第四章 不定积分

习 题 四

4.1

1. 求下列不定积分.

(1) $\int x^2 \sqrt[3]{x}\,\mathrm{d}x$;

$= \int x^{\frac{7}{3}} dx = \frac{1}{\frac{7}{3}+1} x^{\frac{7}{3}+1} + C$

$= \frac{3}{10} x^{\frac{10}{3}} + C$

(2) $\int \frac{(1-x)^2}{\sqrt{x}}\,\mathrm{d}x$;

$= \int (x^{\frac{3}{2}} - 2x^{\frac{1}{2}} + x^{-\frac{1}{2}}) dx$

$= \int x^{\frac{3}{2}} dx - 2\int x^{\frac{1}{2}} dx + \int x^{-\frac{1}{2}} dx$

$= \frac{2}{5} x^{\frac{5}{2}} - \frac{4}{3} x^{\frac{3}{2}} + 2x^{\frac{1}{2}} + C$

(3) $\int \left(2\mathrm{e}^x + \frac{3}{x}\right) \mathrm{d}x$;

$= 2\int e^x dx + 3\int \frac{dx}{x}$

$= 2e^x + 3\ln|x| + C$

(4) $\int \frac{2 \cdot 3^x - 5 \cdot 2^x}{3^x}\,\mathrm{d}x$;

$= \int [2 - 5(\frac{2}{3})^x] dx$

$= 2\int dx - 5\int (\frac{2}{3})^x dx$

$= 2x - \frac{5}{\ln\frac{2}{3}} (\frac{2}{3})^x + C$

(5) $\int \cot^2 x\,\mathrm{d}x$;

$= \int (\csc^2 x - 1) dx$

$= \int \csc^2 x dx - \int dx$

$= -\cot x - x + C$

(6) $\int \frac{1+\cos^2 x}{1+\cos 2x}\,\mathrm{d}x$;

$= \int \frac{1+\cos^2 x}{2\cos^2 x} dx$

$= \frac{1}{2}\int \frac{1}{\cos^2 x} dx + \frac{1}{2}\int dx$

$= \frac{1}{2}\tan x + \frac{x}{2} + C$

(7) $\int \frac{\sqrt{1+x^2}}{\sqrt{1-x^4}}\,\mathrm{d}x$;

$= \int \frac{\sqrt{1+x^2}}{\sqrt{(1-x^2)(1+x^2)}} dx$

$= \int \frac{dx}{\sqrt{1-x^2}} = \arcsin x + C$

(8) $\int \frac{3x^4 + 2x^2}{x^2+1}\,\mathrm{d}x$.

$= \int (3x^2 - 1 + \frac{1}{x^2+1}) dx$

$= 3\int x^2 dx - \int dx + \int \frac{dx}{1+x^2}$

$= x^3 - x + \arctan x + C$

2. 一曲线通过点$(e^2,3)$，且在任一点处的切线的斜率等于该点横坐标的倒数，求该曲线的方程.

由题设得

$$y'=\frac{1}{x}$$

所以 $y=\int\frac{dx}{x}=\ln|x|+C$

又因为曲线过点$(e^2,3)$，所以曲线定义在$x>0$上，且

$$3=\ln e^2+C$$

解得$C=1$，故所求曲线方程为

$$y=\ln x+1$$

3. 一物体由静止开始运动，经过 t s后的速度是 $3t^2$ m/s，问：

(1) 在 3 s后物体离开出发点的距离是多少？

(2) 物体走完 360 m 需要多长时间？

设路程函数为$S(t)$，则

$$S'(t)=3t^2$$

所以

$$S(t)=\int 3t^2dt=t^3+C$$

又$S(0)=0$，所以$C=0$，故$S(t)=t^3$

(1) $S(3)=3^3=27$(m)

(2) 由 $S(t)=t^3=360$得

$$t=\sqrt[3]{360}\approx 7.1\,(s)$$

4. 证明：函数 $\arcsin(2x-1)$，$\arccos(1-2x)$ 和 $2\arctan\sqrt{\frac{x}{1-x}}$ 都是$\frac{1}{\sqrt{x-x^2}}$的原函数.

因为

$$[\arcsin(2x-1)]'=\frac{1}{\sqrt{1-(2x-1)^2}}\cdot 2=\frac{1}{\sqrt{x-x^2}}$$

$$[\arccos(1-2x)]'=-\frac{1}{\sqrt{1-(1-2x)^2}}\cdot(-2)=\frac{1}{\sqrt{x-x^2}}$$

$$\left[2\arctan\sqrt{\frac{x}{1-x}}\right]'$$

$$=2\frac{1}{1+\left(\sqrt{\frac{x}{1-x}}\right)^2}\cdot\frac{1}{2}\frac{1}{\sqrt{\frac{x}{1-x}}}\cdot\frac{1-x-x(-1)}{(1-x)^2}$$

$$=\frac{1}{\sqrt{x-x^2}}$$

所以结论成立.

4.2

1. 用第一类换元积分法计算下列不定积分.

