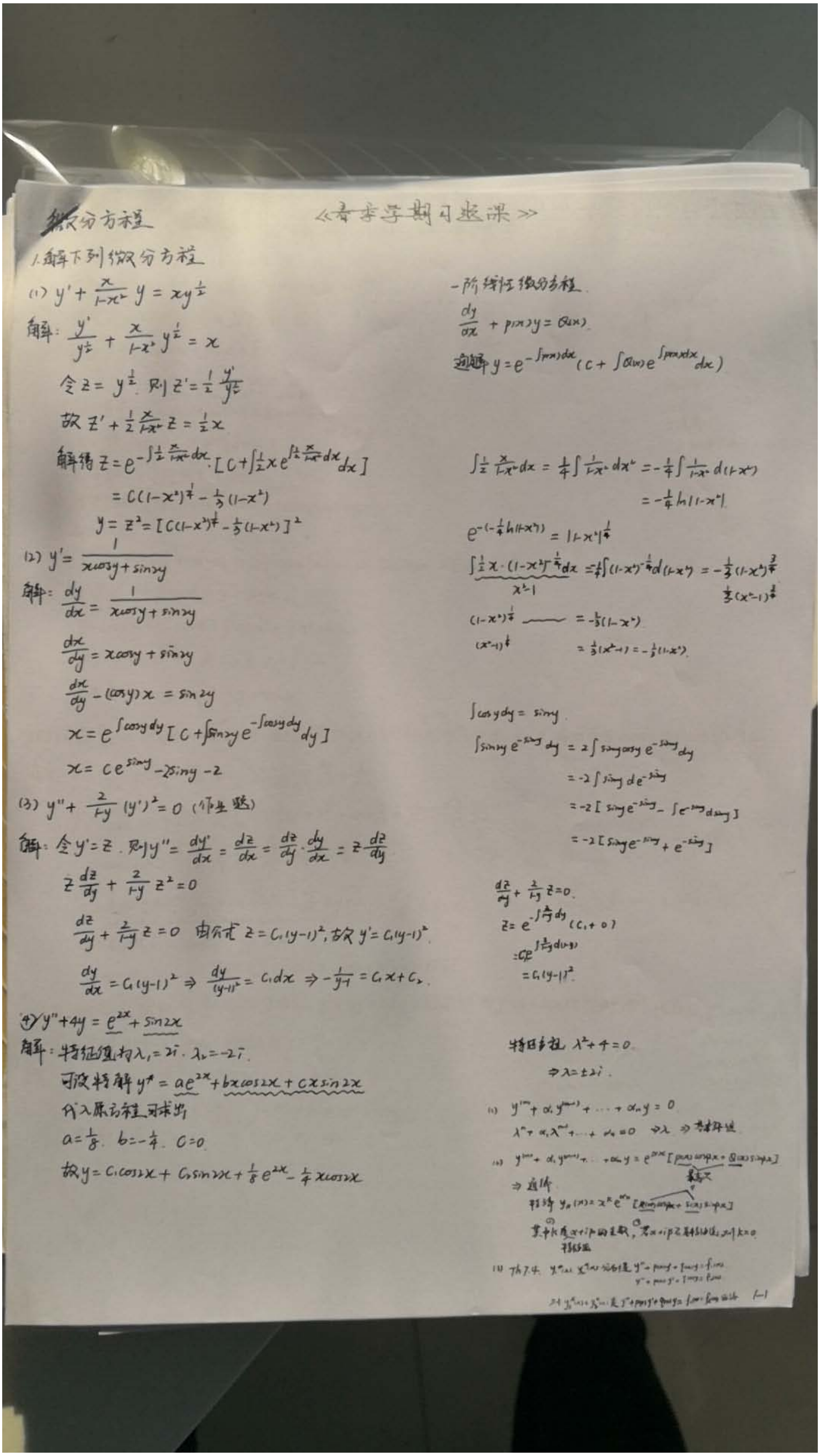

~~微~~分方程

《~~春季学期~~习题课》

1.解下列微分方程

(1) $y'+\frac{x}{1-x^2}y=xy^{\frac{1}{2}}$

解：$\frac{y'}{y^{\frac{1}{2}}}+\frac{x}{1-x^2}y^{\frac{1}{2}}=x$

令 $z=y^{\frac{1}{2}}$，则 $z'=\frac{1}{2}\frac{y'}{y^{\frac{1}{2}}}$

故 $z'+\frac{1}{2}\frac{x}{1-x^2}z=\frac{1}{2}x$

解得 $z=e^{-\int\frac{1}{2}\frac{x}{1-x^2}dx}\cdot[C+\int\frac{1}{2}xe^{\int\frac{1}{2}\frac{x}{1-x^2}dx}dx]$

$=C(1-x^2)^{\frac{1}{4}}-\frac{1}{3}(1-x^2)$

$y=z^2=[C(1-x^2)^{\frac{1}{4}}-\frac{1}{3}(1-x^2)]^2$

(2) $y'=\frac{1}{x\cos y+\sin 2y}$

解：$\frac{dy}{dx}=\frac{1}{x\cos y+\sin 2y}$

$\frac{dx}{dy}=x\cos y+\sin 2y$

$\frac{dx}{dy}-(\cos y)x=\sin 2y$

$x=e^{\int\cos y dy}[C+\int\sin 2y\, e^{-\int\cos y dy}dy]$

$x=Ce^{\sin y}-2\sin y-2$

(3) $y''+\frac{2}{1-y}(y')^2=0$（作业题）

解：令 $y'=z$，则 $y''=\frac{dy'}{dx}=\frac{dz}{dx}=\frac{dz}{dy}\cdot\frac{dy}{dx}=z\frac{dz}{dy}$

$z\frac{dz}{dy}+\frac{2}{1-y}z^2=0$

$\frac{dz}{dy}+\frac{2}{1-y}z=0$ 由公式 $z=C_1(y-1)^2$，故 $y'=C_1(y-1)^2$.

