微积分A期末第二次模拟考答案

一、填空题（每小题2分，共四小题，满分8分）

1. $2(\frac{3}{2})^{\frac{3}{2}}$

$\boldsymbol{y}' = \frac{\boldsymbol{1}}{\boldsymbol{x}}$ $\boldsymbol{y}'' = -\frac{\boldsymbol{1}}{\boldsymbol{x^2}}$ $\boldsymbol{r} = \frac{\boldsymbol{1}}{\boldsymbol{k}} = \frac{(\boldsymbol{1}+\boldsymbol{y}'^{\boldsymbol{2}})^{\frac{\boldsymbol{3}}{\boldsymbol{2}}}}{|\boldsymbol{y}''|}$ 当 x=$\frac{1}{\sqrt{2}}$时 r 取到最小值 $2(\frac{3}{2})^{\frac{3}{2}}$

2. $aln(2\pi + \sqrt{4\pi^2+1})$

极坐标情况下$ds = \sqrt{r(\theta)^2 + r(\theta)'^2}d\theta = a\sqrt{1+\theta^2}d\theta$

$s = \int_0^{2\pi} a\sqrt{1+\theta^2}d\theta = \ aln\left(\theta + \sqrt{\theta^2+1}\right)\Big|_0^{2\pi} = aln(2\pi + \sqrt{4\pi^2+1})$

3. $(\int_0^x cost^4dt + xcosx^4)dx$

积分变量为 t, 将 x 提到积分外进行计算, 原式=$(\int_0^x cost^4dt + xcosx^4)dx$

4. $y^{-2} = -2\ln x + C$

本质为伯努利方程，移项整理后得：$\frac{dy}{dx}y^{-3} - \frac{y^{-2}}{x} = (1 + lnx)$

令$t = y^{-2}$ 得$\frac{dt}{dx} + \frac{2t}{x} = -(1+lnx)$ 解得$t = -2lnx + C$ 即$y^{-2} = -2lnx + C$

二、选择题（每小题2分，共四小题，满分8分）

1. 答案为 B

$cosx = 1 - \frac{x^2}{2} + \frac{x^4}{24} + o(x^4)$ $\ln(1+y) = y - \frac{y^2}{2} + o(y^2)$

则$lncosx = \ln\left[1 - \frac{x^2}{2} + \frac{x^4}{24} + o(x^4)\right]$

$=[-\frac{x^2}{2} + \frac{x^4}{24} + o(x^4)] - \frac{(-\frac{x^2}{2}+\frac{x^4}{24}+o(x^4))^2}{2} + o(x^4)$

$=-\frac{x^2}{2} - \frac{x^4}{12} + o(x^4)$ 故选 B

2. 答案为 A

$e^{\frac{1}{x}}\frac{1}{x^2}dx = -d(1+e^{\frac{1}{x}})$ 原式=$(1+e^{\frac{1}{x}})^{-1}|_{-1}^{1} = \frac{1-e}{1+e}$ 故选 A

3. 答案为 C

A：$\int_1^{+\infty}\frac{dx}{x\sqrt{1+x}} < \int_1^{+\infty}\frac{dx}{x\sqrt{x}} = -2\frac{1}{\sqrt{x}}|_1^{+\infty}$ 收敛

B：$\int_1^{+\infty}\frac{dx}{x^2}=-\frac{1}{x}|_1^{+\infty}$　　收敛

D：$\int_1^{+\infty}e^{-x}sinxdx<\int_1^{+\infty}e^{-x}dx=-e^{-x}|_1^{+\infty}$　　收敛

4. 答案为 A

$F=\frac{mMk}{x^2}$　　由题意知：$dF=\frac{km\mu dx}{(a+x)^2}$　　则 F=$\int_0^l\frac{km\mu dx}{(a+x)^2}$=$\int_{-l}^0\frac{km\mu dx}{(a-x)^2}$

