frac{t^3}{3} \right]_1^2 = \frac{16}{3}$

(2) $\int_1^4 \sqrt{x^2-1} dx$

令 $t = \sqrt{x^2-1}$, 则 $x = \sqrt{t^2+1}$, $dx = \frac{t}{\sqrt{t^2+1}} dt$
 于是 $\int_1^4 \sqrt{x^2-1} dx = \int_0^3 t \cdot \frac{t}{\sqrt{t^2+1}} dt = \frac{1}{2} \int_0^3 \frac{2t^2}{\sqrt{t^2+1}} dt$
 $= \frac{1}{2} \int_0^3 (1 + \frac{t^2-1}{\sqrt{t^2+1}}) dt = \frac{1}{2} \left[t + \frac{t^2-1}{\sqrt{t^2+1}} \right]_0^3 = \frac{1}{2} \left(3 + \frac{8}{\sqrt{10}} \right)$

(3) $\int_0^{\frac{\pi}{2}} \frac{\sqrt{1-x^2}}{x^2} dx$

令 $x = \sin t$, 则 $dx = \cos t dt$,
 于是 $\int_0^{\frac{\pi}{2}} \frac{\sqrt{1-x^2}}{x^2} dx = \int_0^{\frac{\pi}{2}} \frac{\cos t}{\sin^2 t} \cdot \cos t dt$
 $= \int_0^{\frac{\pi}{2}} (\csc^2 t - \csc^4 t) dt = \left[-\cot t + \frac{\cot t \csc^2 t}{2} \right]_0^{\frac{\pi}{2}} = -\frac{1}{2}$

(4)

令 $t = \sqrt{x}$

(1)

令 $t = \sqrt{x^2-1}$

(2)

令 $x = \sin t$

(1) $\int_1^4 \frac{dx}{x^2 \sqrt{x^2+2}}$

令 $x = \sqrt{2} \tan t$, 则 $dx = \sqrt{2} \sec^2 t dt$,
 于是 $\int_1^4 \frac{dx}{x^2 \sqrt{x^2+2}} = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{\sqrt{2} \sec^2 t}{2 \tan^2 t \cdot \sqrt{2} \sec t} dt = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{1}{\tan^2 t \sec t} dt$
 $= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{\cos t}{\sin^2 t} dt = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \csc^2 t dt = \left[-\cot t \right]_{\frac{\pi}{4}}^{\frac{\pi}{2}} = 1 - \frac{1}{\sqrt{2}}$

上述用积分法:

(1) $\int_1^4 \frac{dx}{1+x^2} = \int_1^4 \frac{dx}{1+x^2} (x > 0)$

令 $t = \frac{1}{1+x^2}$, 则 $dx = -\frac{dt}{2t^2}$
 于是 $\int_1^4 \frac{dx}{1+x^2} = \int_{\frac{1}{5}}^{\frac{1}{17}} \frac{-\frac{dt}{2t^2}}{t} = \frac{1}{2} \int_{\frac{1}{5}}^{\frac{1}{17}} \frac{dt}{t^3} = \frac{1}{2} \left[-\frac{1}{2t^2} \right]_{\frac{1}{5}}^{\frac{1}{17}} = \frac{1}{2} \left(\frac{1}{5^2} - \frac{1}{17^2} \right)$

(2) $\int_0^1 x^2 f(x^2) dx = \frac{1}{2} \int_0^1 x f(x) dx$ (令 $t = x^2$)

令 $t = x^2$, 则 $dx = \frac{dt}{2\sqrt{t}}$
 于是 $\int_0^1 x^2 f(x^2) dx = \frac{1}{2} \int_0^1 \sqrt{t} f(t) dt$
 $\int_0^1 x^2 f(x^2) dx = \frac{1}{2} \int_0^1 \sqrt{t} f(t) dt$

证:

(1) $\int_0^1 \frac{dx}{1+x^2}$

令 $x = \tan t$, 则 $dx = \sec^2 t dt$,
 于是 $\int_0^1 \frac{dx}{1+x^2} = \int_0^{\frac{\pi}{4}} \frac{\sec^2 t}{1+\tan^2 t} dt = \int_0^{\frac{\pi}{4}} \frac{\sec^2 t}{\sec^2 t} dt = \int_0^{\frac{\pi}{4}} 1 dt = \frac{\pi}{4}$

(2) 设 $f(x) = \frac{1}{1+x^2}$, 则 $f'(x) = -\frac{2x}{(1+x^2)^2}$
 于是 $\int_0^1 x^2 f(x^2) dx = \frac{1}{2} \int_0^1 x f(x) dx = \frac{1}{2} \int_0^1 \frac{x}{1+x^2} dx = \frac{1}{4} \int_0^1 \frac{du}{u} = \frac{1}{4} \ln 2$

(3) $\int_0^1 f(x) dx = \int_0^1 f(a-x) dx$ (定值), 并求 $\int_0^1 \frac{\sin^2 x}{\sin x + \cos x} dx$.

令 $y = a - t, dy = -dt$
 $\int_0^1 f(x) dx = \int_0^1 f(a-t) (-dt)$

于是 $\int_0^1 \frac{\sin^2 x}{\sin x + \cos x} dx = \int_0^1 \frac{\sin^2(a-t)}{\sin(a-t) + \cos(a-t)} (-dt)$
 $= \int_0^1 \frac{\sin^2 t}{\sin t + \cos t} dt$

于是 $\int_0^1 \frac{\sin^2 x}{\sin x + \cos x} dx = \frac{1}{2} \int_0^1 \frac{\sin^2 x + \cos^2 x}{\sin x + \cos x} dx = \frac{1}{2} \int_0^1 \frac{1}{\sin x + \cos x} dx$

令 $t = \frac{\pi}{4} - x, dt = -dx$
 $\int_0^1 \frac{1}{\sin x + \cos x} dx = \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \frac{1}{\sin(\frac{\pi}{4}-t) + \cos(\frac{\pi}{4}-t)} (-dt)$

于是 $\int_0^1 \frac{1}{\sin x + \cos x} dx = \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \frac{1}{\sin t} dt = \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \frac{1}{\sin t} dt$

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4. 计算下列定积分.

(1) $\int_0^1 \arctan x dx$

$= \frac{1}{2} \int_0^1 \arctan(x^2) dx$

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(3) $\int_0^{\pi} (\sin x)^2 dx = \int_0^{\pi} x^2 \frac{1-\cos 2x}{2} dx$
 $= \frac{1}{2} \int_0^{\pi} x^2 dx - \frac{1}{4} \int_0^{\pi} x^2 d \sin 2x$
 $= \frac{1}{6} x^3 \Big|_0^{\pi} - \frac{1}{4} \left[x^2 \sin 2x \Big|_0^{\pi} - \int_0^{\pi} \sin 2x \cdot 2x dx \right]$
 $= \frac{\pi^3}{6} - \frac{1}{4} \int_0^{\pi} x d \cos 2x$
 $= \frac{\pi^3}{6} - \frac{1}{4} \left[x \cos 2x \Big|_0^{\pi} - \int_0^{\pi} \cos 2x dx \right]$
 $= \frac{\pi^3}{6} - \frac{1}{4} \left[\pi \cos 2\pi - \frac{1}{2} \sin 2x \Big|_0^{\pi} \right] = \frac{\pi^3}{6} - \frac{\pi^2}{4}$