$\frac{dy}{dx}=C_1(y-1)^2\Rightarrow\frac{dy}{(y-1)^2}=C_1dx\Rightarrow-\frac{1}{y-1}=C_1x+C_2$.

④ $y''+4y=\underline{e^{2x}}+\underline{\sin 2x}$

解：特征值为 $\lambda_1=2i$，$\lambda_2=-2i$.

可设特解 $y^*=ae^{2x}+bx\cos 2x+cx\sin 2x$

代入原方程，可求出

$a=\frac{1}{8}$，$b=-\frac{1}{4}$，$c=0$

故 $y=C_1\cos 2x+C_2\sin 2x+\frac{1}{8}e^{2x}-\frac{1}{4}x\cos 2x$

一阶线性微分方程

$\frac{dy}{dx}+p(x)y=Q(x)$

通解 $y=e^{-\int p(x)dx}(C+\int Q(x)e^{\int p(x)dx}dx)$

$\int\frac{1}{2}\frac{x}{1-x^2}dx=\frac{1}{4}\int\frac{1}{1-x^2}dx^2=-\frac{1}{4}\int\frac{1}{1-x^2}d(1-x^2)$

$=-\frac{1}{4}\ln|1-x^2|$

$e^{-(-\frac{1}{4}\ln|1-x^2|)}=|1-x^2|^{\frac{1}{4}}$

$\int\underbrace{\frac{1}{2}x\cdot(1-x^2)^{-\frac{1}{4}}dx}_{x^2-1}=-\frac{1}{4}\int(1-x^2)^{-\frac{1}{4}}d(1-x^2)=-\frac{1}{3}(1-x^2)^{\frac{3}{4}}$

$\frac{1}{3}(x^2-1)^{\frac{3}{4}}$

$(1-x^2)^{\frac{1}{4}}$ —— $=-\frac{1}{3}(1-x^2)$

$(x^2-1)^{\frac{1}{4}}$ $=\frac{1}{3}(x^2-1)=-\frac{1}{3}(1-x^2)$.