(1) $\int\frac{dx}{1-2x}$；

$$=-\frac{1}{2}\int\frac{1}{1-2x}(1-2x)'dx$$

$$=-\frac{1}{2}\int\frac{1}{1-2x}d(1-2x)$$

$$=-\frac{1}{2}\ln|1-2x|+C$$

(2) $\int(3x+2)^{100}dx$；

$$=\frac{1}{3}\int(3x+2)^{100}\cdot(3x+2)'dx$$

$$=\frac{1}{3}\int(3x+2)^{100}d(3x+2)$$

$$=\frac{1}{3}\cdot\frac{1}{101}(3x+2)^{101}+C$$

$$=\frac{1}{303}(3x+2)^{101}+C$$

(3) $\int x e^{-x^2} dx$;

$= -\frac{1}{2}\int e^{-x^2}(-x^2)' dx$

$= -\frac{1}{2}\int e^{-x^2} d(-x^2)$

$= -\frac{1}{2} e^{-x^2} + C$

(4) $\int \frac{\sin \lg x}{x} dx$;

$= \ln 10 \int \sin \lg x \, (\lg x)' dx$

$= \ln 10 \int \sin \lg x \, d \lg x$

$= -\ln 10 \cos \lg x + C$

(5) $\int \sqrt{\frac{\arcsin x}{1-x^2}} dx$;

$= \int \sqrt{\arcsin x} \cdot (\arcsin x)' dx$

$= \int \sqrt{\arcsin x} \, d \arcsin x$

$= \frac{1}{\frac{1}{2}+1}(\arcsin x)^{\frac{1}{2}+1} + C = \frac{2}{3}(\arcsin x)^{\frac{3}{2}} + C$

(6) $\int \frac{\sqrt{x}}{\sqrt{a^3-x^3}} dx \, (a>0)$;

$= \frac{2}{3}\int \frac{1}{\sqrt{(a^{\frac{3}{2}})^2-(x^{\frac{3}{2}})^2}} (x^{\frac{3}{2}})' dx$

$= \frac{2}{3}\int \frac{1}{\sqrt{(a^{\frac{3}{2}})^2-(x^{\frac{3}{2}})^2}} d(x^{\frac{3}{2}})$

$= \frac{2}{3} \arcsin \frac{x^{\frac{3}{2}}}{a^{\frac{3}{2}}} + C$

$= \frac{2}{3} \arcsin \left(\frac{x}{a}\right)^{\frac{3}{2}} + C$

(7) $\int \frac{3^x}{1+9^x} dx$;

$= \frac{1}{\ln 3}\int \frac{1}{1+(3^x)^2}(3^x)' dx$

$= \frac{1}{\ln 3}\int \frac{1}{1+(3^x)^2} d(3^x)$

$= \frac{1}{\ln 3} \arctan(3^x) + C$

(8) $\int \frac{\sin x + \cos x}{\sqrt[3]{\sin x - \cos x}} dx$;

$= \int \frac{1}{\sqrt[3]{\sin x - \cos x}} (\sin x - \cos x)' dx$

$= \int (\sin x - \cos x)^{-\frac{1}{3}} d(\sin x - \cos x)$

$= \frac{1}{-\frac{1}{3}+1}(\sin x - \cos x)^{-\frac{1}{3}+1} + C = \frac{3}{2}(\sin x - \cos x)^{\frac{2}{3}} + C$

(9) $\int \frac{1+\sin 3x}{\cos^2 3x} dx$;

$= \int \frac{1}{\cos^2 3x} dx + \int \frac{\sin 3x}{\cos^2 3x} dx$

$= \frac{1}{3}\int \frac{1}{\cos^2 3x} d(3x) - \frac{1}{3}\int \frac{1}{\cos^2 3x} d(\cos 3x)$

$= \frac{1}{3}\tan 3x + \frac{1}{3}\frac{1}{\cos 3x} + C$

(10) $\int \frac{dx}{x \ln x \ln \ln x}$;

$= \int \frac{1}{\ln x \ln \ln x} d(\ln x)$

$= \int \frac{1}{\ln \ln x} d(\ln \ln x)$

$= \ln|\ln \ln x| + C$

(11) $\int \frac{\ln \tan x}{\cos x \sin x} dx$;

$= \int \frac{\ln \tan x}{\tan x} \cdot \frac{1}{\cos^2 x} dx$

$= \int \frac{\ln \tan x}{\tan x} d(\tan x)$

$= \int \ln \tan x \, d(\ln \tan x)$

$= \frac{1}{2} (\ln \tan x)^2 + C$

(12) $\int \tan^3 \frac{x}{3} \sec^2 \frac{x}{3} dx$;

$= 3 \int \tan^3 \frac{x}{3} \sec^2 \frac{x}{3} d(\frac{x}{3})$

$= 3 \int \tan^3 \frac{x}{3} d(\tan \frac{x}{3})$

$= \frac{3}{4} \tan^4 \frac{x}{3} + C$

(13) $\int \cos x \cos \frac{x}{2} dx$;

