等价无穷小

$x \to 0$ 时，$\sin x \sim x$ $\qquad 1-\cos x \sim \frac{1}{2}x^2$

$\tan x \sim x$ $\qquad \ln(x+1) \sim x$

$e^x - 1 \sim x$ $\qquad (1+x)^\alpha - 1 \sim \alpha x$

$\arcsin x \sim x$ $\qquad \arctan x \sim x$

$\underset{e^{x\ln\alpha}}{\underline{\alpha^x}} - 1 = x\ln\alpha$

泰勒公式

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots + (-1)^n \frac{x^{2n}}{(2n)!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \cdots + (-1)^{n-1} \frac{x^n}{n!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{(n+1)!} \quad (|x|<1)$$

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)x^2}{2} + \cdots + \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)x^n}{n!} \quad (|x|<1)$$

求导

$(\tan x)' = \sec^2 x$

$(\cot x)' = -\csc^2 x$

$(\ln|\sin x|)' = \cot x$

$(\ln|\cos x|)' = -\tan x$

$(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}$

$(\arccos x)' = -\frac{1}{\sqrt{1-x^2}}$

$(\arctan x)' = \frac{1}{1+x^2}$

$(\text{arccot} x)' = -\frac{1}{1+x^2}$

$(\sec x)' = \sec x \tan x$

$(\csc x)' = -\csc x \cot x$

$d(x^n) = nx^{n-1}dx$

$dx^n = (dx)^n$

幂指型函数求导

$$[u(x)^{v(x)}]' = u(x)^{v(x)} \cdot [v'(x)\ln u(x) + \frac{u'(x)}{u(x)} \cdot v(x)]$$

参数方程求导

$$\begin{cases} x = \varphi(t) \\ y = \psi(t) \end{cases}$$

$$\frac{dy}{dx} = \frac{\psi'(t)}{\varphi'(t)}$$

$$\frac{d^2y}{dx^2} = \frac{\psi''(t)\varphi'(t) - \varphi''(t)\psi'(t)}{[\varphi'(t)]^3}$$

莱布尼茨高阶导

$$(uv)^{(n)} = C_n^0 u^{(n)} v + C_n^1 u^{(n-1)} v' + \cdots + C_n^n u v^{(n)}$$

拐点与极值点导数

$f'(x_0) = f''(x_0) = \cdots = f^{(n-1)}(x_0) = 0$，但 $f^{(n)}(x_0) \neq 0$，

当 n 为奇数时，$(x_0, f(x_0))$ 为拐点

当 n 为偶数时、x_0 为极值点、

弧微分： $\frac{ds}{dx} = \sqrt{1+(\frac{dy}{dx})^2}$

直角坐标： $ds = \sqrt{1+y'^2}\, dx$

参数方程 $\begin{cases} x = \varphi(t) \\ y = \psi(t) \end{cases}$

$$ds = \sqrt{[\varphi'(t)]^2 + [\psi'(t)]^2}\, dt$$

极坐标： $r = r(\theta)$

$$ds = \sqrt{[r(\theta)]^2 + [r'(\theta)]^2}\, dt$$

曲率

$$k = \frac{|y''|}{[1+(y')^2]^{\frac{3}{2}}} \qquad R = \frac{1}{k}$$

参数方程 $\begin{cases} x = \varphi(t) \\ y = \psi(t) \end{cases}$

$$k = \frac{|\psi''(t)\varphi'(t) - \varphi''(t)\psi'(t)|}{\{[\varphi'(t)]^2 + [\psi'(t)]^2\}^{\frac{3}{2}}}$$

极坐标方程 $r = r(\theta)$

$$k = \frac{2[r'(\theta)]^2 + r^2(\theta) - r(\theta)\cdot r''(\theta)}{\{[r(\theta)]^2 + [r'(\theta)]^2\}^{\frac{3}{2}}} \quad \left(\frac{2r'^2 + r^2 - rr''}{(r^2 + r'^2)^{\frac{3}{2}}}\right)$$

曲率中心 (ξ, η)

$$\begin{cases} \xi = x - \dfrac{y'(1+y')}{y''} \\ \eta = y + \dfrac{(1+y'^2)}{y''} \end{cases}$$

积分公式

$$\int x^{\mu}dx = \frac{1}{\mu+1}\cdot x^{\mu+1} + C$$

$$\int \sec^2 x\,dx = \tan x + C$$

$$\int \csc^2 x\,dx = -\cot x + C$$

$$\int \sec x \tan x\,dx = \sec x + C$$

$$\int \csc x \cot x\,dx = -\csc x + C$$

$$\int \frac{1}{\sqrt{1-x^2}}dx = \arcsin x + C$$

$$\int \frac{1}{1+x^2}dx = \arctan x + C$$

$$\int \tan x\,dx = -\ln|\cos x| + C$$

$$\int \cot x\,dx = \ln|\sin x| + C$$

$$\int \sec x\,dx = \ln|\sec x + \tan x| + C$$

$$\int \csc x\,dx = \ln|\csc x - \cot x| + C$$

$$\int \frac{dx}{a^2+x^2} = \frac{1}{a}\arctan\frac{x}{a} + C$$

$$\int \frac{dx}{\sqrt{a^2-x^2}} = \arcsin\frac{x}{a} + C$$

$$\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + C$$

$$\int \frac{dx}{\sqrt{x^2 \pm a^2}} = \ln\left|x+\sqrt{x^2 \pm a^2}\right| + C$$

$$\int \sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\arcsin\frac{x}{a} + C$$

$$\int \sqrt{x^2 \pm a^2}\,dx = \frac{x}{2}\sqrt{x^2 \pm a^2} \pm \frac{a^2}{2}\ln\left|x+\sqrt{x^2 \pm a^2}\right| + C$$

$$\sqrt{a^2-x^2} = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\arcsin$$

递推公式（降高阶分母）

$$\int \frac{dx}{(x^2+a^2)^{n+1}} = \frac{1}{2na^2} \cdot \frac{x}{(x^2+a^2)^n} + \frac{2n-1}{2na^2}\int \frac{dx}{(x^2+a^2)^n}$$