$\int\cos y dy=\sin y$

$\int\sin 2y\, e^{-\sin y}dy=2\int\sin y\cos y\, e^{-\sin y}dy$

$=-2\int\sin y\, de^{-\sin y}$

$=-2[\sin y\, e^{-\sin y}-\int e^{-\sin y}d\sin y]$

$=-2[\sin y\, e^{-\sin y}+e^{-\sin y}]$

$\frac{dz}{dy}+\frac{2}{1-y}z=0$

$z=e^{-\int\frac{2}{1-y}dy}(C_1+0)$

$=Ce^{\int\frac{2}{1-y}d(1-y)}$

$=C_1(y-1)^2$.

特征方程 $\lambda^2+4=0$

$\Rightarrow\lambda=\pm 2i$.

(1) $y^{(n)}+\alpha_1y^{(n-1)}+\cdots+\alpha_ny=0$

$\lambda^n+\alpha_1\lambda^{n-1}+\cdots+\alpha_n=0$ ⇒λ ⇒ 基本解组

(2) $y^{(n)}+\alpha_1y^{(n-1)}+\cdots+\alpha_ny=e^{\alpha x}[\underline{P(x)}\cos\beta x+\underline{Q(x)}\sin\beta x]$

最高次

⇒ 通解

特解 $y_0(x)=x^ke^{\alpha x}[\underline{R(x)}\cos\beta x+\underline{S(x)}\sin\beta x]$

① 其中k是α+iβ的重数，② 若α+iβ不是特征值，则k=0.

特征值

(3) Th7.4. $y_1^*(x)$，$y_2^*(x)$ 分别是 $y''+p(x)y'+q(x)y=f_1(x)$

$y''+p(x)y'+q(x)y=f_2(x)$

则 $y_1^*(x)+y_2^*(x)$ 是 $y''+p(x)y'+q(x)y=f_1(x)+f_2(x)$ 的解 1-1

§4.4

$\lambda^2+1=0.\ \lambda=\pm i$

微分方程 $y''+y=f(x)$ 满足 $y(0)=0,\ y'(0)=1$ 的特解，其中函数 $f(x)$ 满足条件 $\overset{\text{二阶可导}}{}$ $\sin x-f(x)=\int_0^x(x-t)f(t)dt$

由 $\sin x-f(x)=\int_0^x(x-t)f(t)dt$ ①

$\underline{\sin x-f(x)}=\int_0^x xf(t)dt-\int_0^x tf(t)dt=\underline{x\int_0^x f(t)dt-\int_0^x tf(t)dt}$

$'=\int_0^x f(t)dt+\underline{xf(x)-xf(x)}$

$=\int_0^x f(t)dt$

求导得 $\cos x-f'(x)=\int_0^x f(t)dt$ ②

再求导,得 $-\sin x-f''(x)=f(x)$

$$\underset{\lambda^2+1=0,\ \lambda=\pm i}{f''(x)+f(x)}=\underset{\alpha=0,\beta=1}{\underline{-\sin x}}$$

解得 $f(x)=C_1\cos x+C_2\sin x+\frac{1}{2}x\cos x$

由① 得 $f(0)=0$，由② 得 $f'(0)=1$ $\Rightarrow C_1=0.\ C_2=\frac{1}{2}$.

求得 $f(x)=\underset{\alpha=0,\beta=1}{\underline{\frac{1}{2}\sin x}}+\underset{\alpha=0,\beta=1}{\underline{\frac{1}{2}x\cos x}}$

可设特解 $y^*=(ax+b)x\cos x+(cx+d)x\sin x$

代入求得

$(y^*)''+y^*=(-4ax-2b+2c)\sin x+(4cx+2d+2a)\cos x=\frac{1}{2}\sin x+\frac{1}{2}x\cos x$

求得 $y^*=-\frac{x}{8}\cos x+\frac{1}{8}x^2\sin x$

$y=C_1\cos x+C_2\sin x-\frac{x}{8}\cos x+\frac{1}{8}x^2\sin x$

由 $y(0)=0.\ y'(0)=1\Rightarrow C_1=0.\ C_2=\frac{7}{8}$

$y=\frac{7}{8}\sin x-\frac{1}{8}x\cos x+\frac{1}{8}x^2\sin x$.