$= \int \frac{1}{2} (\cos \frac{x}{2} + \cos \frac{3x}{2}) dx$

$= \frac{1}{2} \int \cos \frac{x}{2} dx + \frac{1}{2} \int \cos \frac{3x}{2} dx$

$= \int \cos \frac{x}{2} d(\frac{x}{2}) + \frac{1}{3} \int \cos \frac{3x}{2} d(\frac{3x}{2})$

$= \sin \frac{x}{2} + \frac{1}{3} \sin \frac{3x}{2} + C$

(14) $\int \sec^4 x dx$;

$= \int \sec^2 x \sec^2 x dx$

$= \int (1 + \tan^2 x) d \tan x$

$= \tan x + \frac{1}{3} \tan^3 x + C$

(15) $\int \tan^3 x \sec x dx$;

$= \int \tan^2 x (\sec x \tan x) dx$

$= \int (\sec^2 x - 1) d \sec x$

$= \frac{1}{3} \sec^3 x - \sec x + C$

(16) $\int \frac{\sin 2x}{\sqrt{1 - \cos^4 x}} dx$;

$= \int \frac{2 \sin x \cos x}{\sqrt{1 - \cos^4 x}} dx = - \int \frac{2 \cos x}{\sqrt{1 - \cos^4 x}} d \cos x$

$= - \int \frac{1}{\sqrt{1 - (\cos^2 x)^2}} d(\cos^2 x)$

$= - \arcsin(\cos^2 x) + C$

(17) $\int \frac{1}{1 + \sin x} dx$;

$= \int \frac{1 - \sin x}{1 - \sin^2 x} dx = \int \frac{1 - \sin x}{\cos^2 x} dx$

$= \int \frac{dx}{\cos^2 x} - \int \frac{\sin x}{\cos^2 x} dx$

$= \int \frac{dx}{\cos^2 x} + \int \frac{1}{\cos^2 x} d \cos x$

$= \tan x - \frac{1}{\cos x} + C$

(18) $\int \frac{x - 1}{x^2 + 2x + 3} dx$.

$= \int \frac{x + 1 - 2}{x^2 + 2x + 3} dx$

$= \int \frac{x + 1}{x^2 + 2x + 3} dx - 2 \int \frac{dx}{x^2 + 2x + 3}$

$= \frac{1}{2} \int \frac{1}{x^2 + 2x + 3} d(x^2 + 2x + 3) - 2 \int \frac{1}{(x+1)^2 + (\sqrt{2})^2} dx$

$= \frac{1}{2} \ln(x^2 + 2x + 3) - 2 \cdot \frac{1}{\sqrt{2}} \arctan \frac{x+1}{\sqrt{2}} + C$

$= \frac{1}{2} \ln(x^2 + 2x + 3) - \sqrt{2} \arctan \frac{x+1}{\sqrt{2}} + C$

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2. 用第二类换元积分法计算下列不定积分.

(1) $\int \frac{dx}{1+\sqrt{2x}}$;

令 $t=\sqrt{2x}$，则 $x=\frac{t^2}{2}$，$dx = t\,dt$，

于是 $\int \frac{dx}{1+\sqrt{2x}} = \int \frac{t\,dt}{1+t}$

$= \int \left(1-\frac{1}{1+t}\right)dt = \int dt - \int \frac{1}{1+t}d(1+t)$

$= t - \ln(1+t) + C$

$= \sqrt{2x} - \ln(1+\sqrt{2x}) + C$

(2) $\int \frac{dx}{\sqrt{1+e^x}}$;

令 $t=\sqrt{1+e^x}$，则 $x=\ln(t^2-1)$，$dx=\frac{2t}{t^2-1}dt$，

于是 $\int \frac{dx}{\sqrt{1+e^x}} = \int \frac{\frac{2t}{t^2-1}dt}{t}$

$= 2\int \frac{dt}{t^2-1} = 2\cdot\frac{1}{2}\ln\left|\frac{t-1}{t+1}\right| + C$

$= \ln \frac{\sqrt{1+e^x}-1}{\sqrt{1+e^x}+1} + C$

(3) $\int \frac{x^2}{\sqrt{a^2-x^2}}dx\ (a>0)$;

令 $x = a\sin t\ \left(-\frac{\pi}{2}<t<\frac{\pi}{2}\right)$，则 $dx = a\cos t\,dt$，

于是 $\int \frac{x^2}{\sqrt{a^2-x^2}}dx = \int \frac{(a\sin t)^2}{\sqrt{a^2-(a\sin t)^2}}\cdot a\cos t\,dt$

$= \int \frac{a^2\sin^2 t}{a\cos t}\cdot a\cos t\,dt = a^2\int \sin^2 t\,dt$

$= a^2\int \frac{1-\cos 2t}{2}dt = \frac{a^2}{2}\left(t-\frac{1}{2}\sin 2t\right) + C$