(1) $\int_0^{\pi} e^{2x} \cos x dx = \int_0^{\pi} e^{2x} (-\sin x) dx$
 $= -\frac{1}{2} \int_0^{\pi} e^{2x} \cos x dx - \int_0^{\pi} e^{2x} \sin x dx$
 $= -\frac{1}{2} + \frac{1}{2} \int_0^{\pi} e^{2x} \sin x dx$
 $= -\frac{1}{2} + \frac{1}{4} \int_0^{\pi} e^{2x} \cos x dx - \int_0^{\pi} e^{2x} \cos x dx$
 $= -\frac{1}{2} + \frac{1}{4} \int_0^{\pi} e^{2x} \cos x dx - \int_0^{\pi} e^{2x} \cos x dx$
 $= -\frac{1}{2} + \frac{1}{4} \int_0^{\pi} e^{2x} \cos x dx$
 $= -\frac{1}{2} + \frac{1}{4} \int_0^{\pi} e^{2x} \cos x dx = -\frac{1}{2} (e^{\pi} - 2)$

(3) $\int_0^{\pi} \ln x dx = \int_0^{\pi} e^{2x} \cos x dx = -\frac{1}{2} (e^{\pi} - 2)$
 $= \int_0^{\pi} -2x dx + \int_0^{\pi} 2x dx$
 $= -x^2 \Big|_0^{\pi} + \int_0^{\pi} x \cdot \frac{1}{2} dx + x^2 \Big|_0^{\pi} - \int_0^{\pi} x \cdot \frac{1}{2} dx$
 $= -\frac{1}{2} + (1 - e) + e - (e - 1)$
 $= 2 - e$

5. 计算下列定积分.

(1) $\int_0^1 \frac{x^2 \sin^2 x}{x^2 + x^2 + 1} dx = 0$

(2) $\int_0^1 \frac{1+x}{1+\cos x} dx = \int_0^1 \frac{x}{1+\cos x} dx + \int_0^1 \frac{1}{1+\cos x} dx$
 $= 2 \int_0^{\frac{\pi}{2}} \frac{t \sin t}{1+\cos t} dt + 0$
 $= 2 \int_0^{\frac{\pi}{2}} \frac{t \sin t}{2 \cos^2 t} dt = \int_0^{\frac{\pi}{2}} t \tan^2 t dt = 2 - \frac{\pi}{2} - 0 = 2$

(3) $\int_0^3 (1+x)e^{2x} dx = \int_0^3 1 \cdot e^{2x} dx + \int_0^3 x e^{2x} dx$
 $= \int_0^3 e^{2x} dx + \int_0^3 x e^{2x} dx$
 $= \int_0^3 x^2 e^{2x} dx + 2 \int_0^3 x e^{2x} dx$
 $= 2 \int_0^3 x e^{2x} dx = 2 \int_0^3 x dx$
 $= 2 \left[x^2 \Big|_0^3 - \int_0^3 e^{2x} dx \right]$
 $= 6e^3 - 2e^3 \Big|_0^3 = 6e^3 - 2e^3 - 1$
 $= 4e^3 + 2$

(1) $\int_0^1 x dx = 2$

$= 2 \int_0^1 x dx = 4$

$= 4 \int_0^1 x dx = 4$

$= 4$

4.4

1. 判定下

(1) $\int_0^1 x dx = -3$

$= \lim_{x \rightarrow 0} x^2 = 0$

$= 0 +$

(2) $\int_0^1 x dx = \lim_{x \rightarrow 0} x^2 = 0$

$= \lim_{x \rightarrow 0} x^2 = 0$

$= \lim_{x \rightarrow 0} x^2 = 0$

$= \lim_{x \rightarrow 0} x^2 = 0$

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$= \lim_{x \rightarrow 0} x^2 = 0$

$= \lim_{x \rightarrow 0} x^2 = 0$

$$\begin{aligned}
 (4) \int_{-\infty}^{+\infty} \sin^2 x dx &= \int_0^{2\pi} \sin^2 x dx = \int_0^{\pi} \sin^2 x dx + \int_{\pi}^{2\pi} \sin^2 x dx \\
 &= 2 \int_0^{\pi} \sin^2 x dx = 2 \int_0^{\frac{\pi}{2}} \sin^2 x dx + 2 \int_{\frac{\pi}{2}}^{\pi} \sin^2 x dx \\
 &= 4 \int_0^{\frac{\pi}{2}} \sin^2 x dx = 4 \cdot \frac{\pi}{4} \cdot \frac{1}{2} = \frac{\pi}{2}
 \end{aligned}$$