f(x)通解

⇓

f(x)条件

⇓

f(x)解

⇓

y通解

⇓ 初始条件

y解

(6) 用变量替换 $x=\cos t\ (0<t<\pi)$ 化简微分方程 $(1-x^2)y''-xy'+y=0$，并求满足 $y(0)=1,\ y(0)=2$ 的特解

$(\arccos x)'=-\frac{1}{\sqrt{1-x^2}}$

解: $y'=\frac{dy}{dx}=\frac{dy}{dt}\cdot\frac{dt}{dx}=-\frac{1}{\sqrt{1-x^2}}\frac{dy}{dt}.\ (\frac{dt}{dx}=-\frac{1}{\sqrt{1-x^2}})$

$y''=\frac{dy'}{dx}=(-\frac{1}{\sqrt{1-x^2}}\frac{dy}{dt})'_x=+\frac{1}{2}(1-x^2)^{-\frac{3}{2}}\cdot(-2x)\frac{dy}{dt}\mp\frac{1}{\sqrt{1-x^2}}\frac{d^2y}{dt^2}\underset{-\frac{1}{\sqrt{1-x^2}}}{\underset{\parallel}{\frac{dt}{dx}}}$

$=-x(1-x^2)^{-\frac{3}{2}}\frac{dy}{dt}\pm\frac{1}{1-x^2}\frac{d^2y}{dt^2}$ 代入,得

$+\frac{d^2y}{dt^2}\underset{=0}{\underbrace{\pm x(1-x^2)^{-\frac{1}{2}}\frac{dy}{dt}+x(1-x^2)^{-\frac{1}{2}}\frac{dy}{dt}}}+y=0\Rightarrow\frac{d^2y}{dt^2}+y=0$ $\lambda^2+1=0\Rightarrow\lambda=\pm i\Rightarrow y_1=\cos t.\ y_2=\sin t$

$y=C_1\cos t+C_2\sin t.\quad y=C_1x+C_2\sqrt{1-x^2}$

由 $y(0)=1.\ y'(0)=2$ 可知 $C_2=1.\ C_1=2.\ \Rightarrow y=2x+\sqrt{1-x^2}$

(17) 若二阶常系数线性齐次微分方程 $y''+ay'+by=0$ 的通解为 $y=(C_1+C_2x)e^x$，则非齐次方程 $y''+ay'+by=x$

满足条件 $y(0)=2,\ y'(0)=0$ 的解为 $y=$？

$\lambda^2+a\lambda+b=0$

$(\lambda-\lambda_1)(\lambda-\lambda_2)=0$

$\lambda^2-\underset{a}{\underset{\parallel}{(\lambda_1+\lambda_2)}}\lambda+\underset{b}{\underset{\parallel}{\lambda_1\lambda_2}}$

由齐次解 e^x 和 xe^x

可知二重实根1

$\lambda_1=\lambda_2=1$

解: $\lambda_1=\lambda_2=1\ \Rightarrow a=-(\lambda_1+\lambda_2)=-2.\ b=\lambda_1\lambda_2=1$

$y^*=ax+b$ 代入原方程

得 $a=1.\ b=2$

方程为: $y''-2y'+y=\underset{\alpha=0,\beta=0}{\underline{x}}.\quad y=(C_1+C_2x)e^x+\underline{x+2}$

由 $y(0)=2,\ y'(0)=0\ \Rightarrow C_1=0.\ C_2=-1$

$y=-xe^x+x+2$

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微分方程 $x^3y''-x^2y'+xy=x^2+1$ 的通解　$x\neq0$时 求解