$= \frac{a^2}{2}\arcsin\frac{x}{a} - \frac{x}{2}\sqrt{a^2-x^2} + C$

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(4) $\int \frac{\sqrt{x^2+a^2}}{x^2}dx\ (a>0)$;

令 $x = a\tan t\ \left(-\frac{\pi}{2}<t<\frac{\pi}{2}\right)$，则 $dx = a\sec^2 t\,dt$，

于是 $\int \frac{\sqrt{x^2+a^2}}{x^2}dx = \int \frac{\sqrt{(a\tan t)^2+a^2}}{(a\tan t)^2}\cdot a\sec^2 t\,dt$

$= \int \frac{a\sec t}{a^2\tan^2 t}\cdot a\sec^2 t\,dt = \int \frac{1}{\sin^2 t\cos t}dt$

$= \int \frac{\sin^2 t+\cos^2 t}{\sin^2 t\cos t}dt = \int\left(\frac{1}{\cos t}+\frac{\cos t}{\sin^2 t}\right)dt$

$= \int \sec t\,dt + \int \frac{1}{\sin^2 t}d\sin t$

$= \ln|\sec t+\tan t| - \frac{1}{\sin t} + C$

$= \ln\left|\frac{\sqrt{x^2+a^2}}{a}+\frac{x}{a}\right| - \frac{\sqrt{x^2+a^2}}{x} + C_1$

$= \ln|x+\sqrt{x^2+a^2}| - \frac{\sqrt{x^2+a^2}}{x} + C$ (C= [illegible]

(5) $\int \frac{\sqrt{x^2-9}}{x}dx$;

令 $x = 3\sec t$ $\left(0\le t<\frac{\pi}{2}\right.$ 或 $\left.\pi\le t<\frac{3\pi}{2}\right)$，则 $dx = 3\sec t\tan t\,dt$，

于是 $\int \frac{\sqrt{x^2-9}}{x}dx = \int \frac{\sqrt{(3\sec t)^2-9}}{3\sec t}\cdot 3\sec t\tan t\,dt$

$= \int \frac{3\tan t}{3\sec t}\cdot 3\sec t\tan t\,dt = 3\int \tan^2 t\,dt$

$= 3\int(\sec^2 t-1)dt = 3\tan t - 3t + C$

$= \sqrt{x^2-9} - 3\arccos\frac{3}{x} + C$

(6) $\int \frac{dx}{x+\sqrt{1-x^2}}$;

令 $x = \sin t\ \left(-\frac{\pi}{2}\le t\le\frac{\pi}{2}\right)$，则 $dx = \cos t\,dt$，

于是 $\int \frac{dx}{x+\sqrt{1-x^2}} = \int \frac{\cos t\,dt}{\sin t+\sqrt{1-\sin^2 t}}$

$= \int \frac{\cos t}{\sin t+\cos t}dt = \frac{1}{2}\int \frac{\cos t+\sin t+\cos t-\sin t}{\sin t+\cos t}dt$

$= \frac{1}{2}\int\left(1+\frac{\cos t-\sin t}{\sin t+\cos t}\right)dt$

$= \frac{1}{2}\int dt + \frac{1}{2}\int \frac{1}{\sin t+\cos t}d(\sin t+\cos t)$

$= \frac{t}{2} + \frac{1}{2}\ln|\sin t+\cos t| + C$

$= \frac{1}{2}\arcsin x + \frac{1}{2}\ln|x+\sqrt{1-x^2}| + C$

(7) $\int \frac{1}{1+\sqrt{x^2+2x+2}}\mathrm{d}x$.

$= \int \frac{1}{1+\sqrt{(x+1)^2+1}}dx$

令 $x+1=\tan t\ (-\frac{\pi}{2}<t<\frac{\pi}{2})$，则 $dx=\sec^2 t\,dt$

于是 $\int \frac{dx}{1+\sqrt{x^2+2x+2}} = \int \frac{\sec^2 t\,dt}{1+\sqrt{\tan^2 t+1}}$

$= \int \frac{1}{\cos t(1+\cos t)}dt = \int\left(\frac{1}{\cos t} - \frac{1}{1+\cos t}\right)dt$

$= \int \sec t\,dt - \int \frac{1}{\cos^2\frac{t}{2}}d\left(\frac{t}{2}\right)$

$= \ln|\sec t+\tan t| - \tan\frac{t}{2} + C$

$= \ln|x+1+\sqrt{x^2+2x+2}| - \frac{\sqrt{x^2+2x+2}-1}{x+1} + C$

4.3

1. 用分部积分法计算下列不定积分.

