声明：本人绝对未在考试中实施任何作弊行为，也绝对未将试卷带出考场，以下试题仅是凭记忆整理，可能不尽准确，仅供参考。请不要将试题和答案传到工大以外。
以下参考答案为本人根据回忆版本整理，汇总了自己和其他人的解法，仅供参考。

哈尔滨工业大学（深圳）2021/2022 学年秋季学期

高等数学 A 期末试题 参考答案

一、填空题（每题 2 分，共 8 分）

1. $\frac{\sqrt{2}}{2}$　　2. $(-\infty, 2)$　　3. $x^2 f(x^3)$　　4. 12

二、选择题（每题 2 分，共 8 分）

1. （C）　　2. （A）（$a=1, b=0, c=-\frac{7}{6}$）

3. （D）$(0<\alpha<2)$　　4. （C）

三、(9 分)

$$1. \int \frac{3x+6}{(x+1)(x^2+x+1)}dx$$

$$= \int\left(\frac{3}{x+1} - \frac{3x-3}{x^2+x+1}\right)dx = 3\ln|x+1| - 3\int\frac{x-\frac{3}{2}+\frac{1}{2}}{x^2+x+1}dx$$

$$= 3\ln|x+1| - 3\int\frac{x+\frac{1}{2}}{x^2+x+1}dx - \frac{9}{2}\int\frac{1}{x^2+x+1}dx$$

$$= 3\ln|x+1| - \frac{3}{2}\int\frac{d(x^2+x+1)}{x^2+x+1}dx - \frac{9}{2}\int\frac{1}{(x+\frac{1}{2})^2+\frac{3}{4}}dx$$

$$= 3\ln|x+1| - \frac{3}{2}\ln(x^2+x+1) - \frac{9}{2}\int\frac{1}{(x+\frac{1}{2})^2+\frac{3}{4}}dx$$

$$= 3\ln|x+1| - \frac{3}{2}\ln(x^2+x+1) + \frac{9}{2}\times\frac{\sqrt{3}}{2}\int\frac{\sec^2 t}{\frac{3}{4}\sec^2 t}dt$$

$$= 3\ln|x+1| - \frac{3}{2}\ln(x^2+x+1) + 3\sqrt{3}\arctan\left(\frac{2\sqrt{3}}{3}x + \frac{\sqrt{3}}{3}\right) + C.$$

$$2.\int \frac{arctan\ e^x}{e^x}dx = -\int arctan\ e^x d(e^{-x})$$

$$= -[e^{-x}\ arctan\ e^x - \int e^{-x}\ \frac{e^x}{1+e^{2x}}dx]$$

$$= -e^{-x}\ arctan\ e^x + \int \frac{1}{1+e^{2x}}dx$$

$$= -e^{-x}\ arctan\ e^x + \int \frac{1}{1+u^2}\frac{1}{u}du$$

$$= -e^{-x}\ arctan\ e^x + \int \left(\frac{1}{u} - \frac{u}{1+u^2}\right)du$$

$$= -e^{-x}\ arctan\ e^x + ln\ u - \frac{1}{2}ln(1+u^2) + C$$

$$= -e^{-x}\ arctan\ e^x + x - \frac{1}{2}ln(1+e^{2x}) + C$$

$$3.\int_1^{\sqrt{3}} \frac{dx}{x^2\sqrt{1+x^2}} = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\sec^2\ tdt}{\tan^2\ t\sec t} = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\sec\ tdt}{\tan^2\ t}$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\cos\ tdt}{\sin^2\ t} = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{d(\sin\ t)}{\sin^2\ t} = \int_{\frac{\sqrt{2}}{2}}^{\frac{\sqrt{3}}{2}} \frac{du}{u^2} = \sqrt{2} - \frac{2\sqrt{3}}{3}.$$

四、(6 分)

$$1. \lim_{x\to 0} \frac{\int_0^x t\ ln(1+t\ \sin\ t)dt}{1-\cos(x^2)} = \lim_{x\to 0} \frac{x\ ln(1+x\ sinx)}{2\ xsin(x^2)}$$

$$= \lim_{x\to 0} \frac{ln(1+x\ sinx)}{2\ sin(x^2)} = \lim_{x\to 0} \frac{x\ sinx}{2x^2} = \frac{1}{2}\lim_{x\to 0} \frac{x^2}{x^2} = \frac{1}{2}$$