5.4
1. 判定下列反常积分的敛散性, 如果收敛, 计算反常积分的值.

$$\begin{aligned}
 (1) \int_1^{+\infty} \frac{dx}{x^2} &= -\frac{1}{x} \Big|_1^{+\infty} = 0 - (-1) = 1 \\
 &= \lim_{x \rightarrow +\infty} \left(-\frac{1}{x}\right) + \frac{1}{1} \\
 &= 0 + \frac{1}{1} = \frac{1}{1}
 \end{aligned}$$

$$\begin{aligned}
 (2) \int_{-\infty}^{+\infty} \frac{dx}{x^2 + 2x + 3} &= \int_{-\infty}^{+\infty} \frac{dx}{(x+1)^2 + 2} \\
 &= \lim_{x \rightarrow +\infty} \arctan\left(\frac{x+1}{\sqrt{2}}\right) - \lim_{x \rightarrow -\infty} \arctan\left(\frac{x+1}{\sqrt{2}}\right) \\
 &= \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) = \pi
 \end{aligned}$$

$$\begin{aligned}
 (1) \int_0^1 \frac{x}{\sqrt{1-x}} dx &= -\frac{2}{3} \int_0^1 (1-x)^{-\frac{3}{2}} dx \\
 &= -\frac{2}{3} \left[-\frac{2}{1} (1-x)^{-\frac{1}{2}} \right]_0^1 \\
 &= -\sqrt{1-x} \Big|_0^1 = 0 - (-1) = 1
 \end{aligned}$$

$$\begin{aligned}
 (1) \int_0^1 \frac{dx}{(1-x)^2} &= -\frac{1}{1-x} \Big|_0^1 \\
 &= \frac{1}{1-x} \Big|_0^1 \\
 &= \lim_{x \rightarrow 1^-} \frac{1}{1-x} - \frac{1}{1-0} \\
 &= +\infty - 1 = +\infty
 \end{aligned}$$

2. 计算反常积分 $\int_0^{+\infty} e^{-x} dx$

$$\begin{aligned}
 &= 0 - \lim_{x \rightarrow +\infty} e^{-x} \\
 &= 0 - \lim_{x \rightarrow +\infty} \frac{e^{-x}}{x} \\
 &= 0 - 0 = 0
 \end{aligned}$$

$$\begin{aligned}
 (3) \int_0^1 \frac{x}{\sqrt{1-x^2}} dx &= -\frac{1}{2} \int_0^1 \frac{1}{\sqrt{1-x^2}} d(1-x^2) \\
 &= -\frac{1}{2} \left[\arcsin(1-x^2) \right]_0^1 = 0
 \end{aligned}$$

$$\begin{aligned}
 (1) \int_0^1 \frac{dx}{(1-x)^2} &= \frac{1}{1-x} \Big|_0^1 \\
 &= \lim_{x \rightarrow 1^-} \frac{1}{1-x} - \frac{1}{1-0} \\
 &= +\infty - 1 = +\infty
 \end{aligned}$$

2. 计算反常积分 $\int_0^{+\infty} \ln x dx$

$$\begin{aligned}
 \int_0^{+\infty} \ln x dx &= x \ln x \Big|_0^{+\infty} - \int_0^{+\infty} x \cdot \frac{1}{x} dx \\
 &= 0 - \lim_{x \rightarrow +\infty} x \ln x - \int_0^{+\infty} 1 dx \\
 &= -\lim_{x \rightarrow +\infty} \frac{x \ln x}{x} - \lim_{x \rightarrow +\infty} \frac{x}{x} \\
 &= 0 - 1 = -1
 \end{aligned}$$

总习题五

1. 设 $I = \int_1^x \frac{x^2}{\sqrt{1+x^2}} dx$, 则估计了值的大致范围是 (B).