(1) $\int x\sin x\,\mathrm{d}x$;

$= -\int x\,d\cos x$

$= -x\cos x + \int \cos x\,dx$

$= -x\cos x + \sin x + C$

(2) $\int x e^{-x}\mathrm{d}x$;

$= -\int x\,de^{-x}$

$= -xe^{-x} + \int e^{-x}dx$

$= -xe^{-x} - e^{-x} + C$

(3) $\int \arctan x\,\mathrm{d}x$;

$= x\arctan x - \int x\,d(\arctan x)$

$= x\arctan x - \int x\cdot\frac{1}{1+x^2}dx$

$= x\arctan x - \frac{1}{2}\int \frac{1}{1+x^2}d(1+x^2)$

$= x\arctan x - \frac{1}{2}\ln(1+x^2) + C$

(4) $\int x\ln(x-1)\mathrm{d}x$;

$= \frac{1}{2}\int \ln(x-1)\,d(x^2-1)$

$= \frac{1}{2}\left[(x^2-1)\ln(x-1) - \int (x^2-1)\,d[\ln(x-1)]\right]$

$= \frac{1}{2}(x^2-1)\ln(x-1) - \frac{1}{2}\int (x^2-1)\cdot\frac{1}{x-1}dx$

$= \frac{1}{2}(x^2-1)\ln(x-1) - \frac{1}{2}\int (x+1)dx$

$= \frac{1}{2}(x^2-1)\ln(x-1) - \frac{x^2}{4} - \frac{x}{2} + C$

(5) $\int \ln^2 x\,\mathrm{d}x$;

$= x\ln^2 x - \int x\,d(\ln^2 x)$

$= x\ln^2 x - \int x\cdot 2\ln x\cdot\frac{1}{x}dx$

$= x\ln^2 x - 2\int \ln x\,dx$

$= x\ln^2 x - 2\left[x\ln x - \int x\,d\ln x\right]$

$= x\ln^2 x - 2x\ln x + 2\int x\cdot\frac{1}{x}dx$

$= x\ln^2 x - 2x\ln x + 2x + C$

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(6) $\int \frac{\arcsin x}{\sqrt{1+x}}\mathrm{d}x$；

$= 2\int \arcsin x \, d\sqrt{1+x}$

$= 2\left[\sqrt{1+x}\arcsin x - \int \sqrt{1+x}\, d(\arcsin x)\right]$

$= 2\sqrt{1+x}\arcsin x - 2\int \sqrt{1+x}\frac{1}{\sqrt{1-x^2}}dx$

$= 2\sqrt{1+x}\arcsin x - 2\int \frac{1}{\sqrt{1-x}}dx$

$= 2\sqrt{1+x}\arcsin x + 2\int \frac{1}{\sqrt{1-x}}d(1-x)$

$= 2\sqrt{1+x}\arcsin x + 4\sqrt{1-x} + C$

(7) $\int \frac{x}{\sin^2 x}\mathrm{d}x$；

$= -\int x \, d\cot x$

$= -x\cot x + \int \cot x \, dx$

$= -x\cot x + \ln|\sin x| + C$

(8) $\int \mathrm{e}^{-2x}\sin\frac{x}{2}\mathrm{d}x$；

$= -\frac{1}{2}\int \sin\frac{x}{2}\, d(e^{-2x})$

$= -\frac{1}{2}\left[e^{-2x}\sin\frac{x}{2} - \int e^{-2x}\cdot\cos\frac{x}{2}\cdot\frac{1}{2}dx\right]$

$= -\frac{1}{2}e^{-2x}\sin\frac{x}{2} - \frac{1}{8}\int \cos\frac{x}{2}\, d(e^{-2x})$

$= -\frac{1}{2}e^{-2x}\sin\frac{x}{2} - \frac{1}{8}\left[e^{-2x}\cos\frac{x}{2} - \int e^{-2x}\left(-\sin\frac{x}{2}\right)\cdot\frac{1}{2}dx\right]$

$= -\frac{1}{8}\left(4\sin\frac{x}{2} + \cos\frac{x}{2}\right)e^{-2x} - \frac{1}{16}\int e^{-2x}\sin\frac{x}{2}dx$

解得

$\int e^{-2x}\sin\frac{x}{2}dx = -\frac{2}{17}\left(4\sin\frac{x}{2} + \cos\frac{x}{2}\right)e^{-2x} + C$

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(9) $\int x^2\cos^2\frac{x}{2}\mathrm{d}x$；

$= \int x^2\cdot\frac{1+\cos x}{2}dx = \frac{1}{2}\int x^2 dx + \frac{1}{2}\int x^2\cos x\, dx$

$= \frac{x^3}{6} + \frac{1}{2}\int x^2 \, d\sin x$

$= \frac{x^3}{6} + \frac{1}{2}\left[x^2\sin x - \int \sin x\cdot 2x\, dx\right]$

$= \frac{x^3}{6} + \frac{x^2}{2}\sin x + \int x\, d\cos x$

$= \frac{x^3}{6} + \frac{x^2}{2}\sin x + x\cos x - \int \cos x\, dx$

$= \frac{x^3}{6} + \frac{x^2}{2}\sin x + x\cos x - \sin x + C$

(10) $\int \mathrm{e}^{\sqrt[3]{x}}\mathrm{d}x$；

令 $t=\sqrt[3]{x}$，则 $x=t^3$，$dx=3t^2dt$，于是

$\int e^{\sqrt[3]{x}}dx = \int e^{t}\cdot 3t^2 dt = \int 3t^2 de^{t}$

$= 3t^2e^t - \int e^t\cdot 6t\, dt = 3t^2e^t - \int 6t\, de^t$

$= 3t^2e^t - \left[6te^t - \int e^t\cdot 6dt\right]$

$= 3t^2e^t - 6te^t + 6e^t + C$

$= 3e^{\sqrt[3]{x}}\left(x^{\frac{2}{3}} - 2x^{\frac{1}{3}} + 2\right) + C$

(11) $\int \cos\ln x\,\mathrm{d}x$.