2.原极限$=\frac{1}{2\pi}\lim_{n\to\infty}\sum_{k=1}^{n}\frac{2\pi}{n}\sqrt{1+\cos\left(k\frac{2\pi-0}{n}\right)}$

$$= \frac{1}{2\pi}\int_0^{2\pi}\sqrt{1+\cos\ x}dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi}\sqrt{2\cos^2\frac{x}{2}}dx$$

$$= \frac{1}{2\pi}2\sqrt{2}\int_0^{\pi}\sqrt{\cos^2\ u}du$$

$$= \frac{1}{2\pi}2\sqrt{2}\left(\int_0^{\frac{\pi}{2}}\cos\ udu - \int_{\frac{\pi}{2}}^{\pi}\cos\ udu\right) = \frac{2\sqrt{2}}{\pi}$$

五、(6 分)

1. （1）$S = \int_0^2 3x^2dx = x^3 \Big|_0^2 = 8.$

（2）$V_1 = 2\pi\int_0^2 xydx = 2\pi \times 3\int_0^2 x^3dx = \frac{6\pi}{4} x^4 \Big|_0^2 = 24\pi$

$V_2 = 2\pi\int_0^2 (3-x)ydx = 2\pi \times 3\int_0^2 (3-x)x^2dx = 6\pi\int_0^2 (3x^2 - x^3)dx = 6\pi(x^3 \Big|_0^2 - \frac{1}{4} x^4 \Big|_0^2) = 24\pi$

2. $dV = \pi x^2dy = \pi ydy, dF = \pi\rho gydy$，$dW = (H-y)\pi\rho gydy$

$\therefore W = \int_0^{\frac{H}{2}} (H-y)\pi\rho gydy = \pi\rho g\int_0^{\frac{H}{2}} (Hy - y^2)dy = \pi\rho g(\frac{1}{2} Hy^2 \Big|_0^{\frac{H}{2}} - \frac{1}{3} y^3 \Big|_0^{\frac{H}{2}}) = \frac{\pi\rho gH^3}{12}.$

六、(6 分)

1. $5x\int_0^1 f(xt)\,dt = 5\int_0^1 f(xt)\,d(xt) = 5\int_0^x f(t)\,d(t)$

原方程化为$3\int_0^x f(t)\,dt = xf(x) + x^3$，两边求导得$3f(x) = f(x) + xf'(x) + 3x^2$

化为$2f(x) = xf'(x) + 3x^2$，即$-3x = f'(x) - \frac{2}{x} f(x)$

通解为

$$f(x) = e^{\int \frac{2}{x}dx}(C + \int (-3x)e^{-\int \frac{2}{x}dx}dx) = x^2(C + \int (-3x)\frac{1}{x^2}dx) = x^2(C - 3\int \frac{1}{x}dx) = Cx^2 - 3x^2 \ln x.$$

又 f(1)=1，因此特解为$f(x) = x^2 - 3x^2 \ln x.$

2. $p = y', y'' = p\frac{dp}{dy}$，回代有$2yp\frac{dp}{dy} = 1 + p^2$，得$\frac{2pdp}{1+p^2} = \frac{dy}{y}$

$\ln(1+p^2) = \ln y + \ln C$，即得$1 + p^2 = C_1 y$，代入初始条件，有$1 + p^2 = 2y$

$p = \pm\sqrt{2y-1}$。由初始条件，进一步可确定应取负号。$\frac{dy}{dx} = -\sqrt{2y-1},$

$-\frac{dy}{\sqrt{2y-1}} = dx,$即得$\sqrt{2y-1} = -x + C_2,$代入初始条件，有$C_2 = 2$，则

方程特解是$x + \sqrt{2y-1} = 2$。

七、(4 分)

1. $F(x) = 2x - \sin\frac{\pi}{2}x, F'(x) = 2 - \frac{\pi}{2}\cos\frac{\pi}{2}x > 2 - \frac{\pi}{2} > 0$，所以$[0,1]$上 $F(x)$单调递增，

$F(x) \le F(1) = 2 - 1 = 1$，得证。

2. $\lim\limits_{n\to\infty}\left(\int_0^1\left(1+\sin\frac{\pi}{2}x\right)^n dx\right)^{\frac{1}{n}} \ge \lim\limits_{n\to\infty}\left(\int_0^1 2^n x^n dx\right)^{\frac{1}{n}} = \lim\limits_{n\to\infty}\left(\frac{1}{n+1}2^n\right)^{\frac{1}{n}} = 2\lim\limits_{n\to\infty}\left(\frac{1}{n+1}\right)^{\frac{1}{n}}$