- (A) $0 < I < \sqrt{2}$
- (B) $\frac{\sqrt{2}}{2} < I < \frac{1}{2}$
- (C) $\frac{1}{2} < I < 1$
- (D) $I > 1$

在 $[0, 1]$ 上, $\frac{x^2}{\sqrt{1+x^2}} \leq \frac{x^2}{1+x^2} \leq x^2$,
 由定积分性质得 $\int_0^1 \frac{x^2}{\sqrt{1+x^2}} dx \leq \int_0^1 x^2 dx = \frac{1}{3}$, (B)
 又 $\frac{x^2}{\sqrt{1+x^2}} \geq \frac{x^2}{1+x^2} \geq \frac{x^2}{2}$, (B)
 故 $I = \int_1^x \frac{x^2}{\sqrt{1+x^2}} dx = \int_1^x \frac{x^2}{1+x^2} dx + \int_1^x \frac{x^2}{\sqrt{1+x^2}} dx - \int_1^x \frac{x^2}{1+x^2} dx$
 且 $f(1) = 0, f(x) = \int_1^x \frac{x^2}{\sqrt{1+x^2}} dx$, 故

令 $u = x - \frac{1}{x}$, 则 $du = 1 + \frac{1}{x^2} dx$, $\frac{1}{x^2} dx = -\frac{1}{x^2} du$
 当 $x \rightarrow 0^+$ 时, $u \rightarrow -\infty$; 当 $x \rightarrow +\infty$ 时, $u \rightarrow +\infty$
 $\lim_{x \rightarrow 0^+} \frac{1}{x^2} \int_0^x \frac{1}{\sqrt{1+t^2}} dt = \lim_{x \rightarrow 0^+} \frac{1}{x^2} \int_0^x \frac{1}{\sqrt{1+t^2}} dt = \lim_{x \rightarrow 0^+} \frac{1}{x^2} \ln \frac{1+\sqrt{1+x^2}}{1}$
 $= \lim_{x \rightarrow 0^+} \frac{\ln(1+\sqrt{1+x^2})}{x^2} = \lim_{x \rightarrow 0^+} \frac{\frac{1}{2} \cdot \frac{2x}{\sqrt{1+x^2}}}{2x} = \lim_{x \rightarrow 0^+} \frac{1}{2\sqrt{1+x^2}} = \frac{1}{2}$

1. 求极限

(1) $\lim_{n \rightarrow \infty} \left[\frac{(2n)!}{n! n!} \right]^{\frac{1}{n}}$
 $= \lim_{n \rightarrow \infty} \left[\left(1 + \frac{1}{n}\right) \left(1 + \frac{1}{2n}\right) \cdots \left(1 + \frac{1}{n}\right) \right]^{\frac{1}{n}}$
 $= \lim_{n \rightarrow \infty} e^{\frac{1}{n} \ln \left[\left(1 + \frac{1}{n}\right) \left(1 + \frac{1}{2n}\right) \cdots \left(1 + \frac{1}{n}\right) \right]}$
 $= \lim_{n \rightarrow \infty} e^{\frac{1}{n} \left[\ln \left(1 + \frac{1}{n}\right) + \ln \left(1 + \frac{1}{2n}\right) + \cdots + \ln \left(1 + \frac{1}{n}\right) \right]}$
 $= e^{\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \ln \left(1 + \frac{1}{kn}\right)}$
 $= e^{\lim_{n \rightarrow \infty} \int_0^1 \ln(1+x) dx} = e^{\int_0^1 \ln(1+x) dx} = e^{\frac{1}{2}}$

其中 $\int_0^1 \ln(1+x) dx = x \ln(1+x) - \int_0^1 x \cdot \frac{1}{1+x} dx$
 $= x \ln(1+x) - \int_0^1 \frac{x}{1+x} dx = x \ln(1+x) - \int_0^1 \frac{x+1-1}{1+x} dx$
 $= x \ln(1+x) - \int_0^1 (1 - \frac{1}{1+x}) dx = x \ln(1+x) - x + \ln(1+x) \Big|_0^1$
 $= 1 \ln 2 - 1 + \ln 2 = 2 \ln 2 - 1$