$x^2y''-xy'+y=x+\frac{1}{x}$

(1) 当 $x>0$ 时．令 $t=\ln x$．$D=\frac{d}{dt}$．则

$xy'=Dy$　$x^2y''=D(D-1)y$

$D(D-1)y-Dy+y=e^t+e^{-t}$

$\frac{d^2y}{dt^2}-2\frac{dy}{dt}+y=e^t+e^{-t}$

$\Rightarrow y=C_1e^t+C_2te^t+\frac{1}{2}t^2e^t+\frac{1}{4}e^{-t}$

$y=C_1x+C_2x\ln x+\frac{1}{2}x\ln^2x+\frac{1}{4}\frac{1}{x}$

(2) 当 $x<0$ 时．令 $t=\ln(-x)$　$D=\frac{d}{dt}$，则

答：其 $\frac{dy}{dx}=-(-\frac{1}{x})\frac{dy}{dt}=\frac{1}{x}\frac{dy}{dt}=\frac{1}{x}Dy$

$xy'=Dy$．$x^2y''=D(D-1)y$

$D(D-1)y-Dy+y=-(e^t+e^{-t})$

$\frac{d^2y}{dt^2}-2\frac{dy}{dt}+y=-(e^t+e^{-t})$

$\Rightarrow y=-C_1x-C_2x\ln(-x)-\frac{1}{2}x\ln^2(-x)-\frac{1}{4}\frac{1}{x}$

欧拉方程 上册 P341

$x^ny^{(n)}+a_1x^{n-1}y^{(n-1)}+\cdots+a_{n-1}xy'+a_ny=f(x)$

$x^3y''-x^2y'+xy=x^2+1$

$\Rightarrow x^2y''-xy'+y=x+\frac{1}{x}$

(1) $t=\ln x$．$x=e^t$

$x^2y''-xy'+y=e^t+e^{-t}$

$D=\frac{d}{dt}$

$D(D-1)y-Dy+y=e^t+e^{-t}$

$\lambda^2-2\lambda+1=0\Rightarrow\lambda=1$（2重）$\Rightarrow e^t$，$te^t$

$\frac{d^2y}{dt^2}-2\frac{dy}{dt}+y=e^t$　设 $y_1^*=at^2e^t$　代入 $\Rightarrow a=\frac{1}{2}$

$\frac{d^2y}{dt^2}-2\frac{dy}{dt}+y=e^{-t}$　设 $y_2^*=be^{-t}$　代入 $\Rightarrow b=\frac{1}{4}$

(2) $t=\ln(-x)$　$x=-e^t$

2. 设 y_1，y_2 是一阶线性非齐次微分方程 $y'+p(x)y=q(x)$ 的两个特解，若常数 λ，μ 使 $\lambda y_1+\mu y_2$ 是该方程的解 $\lambda y_1-\mu y_2$ 是该方程对应的齐次方程的解，则

(A) $\lambda=\frac{1}{2}$．$\mu=\frac{1}{2}$．　(B) $\lambda=\frac{1}{2}$．$\mu=-\frac{1}{2}$．　(C) $\lambda=\frac{2}{3}$　$\mu=\frac{1}{3}$．　(D) $\lambda=\frac{2}{3}$．$\mu=\frac{2}{3}$

$\begin{cases}\lambda+\mu=1\\ \lambda-\mu=0\end{cases}\Rightarrow\lambda=\mu=\frac{1}{2}$．选(A)

$\begin{cases}y_1'+p(x)y_1=q(x)\\ y_2'+p(x)y_2=q(x)\\ \lambda y_1'+\mu y_2'+p(x)(\lambda y_1+\mu y_2)=q(x)\\ \lambda y_1'-\mu y_2'+p(x)(\lambda y_1-\mu y_2)=0\end{cases}\Rightarrow\begin{cases}\lambda+\mu=1\\ \lambda-\mu=0\end{cases}$

3. 设函数 $f(u)$ 具有二阶连续导数．$z=f(e^x\cos y)$ 满足 $\frac{\partial^2z}{\partial x^2}+\frac{\partial^2z}{\partial y^2}=(4z+e^x\cos y)e^{2x}$ 若 $f(0)=0$，$f'(0)=0$，求 $f(u)$ 的表达式．

解：$\frac{\partial z}{\partial x}=f'(e^x\cos y)e^x\cos y$　$\frac{\partial z}{\partial y}=f'(e^x\cos y)(-e^x\sin y)$