$\lim\limits_{n\to\infty}\left(\frac{1}{n+1}\right)^{\frac{1}{n}} = e^{\lim\limits_{n\to\infty}\frac{1}{n}ln\left(\frac{1}{n+1}\right)}$

$\lim\limits_{n\to\infty}\frac{1}{n}ln\left(\frac{1}{n+1}\right) = \lim\limits_{n\to\infty}\frac{ln\left(\frac{1}{n+1}\right)}{n} = -\lim\limits_{n\to\infty}\frac{(\frac{1}{n+1})^2(n+1)}{1} = -\lim\limits_{n\to\infty}\frac{1}{n+1} = 0$，

$\therefore \lim\limits_{n\to\infty}\left(\frac{1}{n+1}\right)^{\frac{1}{n}} = 1$，$\lim\limits_{n\to\infty}\left(\int_0^1\left(1+\sin\frac{\pi}{2}x\right)^n dx\right)^{\frac{1}{n}} \ge 2\lim\limits_{n\to\infty}\left(\frac{1}{n+1}\right)^{\frac{1}{n}}=2$

又 $\lim\limits_{n\to\infty}\left(\int_0^1\left(1+\sin\frac{\pi}{2}x\right)^n dx\right)^{\frac{1}{n}} \le \lim\limits_{n\to\infty}\left(\int_0^1 2^n dx\right)^{\frac{1}{n}} = \lim\limits_{n\to\infty}\left(2^n\right)^{\frac{1}{n}} = 2$，由夹逼准则，原极限 $= 2$.

八、(3 分)

证明：若 $f''(x) \ge 0$，则将 $f(x)$在 $x = \frac{a+b}{2}$处展开：

$f(x) = f\left(\frac{a+b}{2}\right) + f'\left(\frac{a+b}{2}\right)\left(x - \frac{a+b}{2}\right) + \frac{f''(h)}{2}\left(x - \frac{a+b}{2}\right)^2$ (h介于x和 $\frac{a+b}{2}$之间)

由 $f''(\mathrm{h}) \ge 0$，则 $f(x) \ge f\left(\frac{a+b}{2}\right) + f'\left(\frac{a+b}{2}\right)\left(x - \frac{a+b}{2}\right)$，则

$\int_a^b f(t)dt \ge \int_a^b\left[f\left(\frac{a+b}{2}\right) + f'\left(\frac{a+b}{2}\right)\left(x - \frac{a+b}{2}\right)\right]dx = (b-a)f\left(\frac{a+b}{2}\right)$，

则 $f(\frac{a+b}{2}) \le \frac{1}{b-a}\int_a^b f(x)dx$；

若 $f(\frac{a+b}{2}) \le \frac{1}{b-a}\int_a^b f(x)dx$，则设 $\exists x_0 \in R, f''(x_0) < 0$，由于 $f''(x)$ 连续，则

$\lim\limits_{x\to x_0} f''(x) = f''(x_0) < 0$，由极限局部保号性，$\exists \delta > 0, x \in (x_0 - \delta, x_0 + \delta)$ 时，有

$f''(x_0) < 0$。现令 $a = x_0 - \delta, b = x_0 + \delta$，将 $f(x)$ 在 $x = \frac{a+b}{2}$ 处展开：

$$f(x) = f\left(\frac{a+b}{2}\right) + f'\left(\frac{a+b}{2}\right)\left(x - \frac{a+b}{2}\right) + \frac{f''(\xi)}{2}\left(x - \frac{a+b}{2}\right)^2 \ (\xi\text{介于}x\text{和}\frac{a+b}{2}\text{之间})$$

$x \in [a,b]$ 时有 $f(x) \le f\left(\frac{a+b}{2}\right) + f'\left(\frac{a+b}{2}\right)\left(x - \frac{a+b}{2}\right)$ 且仅在 $x=\frac{a+b}{2}$ 一点取等号，

则有 $\int_a^b f(t)dt < \int_a^b \left[f\left(\frac{a+b}{2}\right) + f'\left(\frac{a+b}{2}\right)\left(x - \frac{a+b}{2}\right)\right]dx = (b-a)f\left(\frac{a+b}{2}\right)$

即 $f\left(\frac{a+b}{2}\right) > \frac{1}{b-a}\int_a^b f(t)dt$，矛盾，故有 $f''(x) \ge 0$，证毕。