$= x\cos\ln x - \int x\cdot(-\sin\ln x)\cdot\frac{1}{x}dx$

$= x\cos\ln x + \int \sin\ln x\, dx$

$= x\cos\ln x + x\sin\ln x - \int x\cos\ln x\cdot\frac{1}{x}dx$

$= x(\cos\ln x + \sin\ln x) - \int \cos\ln x\, dx$

解得

$\int \cos\ln x\, dx = \frac{x}{2}(\cos\ln x + \sin\ln x) + C$

2. 已知 $(1+\sin x)\ln x$ 是 $f(x)$ 的一个原函数，求 $\int x f'(x)\mathrm{d}x$.

由题设知

$f(x)=[(1+\sin x)\ln x]'=\frac{1}{x}(1+\sin x)+\cos x\ln x$

于是 $\int x f'(x)dx=\int x\,df(x)$

$=xf(x)-\int f(x)dx$

$=x\left[\frac{1}{x}(1+\sin x)+\cos x\ln x\right]-(1+\sin x)\ln x+C$

$=x\cos x\ln x+(1+\sin x)(1-\ln x)+C$

3. 当 $x\geqslant 0$ 时，$F(x)$ 是 $f(x)$ 的一个原函数，已知 $f(x)F(x)=\sin^2 2x$，且 $F(0)=1$，$F(x)\geqslant 0$，求 $f(x)$.

由题设知 $F'(x)=f(x)$，所以

$f(x)F(x)=F'(x)F(x)=\sin^2 2x$

于是 $\int F'(x)F(x)dx=\int \sin^2 2x\,dx$

$\Rightarrow \frac{1}{2}F^2(x)=\int\frac{1-\cos 4x}{2}dx$

$=\frac{1}{2}\int dx-\frac{1}{2}\int\cos 4x\,dx$

$=\frac{x}{2}-\frac{1}{8}\sin 4x+C$

由 $F(0)=1$ 得 $C=\frac{1}{2}$，所以

$F^2(x)=x-\frac{1}{4}\sin 4x+1$

解得 $F(x)=\frac{\sqrt{4x-\sin 4x+4}}{2}$

从而

$f(x)=\frac{\sin^2 2x}{F(x)}=\frac{2\sin^2 2x}{\sqrt{4x-\sin 4x+4}}$

4.4

1. 计算下列有理函数的不定积分.

(1) $\int\frac{x^3}{x+3}\mathrm{d}x$;

$=\int\left(x^2-3x+9-\frac{27}{x+3}\right)dx$

$=\frac{x^3}{3}-\frac{3}{2}x^2+9x-27\ln|x+3|+C$

(2) $\int\frac{x^2+1}{(x+1)^2(x-1)}\mathrm{d}x$;

$=\int\left[\frac{1}{2}\frac{1}{x-1}+\frac{1}{2}\frac{1}{x+1}-\frac{1}{(x+1)^2}\right]dx$

$=\frac{1}{2}\ln|x-1|+\frac{1}{2}\ln|x+1|+\frac{1}{x+1}+C$

$=\frac{1}{2}\ln|x^2-1|+\frac{1}{x+1}+C$

(3) $\int\frac{\mathrm{d}x}{x^4-1}$;

$=\int\frac{dx}{(x-1)(x+1)(x^2+1)}$

$=\int\left(\frac{1}{4}\frac{1}{x-1}-\frac{1}{4}\frac{1}{x+1}-\frac{1}{2}\frac{1}{x^2+1}\right)dx$

$=\frac{1}{4}\ln|x-1|-\frac{1}{4}\ln|x+1|-\frac{1}{2}\arctan x+C$

$=\frac{1}{4}\ln\left|\frac{x-1}{x+1}\right|-\frac{1}{2}\arctan x+C$

(4) $\int\frac{-x^2-2}{(x^2+x+1)^2}\mathrm{d}x$.