$\frac{\partial^2z}{\partial x^2}=f''(e^x\cos y)e^{2x}\cos^2y+f'(e^x\cos y)e^x\cos y$

$\frac{\partial^2z}{\partial y^2}=f''(e^x\cos y)e^{2x}\sin^2y+f'(e^x\cos y)\cdot(-e^x\cos y)$

$\frac{\partial^2z}{\partial x^2}+\frac{\partial^2z}{\partial y^2}=f''(e^x\cos y)e^{2x}=(4z+e^x\cos y)e^{2x}\Rightarrow f''(\underbrace{e^x\cos y}_{u})=4f(\underbrace{e^x\cos y}_{u})+\underbrace{e^x\cos y}_{u}$

即 $f''(u)=4f(u)+u$　$f''(u)-4f(u)=u$．　$\lambda^2-4=0\Rightarrow\lambda=\pm2$　设 $y^*=au+b$　代入 $\Rightarrow a=-\frac{1}{4}$　$b=0$　$\Rightarrow y^*=-\frac{1}{4}u$

解得 $f(u)=C_1e^{-2u}+C_2e^{2u}-\frac{1}{4}u$．

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函数的微分

断下列极限是否存在，若存在请求极限值

(1) $\lim\limits_{\substack{x\to0\\y\to0}} \frac{x^2\sin y}{(x^2+y^2)}$.

解：由于当$(x,y)\to(0,0)$时，$\frac{x^2}{x^2+y^2}$有界，$\sin y$为无穷小量，故 $\lim\limits_{\substack{x\to0\\y\to0}} \frac{x^2\sin y}{x^2+y^2}=0$

(2) $\lim\limits_{\substack{x\to0\\y\to0}} \frac{x^2y}{x+y}$

解 由于 $\lim\limits_{\substack{y=-x+x^4\\x\to0}} \frac{x^2y}{x+y} = \lim\limits_{x\to0} \frac{x^2(-x+x^4)}{x^4} = \infty$ 不存在，故 $\lim\limits_{\substack{x\to0\\y\to0}} \frac{x^2y}{x+y}$ 不存在

2. 求下列函数在原点处的一阶偏导数

(1) $u = ze^{2x+y} + e^{4xyz} + zy^2$

解：$u(x,0,0)=1 \Rightarrow u'_x(0,0,0)=0$

$u(0,y,0)=1 \Rightarrow u'_y(0,0,0)=0$

$u(0,0,z)=z+1 \Rightarrow u'_z(0,0,0)=1$

(2) $z=\begin{cases}\frac{x^2y}{x^4+y^2}, & (x,y)\neq(0,0);\\ 0, & (x,y)=(0,0)\end{cases}$

解：$z'_x(0,0)=\lim\limits_{x\to0}\frac{z(x,0)-z(0,0)}{x-0}=\lim\limits_{x\to0}\frac{0-0}{x}=0$

同理 $z'_y(0,0)=0$

$\lim\limits_{\rho\to0}\frac{\Delta z-\frac{\partial z}{\partial x}\Delta x-\frac{\partial z}{\partial y}\Delta y}{\rho}$

3. 判断函数的可微性

$z=\begin{cases}xy\frac{x^2-y^2}{x^2+y^2}, & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0)\end{cases}$

解：$z'_x(0,0)=(z(x,0))'_{x=0}=0$

$z'_y(0,0)=(z(0,y))'_{y=0}=0$

$\lim\limits_{\substack{x\to0\\y\to0}}\frac{z(x,y)-z(0,0)-z'_x(0,0)x-z'_y(0,0)y}{\sqrt{x^2+y^2}}$

$=\lim\limits_{\substack{x\to0\\y\to0}}\frac{xy(x^2-y^2)}{(x^2+y^2)^{\frac{3}{2}}}$ （由于$\frac{|x^3|}{(x^2+y^2)^{\frac{3}{2}}}\leq1$，$\frac{|y^3|}{(x^2+y^2)^{\frac{3}{2}}}\leq1$）