三、计算题（每题 2 分，共 4 题，满分 8 分）

1. 4π

原式=$\int_0^4 x\sqrt{4-(x-2)^2}\,dx$

令 t=x-2　则有：$\int_{-2}^2(t+2)\sqrt{4-t^2}\,dt=4\int_0^2\sqrt{4-t^2}=4\pi$

2. $\frac{1}{2}lntan\frac{x}{2}+tan\frac{x}{2}+\frac{tan\frac{x^2}{2}}{4}+C$

令$tan\frac{x}{2}=u$

则原式变为：$\int\frac{(1+u)^2}{2u}du=\frac{1}{2}lnu+u+\frac{u^2}{4}+C=\frac{1}{2}lntan\frac{x}{2}+tan\frac{x}{2}+\frac{tan\frac{x^2}{2}}{4}+C$

3. $\frac{35\pi^2}{64}$

令$t=x-\pi$　原式为：$\int_{-\pi}^{\pi}(t+\pi)cos^8t\,dt=4\pi\int_0^{\frac{\pi}{2}}cos^8t\,dt=\frac{35\pi^2}{64}$

4. 2

$$\int_0^{\pi}f(x)dx=xf(x)|_0^{\pi}-\int_0^{\pi}\frac{xsinx}{\pi-x}dx=\int_0^{\pi}\frac{(\pi-x)sinx}{\pi-x}dx=\int_0^{\pi}sinxdx=2$$

四、（4 分）

（1）略.　（令$f(x)=x-\ln(x+1)\ g(x)=\ln(x+1)-x+\frac{x^2}{2}$）

（2）$\lim_{n\to+\infty}\left[\left(1+\frac{1}{n^2}\right)\left(1+\frac{2}{n^2}\right)\cdots\left(1+\frac{n}{n^2}\right)\right]=e^{\lim_{n\to+\infty}ln\left[\left(1+\frac{1}{n^2}\right)\left(1+\frac{2}{n^2}\right)\cdots\left(1+\frac{n}{n^2}\right)\right]}=S$

由第一问知：有$\frac{i}{n^2}-\frac{i^2}{2n^4}<\ln\left(1+\frac{i}{n^2}\right)<\frac{i}{n^2}\ i=1,2,3\ldots n$

对两边求和得：$\frac{n+1}{2n}-\frac{(n+1)(2n+1)}{12n^3}<ln\left[\left(1+\frac{1}{n^2}\right)\left(1+\frac{2}{n^2}\right)\cdots\left(1+\frac{n}{n^2}\right)\right]<\frac{n+1}{2n}$

解得 S=$\sqrt{e}$

五、（6 分）

（1）$dV_n = \pi y^2 \, dx$ 则$V_n = \pi \int_{(2n-2)\pi}^{(2n-1)\pi} e^{-x} \sin x \, dx = \frac{\pi e^{(1-2n)\pi}}{2}(1 + e^{\pi})$

（2）总时间为$\frac{s}{v_0}$ 在任意时刻 t，冰块质量为 M-mt

$dW = \mu(M - mt)gv_0 dt$

$W = \int_0^{\frac{s}{v_0}} \mu g v_0 (M - mt)$

$= \left[\mu g M v_0 t - \frac{1}{2}\mu g v_0 m t^2\right]_0^{\frac{s}{v_0}}$

$= \mu g M s - \frac{\mu g m s^2}{2v_0}$

六、(7 分)

解：$f(x) = x^2 \int_1^{x^2} e^{-t^2} \, dt - \int_1^{x^2} t e^{-t^2} dt$

$f'(x) = 2x \int_1^{x^2} e^{-t^2} \, dt + 2x^3 e^{-x^4} - 2x^3 e^{-x^4} = 2x \int_1^{x^2} e^{-t^2} dt$