(2) $\lim_{n \rightarrow \infty} \frac{\sin \frac{\pi}{n+1} + \sin \frac{\pi}{n} + \cdots + \sin \frac{\pi}{n+1}}{n+1}$
 $= \lim_{n \rightarrow \infty} \frac{1}{n+1} \sum_{k=1}^n \sin \frac{\pi}{k+1}$
 $= \lim_{n \rightarrow \infty} \int_0^1 \sin \pi x dx = \int_0^1 \sin \pi x dx = \frac{1}{\pi} [-\cos \pi x]_0^1 = \frac{1}{\pi} (1 - \cos \pi) = \frac{2}{\pi}$

因为 $\lim_{n \rightarrow \infty} \frac{1}{n+1} \sum_{k=1}^n \sin \frac{\pi}{k+1} = \int_0^1 \sin \pi x dx = \frac{2}{\pi}$
 $\lim_{n \rightarrow \infty} \frac{1}{n+1} \sum_{k=1}^n \sin \frac{\pi}{k+1} = \frac{2}{\pi}$
 $\lim_{n \rightarrow \infty} \frac{1}{n+1} \sum_{k=1}^n \sin \frac{\pi}{k+1} = \frac{2}{\pi}$

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4. 计算下列积分.

(1) $\int_0^1 \max\{1, x^2\} dx$
 $= 2 \int_0^1 \max\{1, x^2\} dx$
 $= 2 \left[\int_0^1 1 dx + \int_1^2 x^2 dx \right]$
 $= 2 \left[x \Big|_0^1 + \frac{x^3}{3} \Big|_1^2 \right]$
 $= 2 \left[(1-0) + \left(\frac{8}{3} - \frac{1}{3} \right) \right]$
 $= 6\frac{2}{3}$

(2) $\int_0^{+\infty} \frac{e^{-x} + e^{-2x}}{e^{2x} + 1} dx$
 $= \int_0^{+\infty} \frac{e^{-3x} + e^{-x}}{1 + (e^{-x})^2} dx$
 $= e^{-2} \arctan e^{-x} \Big|_0^{+\infty} - \arctan e^{-x} \Big|_0^{+\infty}$
 $= e^{-2} (\frac{\pi}{2} - \arctan e^{-1})$

(3) $\int_0^1 \sqrt{x} dx$
 $= \frac{2}{3} x^{\frac{3}{2}} \Big|_0^1 = \frac{2}{3}$

(3) $\int_0^{\frac{\pi}{2}} x^2 \cos x dx$
 $= \int_0^{\frac{\pi}{2}} x^2 \cos x dx - \int_0^{\frac{\pi}{2}} x^2 \sin x dx$
 $= \int_0^{\frac{\pi}{2}} x^2 d \sin x - \int_0^{\frac{\pi}{2}} x^2 d \cos x$
 $= x^2 \sin x \Big|_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} 2x dx \cos x - [x^2 \cos x - \int_0^{\frac{\pi}{2}} 2x dx \sin x]$
 $= \frac{\pi^2}{4} + \int_0^{\frac{\pi}{2}} 2x d \cos x + \frac{\pi^2}{4} - \int_0^{\frac{\pi}{2}} 2x d \cos x$
 $= \frac{\pi^2}{2} + [2x \cos x \Big|_0^{\frac{\pi}{2}} - 2 \int_0^{\frac{\pi}{2}} \cos x dx] - [2x \cos x \Big|_0^{\frac{\pi}{2}} - 2 \int_0^{\frac{\pi}{2}} \sin x dx]$
 $= \frac{\pi^2}{2} - 2 \cos x \Big|_0^{\frac{\pi}{2}} + 2x \Big|_0^{\frac{\pi}{2}} + 2 \sin x \Big|_0^{\frac{\pi}{2}} = \frac{\pi^2}{2} + 2\pi - 4$