故 $z(x,y)$在$(0,0)$处可微. $\Rightarrow$ $z(x,y)$处处可微

显然，当$(x,y)\neq(0,0)$时，$z(x,y)$可微

下册 P10

z可微

$\Uparrow$

结论 $\underbrace{\Delta z = A\Delta x + B\Delta y + o(\rho)}$ $\rho=\sqrt{(\Delta x)^2+(\Delta y)^2}$

若$z=f(x,y)$在点$P(x,y)$处可微，则在点P处偏导数$\frac{\partial z}{\partial x}$与$\frac{\partial z}{\partial y}$都存在且

$\frac{\partial z}{\partial x}=A$ $\frac{\partial z}{\partial y}=B$

全微分 $dz=A\Delta x+B\Delta y$

若$\frac{\partial z}{\partial x}$ $\frac{\partial z}{\partial y}$存在，可通过$\lim\limits_{\substack{\Delta x\to0\\\Delta y\to0}}\frac{\Delta z-\frac{\partial z}{\partial x}\Delta x-\frac{\partial z}{\partial y}\Delta y}{\rho}=0$来说明z可导

$z'_x(0,0)=\lim\limits_{\Delta x\to0}\frac{z(0+\Delta x,0)-z(0,0)}{\Delta x}=\lim\limits_{x\to0}\frac{z(x,0)-z(0,0)}{x}$

$\overset{推导}{=}[z(x,0)]'_{x=0}$

4. 设$\frac{\partial^2 f}{\partial x\partial y}=x+y$，且$f(x,0)=x^2$，$f(0,y)=y$，求$f(x,y)$.

解：由$\frac{\partial^2 f}{\partial x\partial y}=x+y$可知$\frac{\partial f}{\partial x}=xy+\frac{1}{2}y^2+g(x)$

设$f(x,y)=\frac{1}{2}x^2y+\frac{1}{2}xy^2+G(x)+H(y)$

由于$f(x,0)=x^2 \Rightarrow G(x)+H(0)=x^2$

$f(0,y)=y \Rightarrow G(0)+H(y)=y$

$G(x)+H(y)+G(0)+H(0)=x^2+y$ 令$x=0$，$y=0 \Rightarrow H(0)+G(0)=0$

则$G(x)+H(y)=x^2+y$ $f(x,y)=\frac{x^2y+xy^2}{2}+x^2+y$

§4.4

列函数的一阶偏导数或导函数

$y-x+xe^{z-x-y}=0$

$dz-dy-dx+d(xe^{z-x-y})=0$

$dz-dy-dx+e^{z-x-y}dx+xe^{z-x-y}(dz-dx-dy)=0$

$$dz=\frac{1-e^{z-x-y}+xe^{z-x-y}}{1+xe^{z-x-y}}dx+\frac{1+xe^{z-x-y}}{1+xe^{z-x-y}}dy$$

$dz=\frac{\partial z}{\partial x}dx+\frac{\partial z}{\partial y}dy$

$$\frac{\partial z}{\partial x}=\frac{1-e^{z-x-y}+xe^{z-x-y}}{1+xe^{z-x-y}}$$

$$\frac{\partial z}{\partial y}=\frac{1+xe^{z-x-y}}{1+xe^{z-x-y}}=1.$$

(2) $F(x+\frac{z}{y},y+\frac{z}{x})=0$ 决定了 $z=z(x,y)$

解：$$\frac{\partial z}{\partial x}=-\frac{[F(x+\frac{z}{y},y+\frac{z}{x})]'_x}{[F(x+\frac{z}{y},y+\frac{z}{x})]'_z}=-\frac{F_1'+F_2'\cdot(-\frac{z}{x^2})}{F_1'\cdot\frac{1}{y}+F_2'\cdot\frac{1}{x}}$$

$$\frac{\partial z}{\partial y}=-\frac{[F(x+\frac{z}{y},y+\frac{z}{x})]'_y}{[F(x+\frac{z}{y},y+\frac{z}{x})]'_z}=-\frac{F_1'(-\frac{z}{y^2})+F_2'}{F_1'\cdot\frac{1}{y}+F_2'\cdot\frac{1}{x}}$$