当 $x > 1$ 时，$\int_1^{x^2} e^{-t^2} dt > 0$ $f'(x) > 0$

当 $0 < x < 1$ 时，$\int_1^{x^2} e^{-t^2} dt < 0$ $f'(x) < 0$

当$-1 < x < 0$ 时，$\int_1^{x^2} e^{-t^2} dt < 0$ $f'(x) > 0$

当 $x <- 1$ 时，$\int_1^{x^2} e^{-t^2} dt > 0$ $f'(x) < 0$

$f(x)$在$(-1,0), (1, +\infty)$上单调递增，在$(-\infty, -1), (0,1)$上单调递减

极小值为 $f(-1) = f(1) = 0$

极大值为 $f(0) = \int_1^0 -t e^{-t^2} dt = \frac{1}{2} - \frac{1}{2e}$

七、(5 分)

解：（1）记$f_n(x) = e^x + x^{2n+1}$ $f_n'(x) = e^x + (2n + 1)x^{2n} > 0$

$f_n(0) = 1 > 0$ $f_n(-1) = e^{-1} - 1 < 0$

$\therefore$ 存在唯一的x_n使得$f_n(x_n) = e^{x_n} + {x_n}^{2n+1} = 0$

即方程$e^x + x^{2n+1} = 0$ 的解x_n是唯一的，且$x_n \epsilon (-1,0)$

（2） $f_n(x_n) = e^{x_n} + {x_n}^{2n+1} = 0$

$f_{n+1}(x_{n+1}) = e^{x_{n+1}} + {x_{n+1}}^{2n+3} = 0$ 用（1）的方法知$x_{n+1} \epsilon (-1,0)$

$f_n(x_{n+1}) = e^{x_{n+1}} + {x_{n+1}}^{2n+1} < e^{x_{n+1}} + {x_{n+1}}^{2n+1} \cdot {x_{n+1}}^2 = 0 = f_n(x_n)$

$\therefore -1 < x_{n+1} < x_n < 0$ $\lim\limits_{n\to\infty} x_n$存在

设$\lim\limits_{n\to\infty} x_n = A$ 显然$-1 \le A < 0$ $\lim\limits_{n\to\infty}(e^x + x^{2n+1}) = 0$

若$-1 < A < 0$ 则$\lim\limits_{n\to\infty}(e^x + x^{2n+1}) = e^A = 0$ 矛盾

$\therefore \lim\limits_{n\to\infty} x_n = A = -1$

八、（4 分）

设 $F(x) = \int_{-a}^{x} f(t)dt$ $\dot{F(x)} = f(x)$

由泰勒展开 $F(x) = F(0) + f(0)x + \frac{f(\dot{0})}{2}x^2 + \frac{f(\ddot{\varepsilon})}{6}x^3$

$F(a) = F(0) + f(0)a + \frac{f(\dot{0})}{2}a^2 + \frac{f(\ddot{\varepsilon_1})}{6}a^3 = F(0) + \frac{f(\dot{0})}{2}a^2 + \frac{f(\ddot{\varepsilon_1})}{6}a^3$

$F(-a) = F(0) - f(0)a + \frac{f(\dot{0})}{2}a^2 - \frac{f(\ddot{\varepsilon_2})}{6}a^3 = F(0) + \frac{f(\dot{0})}{2}a^2 - \frac{f(\ddot{\varepsilon_2})}{6}a^3$

$F(a) - F(-a) = \int_{-a}^{a} f(t)dt = \frac{a^3}{3}\left(\frac{f(\ddot{\varepsilon_1})}{2} + \frac{f(\ddot{\varepsilon_2})}{2}\right)$

$\ddot{f}(x)$在$[-a,a]$上连续，则一定可以取得最大值 M 和最小值 m

$m < \ddot{f}(\varepsilon_1) < M$ $m < \ddot{f}(\varepsilon_2) < M$

则 $m < \frac{f(\ddot{\varepsilon_1})}{2} + \frac{f(\ddot{\varepsilon_2})}{2} < M$ 由介值定理，知$\exists \mu \epsilon [-a,a]$, 使$\ddot{f}(\mu) = \frac{f(\ddot{\varepsilon_1})}{2} + \frac{f(\ddot{\varepsilon_2})}{2}$

即$a^3\ddot{f}(\mu)$=$3\int_{-a}^{a} f(x)\,dx$ 证毕