(4) $\int_0^a \frac{x + \sqrt{a^2 - x^2}}{a^2 + x^2} dx$ ($a > 0$)
 $\text{令 } x = a \sin t, dx = a \cos t dt$, 于是
 $\int_0^a \frac{x + \sqrt{a^2 - x^2}}{a^2 + x^2} dx = \int_0^{\frac{\pi}{2}} \frac{a \sin t + a \cos t}{a^2 + a^2 \sin^2 t} a \cos t dt$
 $= \int_0^{\frac{\pi}{2}} \frac{\sin t + \cos t}{1 + \sin^2 t} dt = \int_0^{\frac{\pi}{2}} \frac{\sin t}{1 + \sin^2 t} dt + \int_0^{\frac{\pi}{2}} \frac{\cos t}{1 + \sin^2 t} dt$
 $= \frac{1}{2} \int_0^{\frac{\pi}{2}} \frac{2 \sin t}{1 + \sin^2 t} dt + \int_0^{\frac{\pi}{2}} \frac{\cos t}{1 + \sin^2 t} dt$
 $= \frac{1}{2} \int_0^{\frac{\pi}{2}} \frac{du}{1 + u^2} + \int_0^{\frac{\pi}{2}} \frac{du}{1 + u^2}$
 $= \frac{1}{2} \arctan u \Big|_0^{\frac{\pi}{2}} + \arctan u \Big|_0^{\frac{\pi}{2}}$
 $= \frac{1}{2} \arctan 1 + \arctan 1 = \frac{3}{4} \arctan 1 = \frac{3\pi}{16}$

(5) $\int_0^1 \frac{dx}{\sqrt{1-x^2}}$
 $= \int_0^1 \frac{dx}{\sqrt{1-x^2}} + \int_1^2 \frac{dx}{\sqrt{x^2-1}}$ ($x = \sqrt{1-x^2}$)
 $= \int_0^1 \frac{dx}{\sqrt{1-x^2}} + \int_1^2 \frac{dx}{\sqrt{x^2-1}}$
 $= \arcsin x \Big|_0^1 + \ln |x + \sqrt{x^2-1}| \Big|_1^2$
 $= (\frac{\pi}{2} - 0) + [\ln(2 + \sqrt{3}) - \ln(1 + \sqrt{0})]$
 $= \frac{\pi}{2} + \ln(2 + \sqrt{3})$

1. (积分第一中值定理) 设 $f(x)$ 在区间 $[a, b]$ 上连续, $\varphi(x)$ 在区间 $[a, b]$ 上连续且不恒为零, 证明: 至少存在一点 $\xi \in [a, b]$, 使

$$\int_a^b f(x)\varphi(x)dx = f(\xi) \int_a^b \varphi(x)dx.$$

证: 不妨设 $\varphi(x) \geq 0$. 若 $\varphi(x) \equiv 0$, 则对任意 $\xi \in [a, b]$ 均成立. 若 $\varphi(x) \not\equiv 0$, 则 $\int_a^b \varphi(x)dx > 0$. 因为 $f(x)$ 在 $[a, b]$ 上连续, 所以 $f(x)$ 在 $[a, b]$ 上有最大值 M 和最小值 m , 所以 $m \leq f(x) \leq M, \forall x \in [a, b]$.

由定积分的性质, 有 $\int_a^b m\varphi(x)dx \leq \int_a^b f(x)\varphi(x)dx \leq \int_a^b M\varphi(x)dx$.
 $\Rightarrow m \int_a^b \varphi(x)dx \leq \int_a^b f(x)\varphi(x)dx \leq M \int_a^b \varphi(x)dx$
 $\Rightarrow m \leq \frac{\int_a^b f(x)\varphi(x)dx}{\int_a^b \varphi(x)dx} \leq M$
 记 $\xi = \frac{\int_a^b f(x)\varphi(x)dx}{\int_a^b \varphi(x)dx}$, 则 $m \leq \xi \leq M$.
 根据介值定理, 存在 $\xi \in [a, b]$, 使得 $f(\xi) = \frac{\int_a^b f(x)\varphi(x)dx}{\int_a^b \varphi(x)dx}$.
 从而 $f(\xi) \int_a^b \varphi(x)dx = \int_a^b f(x)\varphi(x)dx$.