隐函数求导 册P20

隐函数存在定理：

设函数 $F(x,y,z)$ 在点 (x_0,y_0,z_0) 的某邻域内具有连续的偏导数，且 $F(x_0,y_0,z_0)=0$，$F_z'(x_0,y_0,z_0)\neq 0$，则方程

$F(x,y,z)=0$ 在点 (x_0,y_0,z_0) 的某邻域内确定唯一一个函数 $z=f(x,y)$，满足

$F(x,y,f(x,y))\equiv 0$，$z_0=f(x_0,y_0)$

函数 $z=f(x,y)$ 在 (x_0,y_0) 的某邻域内单值，有连续的偏导数，且有

偏导数公式 $\frac{\partial z}{\partial x}=-\frac{F_x'(x,y,z)}{F_z'(x,y,z)}$，$\frac{\partial z}{\partial y}=-\frac{F_y'(x,y,z)}{F_z'(x,y,z)}$.

6. 计算下列函数的高阶偏导数

(1) $z=x^3f(xy,\frac{y}{x})$，求 $\frac{\partial^2 z}{\partial x\partial y}$

解：$$\frac{\partial z}{\partial x}=3x^2f(xy,\frac{y}{x})+x^3f_1'(xy,\frac{y}{x})\cdot y+x^3f_2'(xy,\frac{y}{x})\cdot(-\frac{y}{x^2})$$

$$\frac{\partial^2 z}{\partial x\partial y}=3x^3f_1'+3xf_2'+x^3f_1'+x^3f_{11}''\cdot xy+x^2f_{12}''y-xf_2'-x^2yf_{21}''-f_{22}''\cdot y$$

(2) $F(x,y+z)=z$ 决定了 $z=z(x,y)$，求 $\frac{\partial^2 z}{\partial x\partial y}$

解：两边对 x 求导，得 $F_1'+F_2'\cdot\frac{\partial z}{\partial x}=\frac{\partial z}{\partial x}\Rightarrow\frac{\partial z}{\partial x}=\frac{F_1'}{1-F_2'}$

两边对 y 求导，得 $F_2'\cdot(1+\frac{\partial z}{\partial y})=\frac{\partial z}{\partial y}\Rightarrow\frac{\partial z}{\partial y}=\frac{F_2'}{1-F_2'}\Rightarrow 1+\frac{\partial z}{\partial y}=1+\frac{F_2'}{1-F_2'}=\frac{1-F_2'+F_2'}{1-F_2'}=\frac{1}{1-F_2'}$ 方便点.

$$\frac{\partial^2 z}{\partial x\partial y}=\frac{F_{12}''\cdot(1+\frac{\partial z}{\partial y})(1-F_2')-F_1'\cdot(-F_{22}'')(1+\frac{\partial z}{\partial y})}{(1-F_2')^2}=\frac{F_{12}''\cdot\frac{1}{1-F_2'}\cdot(1-F_2')-F_1'\cdot(-F_{22}'')\cdot\frac{1}{1-F_2'}}{(1-F_2')^2}$$

$$=\frac{F_{12}''+F_1'\cdot F_{22}''\cdot\frac{1}{1-F_2'}}{(1-F_2')^2}=\frac{F_{12}''-F_2'F_{12}''+F_1'F_{22}''}{(1-F_2')^2}$$

麻烦了点

$$=\frac{F_{12}''\cdot(1-F_2')+F_{12}''\cdot F_2'+F_1'F_{22}''+F_1'F_{22}''\cdot\frac{F_2'}{1-F_2'}}{(1-F_2')^2}$$

$$=\frac{F_{12}''\cdot(1-F_2')+F_1'F_{22}''(1-F_2')+F_1'F_2'F_{22}''}{(1-F_2')^3}$$

$$=\frac{F_{12}''-F_2'F_{12}''+F_1'F_{22}''}{(1-F_2')^3}$$