$=\int\left(-\frac{1}{x^2+x+1}+\frac{x-1}{(x^2+x+1)^2}\right)dx$

$=-\int\frac{1}{(x+\frac{1}{2})^2+(\frac{\sqrt{3}}{2})^2}d(x+\frac{1}{2})+\frac{1}{2}\int\frac{1}{(x^2+x+1)^2}d(x^2+x+1)$

$-\frac{3}{2}\int\frac{1}{[(x+\frac{1}{2})^2+(\frac{\sqrt{3}}{2})^2]^2}d(x+\frac{1}{2})$

$=-\frac{2}{\sqrt{3}}\arctan\frac{x+\frac{1}{2}}{\frac{\sqrt{3}}{2}}-\frac{1}{2}\frac{1}{x^2+x+1}$

$-\frac{3}{2}\left[\frac{2}{3}\frac{x+\frac{1}{2}}{(x+\frac{1}{2})^2+(\frac{\sqrt{3}}{2})^2}+\frac{2}{3}\int\frac{1}{(x+\frac{1}{2})^2+(\frac{\sqrt{3}}{2})^2}d(x+\frac{1}{2})\right]$

$=-\frac{x+1}{x^2+x+1}-\frac{4}{\sqrt{3}}\arctan\frac{2x+1}{\sqrt{3}}+C$

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2. 计算下列三角函数有理式的不定积分.

(1) $\int \frac{dx}{3+\cos x}$;

令 $u=\tan\frac{x}{2}\ (-\pi<x<\pi)$，则 $x=2\arctan u$，

$dx=\frac{2}{1+u^2}du$，$\cos x=\frac{1-u^2}{1+u^2}$，

所以 $\int\frac{dx}{3+\cos x}=\int\frac{\frac{2}{1+u^2}du}{3+\frac{1-u^2}{1+u^2}}$

$=\int\frac{du}{u^2+2}=\frac{1}{\sqrt2}\arctan\frac{u}{\sqrt2}+C$

$=\frac{1}{\sqrt2}\arctan\frac{\tan\frac{x}{2}}{\sqrt2}+C$

(2) $\int \frac{dx}{2\sin x-\cos x+5}$.

令 $u=\tan\frac{x}{2}\ (-\pi<x<\pi)$，则 $x=2\arctan u$，$dx=\frac{2}{1+u^2}du$，

$\sin x=\frac{2u}{1+u^2}$，$\cos x=\frac{1-u^2}{1+u^2}$，

所以 $\int\frac{dx}{2\sin x-\cos x+5}=\int\frac{\frac{2}{1+u^2}du}{2\frac{2u}{1+u^2}-\frac{1-u^2}{1+u^2}+5}$

$=\int\frac{du}{3u^2+2u+2}=\frac{1}{3}\int\frac{1}{(u+\frac13)^2+(\frac{\sqrt5}{3})^2}d(u+\frac13)$

$=\frac{1}{\sqrt5}\arctan\frac{u+\frac13}{\frac{\sqrt5}{3}}+C$

$=\frac{1}{\sqrt5}\arctan\left(\frac{3\tan\frac{x}{2}+1}{\sqrt5}\right)+C$

3. 计算下列无理函数的不定积分.

(1) $\int \frac{x^{\frac13}}{x^{\frac32}+x^{\frac43}}dx$;

令 $x=t^6$，则 $dx=6t^5dt$

于是 $\int\frac{x^{\frac13}}{x^{\frac32}+x^{\frac43}}dx$

$=\int\frac{t^2}{t^9+t^8}\cdot 6t^5dt=6\int\frac{dt}{t(t+1)}$

$=6\int\left(\frac1t-\frac{1}{t+1}\right)dt=6\ln t-6\ln(1+t)+C$

$=6\ln\frac{t}{t+1}+C=6\ln\frac{\sqrt[6]{x}}{\sqrt[6]{x}+1}+C$

(2) $\int\sqrt{\frac{1-x}{1+x}}\frac{dx}{x}$.

令 $t=\sqrt{\frac{1-x}{1+x}}$，则 $x=\frac{1-t^2}{1+t^2}$，$dx=-\frac{4t}{(1+t^2)^2}dt$，

于是 $\int\sqrt{\frac{1-x}{1+x}}\frac{dx}{x}=\int t\cdot\frac{-\frac{4t}{(1+t^2)^2}dt}{\frac{1-t^2}{1+t^2}}=-4\int\frac{t^2}{(1-t^2)(1+t^2)}dt$

$=2\int\left(\frac{1}{1+t^2}-\frac{1}{1-t^2}\right)dt$

$=2\arctan t+\ln\left|\frac{1-t}{1+t}\right|+C$

$=2\arctan\sqrt{\frac{1-x}{1+x}}+\ln\left|\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right|+C$

总习题四

1. 在下列等式中，正确的结果是（ C ）.

(A) $\int f'(x)dx=f(x)$

(B) $\int df(x)=f(x)$

(C) $\frac{d}{dx}\int f(x)dx=f(x)$

(D) $d\int f(x)dx=f(x)$

2. 计算下列不定积分.