三角函数有理式积分

$u = \tan\frac{x}{2}$ $\quad \sin x = \frac{2u}{1+u^2} \quad \cos x = \frac{1-u^2}{1+u^2}$

$dx = \frac{2}{1+u^2}du$

定积分简化计算

$$\int_{-a}^{a} f(x)\,dx = \int_0^a [f(x)+f(-x)]\,dx$$

$f(x)$在$[-a,a]$上连续，用于化简$f(x)$与$f(-x)$可以消去的计算

当$f(x)$为偶函数时，$\int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx$

当$f(x)$为奇函数时，$\int_{-a}^{a} f(x)dx = 0$

$$\int_a^{a+T} f(x)\,dx = \int_0^T f(x)\,dx$$

（$f(x)$在$(-\infty,+\infty)$以T为周期的连续函数）

$$\int_0^{\frac{\pi}{2}} f(\sin x)\,dx = \int_0^{\frac{\pi}{2}} f(\cos x)\,dx$$

$$\int_0^{\pi} x f(\sin x)\,dx = \frac{\pi}{2}\int_0^{\pi} f(\sin x)\,dx = \pi\int_0^{\frac{\pi}{2}} f(\sin x)\,dx$$

$$I_n = \int_0^{\frac{\pi}{2}} \sin^n x\,dx = \int_0^{\frac{\pi}{2}} \cos^n x\,dx$$

（$n>1$，要注意上限）
$$= \begin{cases} \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdots\frac{2}{3} & n\text{奇数} \\ \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdots\frac{1}{2}\cdot\frac{\pi}{2} & \text{偶数} \end{cases}$$

$$\int_0^{\pi} \sin^n x\,dx = 2\int_0^{\frac{\pi}{2}} \sin^n x\,dx$$

$$\int_0^{\pi} \cos^n x\,dx = \begin{cases} 2\int_0^{\frac{\pi}{2}} \cos^n x\,dx & n\text{偶} \\ 0 & n\text{奇} \end{cases}$$

$$\int_0^{2\pi} \sin^n x\,dx = \int_0^{2\pi} \cos^n x\,dx = \begin{cases} 4\int_0^{\frac{\pi}{2}} \sin^n x\,dx & n\text{偶} \\ 0 & n\text{奇} \end{cases}$$

一阶线性非齐次方程

$$\frac{dy}{dx}+P(x)y=Q(x)$$

通解：

$$y=e^{-\int P(x)dx}\left(\int Q(x)e^{\int P(x)dx}+C\right)$$

伯努利 $\frac{dy}{dx}+P(x)y=Q(x)y^n \ (n\neq 0,1)$

令 $z=y^{1-n}$，得：

$$\frac{dz}{dx}+(1-n)P(x)z=(1-n)Q(x)$$

可降阶的高阶微分方程

① 缺x

令 $z=y'$，$y''=\frac{dz}{dx}=\frac{dz}{dy}\cdot\frac{dy}{dx}=z\frac{dz}{dy}$

也有可能是用常系数微分方程的齐次通解的方法

②缺y的

令 $z=y'$，$y''=\frac{dz}{dx}$

常系数微分方程

齐次通解：

①求特征方程

②根的情况：

(1) 实单根 $e^{\lambda x}$

(2) k重实根 $e^{\lambda x}, xe^{\lambda x}, \cdots x^{k-1}e^{\lambda x}$

(注意，若 $x^{m}e^{\lambda x}$ 为齐次方程的解，则 $x^{n}e^{\lambda x}(n<m)$ 均为齐次方程解，反之则不一定成立)

(3) 单重共轭复根 $\lambda_{1,2}=\alpha\pm\beta i$

对应两个解 $e^{\alpha x}\cos\beta x$，$e^{\alpha x}\sin\beta x$

(4) k重共轭复根 $\lambda_{1,2}=\lambda_{3,4}=\lambda_{2k-1,2k}=\alpha\pm\beta i$

$\lambda_1=\lambda_3=\cdots=\lambda_{2n-1}=\alpha+\beta i$，$\lambda_2=\lambda_4=\cdots=\lambda_{2k}=\alpha-\beta i$

对应2k个解：

$e^{\alpha x}\cos\beta x$，$xe^{\alpha x}\cos\beta x$，$\cdots$，$x^{k-1}e^{\alpha x}\cos\beta x$，

$e^{\alpha x}\sin\beta x$，$xe^{\alpha x}\sin\beta x$，$\cdots$，$x^{k-1}e^{\alpha x}\sin\beta x$

找非齐次特解的形式：

$$y^{(n)}+a_1y^{(n-1)}+a_2y^{(n-2)}\cdots+a_ny=f(x)$$

只有$f(x)$为 $e^{\alpha x}(P(x)\cos\beta x+Q(x)\sin\beta x)$这种形式的线性组合才可以用这种方法做

特解形式为 $y^*=x^ke^{\alpha x}(H(x)\cos\beta x+S(x)\sin\beta x)$

k为 $\alpha+\beta i$为特征方程根的重数（一般无α，或无β，否则极其复杂），$H(x)$与$S(x)$为次数为 m（m为$P(x)$与$Q(x)$中较大次数的值）的待定多项式