证明: 存在 $\xi \in (a, b)$, 使得 $\int_a^b f(x)dx = f(\xi)(b-a)$.

证: 令 $F(x) = \int_a^x f(t)dt$, 则 $F(x)$ 在 $[a, b]$ 上连续, 在 (a, b) 内可导, 且 $F'(x) = f(x)$.
 由介值定理, 存在 $\xi \in (a, b)$, 使得 $F(b) - F(a) = f(\xi)(b-a)$.

$$\frac{F(b) - F(a)}{b - a} = f(\xi)$$

$$\frac{\int_a^b f(x)dx}{b - a} = f(\xi)$$

$$\int_a^b f(x)dx = f(\xi)(b-a)$$

2. 设 $f(x)$ 在 $[a, b]$ 上连续, 证明: $\int_a^b f(x)dx \leq \int_a^b |f(x)|dx$.

证: 在 $[a, b]$ 上, $|f(x)| \geq f(x)$, 所以 $\int_a^b |f(x)|dx \geq \int_a^b f(x)dx$.

$$\int_a^b |f(x)|dx - \int_a^b f(x)dx \geq 0$$

$$\int_a^b (|f(x)| - f(x))dx \geq 0$$

在 $[a, b]$ 上, $|f(x)| - f(x) \geq 0$, 所以 $\int_a^b (|f(x)| - f(x))dx \geq 0$.

因此 $\int_a^b f(x)dx \leq \int_a^b |f(x)|dx$.

$$\int_a^b f(x)dx \leq \int_a^b |f(x)|dx$$

$$\int_a^b f(x)dx \leq \int_a^b |f(x)|dx$$

1. 设 $f(x)$ 在区间 $[a, b]$ 上有连续的导数, 且 $f'(a) = f'(b) = 0$, 证明: $\int_a^b f(x)dx \leq \frac{(b-a)^2}{4} \max_{x \in [a, b]} |f''(x)|$.

证: 在 $[a, b]$ 上, $f(x) = \int_a^x f'(t)dt$. 因为 $f'(a) = 0$, 所以 $f(x) = \int_a^x f''(\xi)(x-\xi)dt$.

$$|f(x)| = \left| \int_a^x f''(\xi)(x-\xi)dt \right| \leq \int_a^x |f''(\xi)|(x-\xi)dt$$

$$\leq \int_a^x M(x-\xi)dt = M \int_a^x (x-\xi)dt = M \left[x\xi - \frac{\xi^2}{2} \right]_a^x$$

$$= M \left(x^2 - \frac{x^2}{2} - ax + \frac{a^2}{2} \right) = M \left(\frac{x^2}{2} - ax + \frac{a^2}{2} \right)$$

因为 $f'(a) = 0$, 所以 $f'(x) = \int_a^x f''(\xi)dt$. 因为 $f'(b) = 0$, 所以 $\int_a^b f''(\xi)dt = 0$.

$$\int_a^b f''(\xi)dt = 0 \Rightarrow \int_a^b f''(\xi)dt = 0$$

$$\int_a^b f''(\xi)dt = 0 \Rightarrow \int_a^b f''(\xi)dt = 0$$

1. 求出下列函数的极值:

(1) $y = x^2 - 2x + 1$

$$y' = 2x - 2 = 0 \Rightarrow x = 1$$

$$y'' = 2 > 0$$

$$y(1) = 0$$

$$y = x^2 - 2x + 1 = (x-1)^2$$

$$= (x-1)^2 \geq 0$$

$$= 0$$