(1) $\int\frac{\cos\sqrt{x}-1}{\sqrt{x}\sin^2\sqrt{x}}dx$;

$=2\int\frac{\cos\sqrt x-1}{\sin^2\sqrt x}d\sqrt x$

$=2\int\frac{\cos\sqrt x}{\sin^2\sqrt x}d\sqrt x-2\int\frac{1}{\sin^2\sqrt x}d\sqrt x$

$=2\int\frac{1}{\sin^2\sqrt x}d\sin\sqrt x-2\int\frac{1}{\sin^2\sqrt x}d\sqrt x$

$=-\frac{2}{\sin\sqrt x}+2\cot\sqrt x+C$

(2) $\int \frac{x\ln x}{(1+x^2)^2}dx$;

$$=\frac{1}{2}\int\frac{\ln x}{(1+x^2)^2}d(1+x^2)$$

$$=-\frac{1}{2}\int\ln x\, d\left(\frac{1}{1+x^2}\right)$$

$$=-\frac{1}{2}\left[\frac{\ln x}{1+x^2}-\int\frac{1}{1+x^2}\cdot\frac{1}{x}dx\right]$$

$$=-\frac{1}{2}\frac{\ln x}{1+x^2}+\frac{1}{2}\int\left(\frac{1}{x}-\frac{x}{1+x^2}\right)dx$$

$$=-\frac{1}{2}\frac{\ln x}{1+x^2}+\frac{1}{2}\ln|x|-\frac{1}{4}\ln(1+x^2)+C$$

(3) $\int\sqrt{x}\sin\sqrt{x}\,dx$;

令 $t=\sqrt{x}$，则 $x=t^2$，$dx=2tdt$，

于是 $\int\sqrt{x}\sin\sqrt{x}dx=\int t\sin t\cdot 2tdt$

$$=-2\int t^2 d\cos t=-2\left[t^2\cos t-\int\cos t\cdot 2tdt\right]$$

$$=-2t^2\cos t+4\int t\,d\sin t$$

$$=-2t^2\cos t+4\left[t\sin t-\int\sin t\,dt\right]$$

$$=-2t^2\cos t+4t\sin t+4\cos t+C$$

$$=-2x\cos\sqrt{x}+4\sqrt{x}\sin\sqrt{x}+4\cos\sqrt{x}+C$$

(4) $\int\frac{\ln x-1}{\ln^2 x}dx$;

$$=\int\frac{1}{\ln x}dx-\int\frac{1}{\ln^2 x}dx$$

$$=\frac{1}{\ln x}\cdot x-\int x\,d\left(\frac{1}{\ln x}\right)-\int\frac{1}{\ln^2 x}dx$$

$$=\frac{x}{\ln x}-\int x\left(-\frac{1}{\ln^2 x}\right)\cdot\frac{1}{x}dx-\int\frac{dx}{\ln^2 x}$$

$$=\frac{x}{\ln x}+\int\frac{dx}{\ln^2 x}-\int\frac{dx}{\ln^2 x}$$

$$=\frac{x}{\ln x}+C$$

(5) $\int\frac{x^3\arccos x}{\sqrt{1-x^2}}dx$.

令 $x=\cos t\ (0<t<\pi)$，则 $t=\arccos x$，

$dx=-\sin t\,dt$，于是

$$\int\frac{x^3\arccos x}{\sqrt{1-x^2}}dx=\int\frac{\cos^3 t\cdot t}{\sin t}\cdot(-\sin t)dt$$

$$=-\int t\cos^3 t\,dt=-\int t(1-\sin^2 t)d\sin t$$

$$=-\int t\,d\left(\sin t-\frac{\sin^3 t}{3}\right)$$

$$=-t\left(\sin t-\frac{\sin^3 t}{3}\right)+\int\left(\sin t-\frac{\sin^3 t}{3}\right)dt$$

$$=-t\left(\sin t-\frac{\sin^3 t}{3}\right)-\frac{2}{3}\cos t-\frac{1}{9}\cos^3 t+C$$

$$=-\frac{1}{3}\sqrt{1-x^2}(2+x^2)\arccos x-\frac{1}{9}x(x^2+6)+C$$

3. 设 n 为正整数，证明：递推公式 $\int\sec^n x\,dx=\frac{1}{n-1}\tan x\sec^{n-2}x+\frac{n-2}{n-1}\int\sec^{n-2}x\,dx\ (n\geqslant 2)$.

$$\int\sec^n x\,dx=\int\sec^{n-2}x\,d\tan x$$

$$=\sec^{n-2}x\tan x-\int\tan x\,d(\sec^{n-2}x)$$

$$=\sec^{n-2}x\tan x-\int\tan x(n-2)\sec^{n-3}x\cdot(\sec x\tan x)dx$$

$$=\sec^{n-2}x\tan x-(n-2)\int\tan^2 x\sec^{n-2}x\,dx$$

$$=\sec^{n-2}x\tan x-(n-2)\int(\sec^2 x-1)\sec^{n-2}x\,dx$$

$$=\sec^{n-2}x\tan x-(n-2)\int\sec^n x\,dx+(n-2)\int\sec^{n-2}x\,dx$$

解得

$$\int\sec^n x\,dx=\frac{1}{n-1}\sec^{n-2}x\tan x+\frac{n-2}{n-1}\int\sec^{n-2}x\,dx$$