第十章重积分

习题十

10.1

1. 设有一平面薄板（不计其厚度）占有 xOy 平面上的闭区域 D，薄板上分布着面密度为 $\mu=\mu(x,y)$ 的电荷，且 $\mu(x,y)$ 在 D 上连续，试用二重积分表达该薄板上的全部电荷 Q.

解 将 D 分成 n 个小闭区域 $D_1,D_2,\cdots,D_n$，其面积记为 $\Delta\sigma_1,\Delta\sigma_2,\cdots,\Delta\sigma_n$，任取 $(\xi_i,\ \eta_i)\in D_i$，则全部电荷为

$$Q=\lim_{\lambda\to 0}\sum_{i=1}^{n}\mu(\xi_i,\eta_i)\Delta\sigma_i=\iint_D \mu(x,y)\mathrm{d}\sigma$$

其中 $\lambda=\max\limits_{1\le i\le n}\{\Delta D_i\text{的直径}\}$

2. 根据二重积分的性质，比较下列积分的大小.

(1) $\iint_D \ln(x+y)\mathrm{d}\sigma$ 与 $\iint_D [\ln(x+y)]^2 d\sigma$，其中 D 是三角形闭区域，三个顶点分别为 $(1,0),(1,1),(2,0)$；

解 因为积分区域 D 位于区域 $\{(x,y)\,|\,1\le x+y\le 2\}$ 内，所以在 D 上

$$0\le\ln(x+y)\le 1$$

可知

$$[\ln(x+y)]^2\le\ln(x+y)$$

且不恒等, 由二重积分性质

$$\iint_D [\ln(x+y)]^2\mathrm{d}\sigma<\iint_D \ln(x+y)\mathrm{d}\sigma$$

(2) $\iint_D (x+y)^2\mathrm{d}\sigma$ 与 $\iint_D (x+y)^3\mathrm{d}\sigma$，其中 D 是由圆周 $(x-2)^2+(y-1)^2=2$ 所围成的闭区域.

解 因为积分区域 D 位于半平面 $\{(x,y)\,|\,x+y\ge 1\}$ 内，所以在 D 上

$$(x+y)^2\le(x+y)^3$$

且不恒等, 由二重积分性质

$$\iint_D (x+y)^2 \mathrm{d}\sigma < \iint_D (x+y)^3 \mathrm{d}\sigma$$

3. 利用二重积分的性质估计下列积分的值.

(1) $\iint_D xy(x+y)\mathrm{d}\sigma$，其中 $D=\{(x,y)\mid 0\le x\le 1, 0\le y\le 1\}$；

解 在 D 上

$$0 \le xy(x+y) \le 2$$

且不恒等, 又 D 的面积等于 1，由二重积分性质

$$0 < \iint_D xy(x+y)\mathrm{d}\sigma < 2$$

(2) $\iint_D (x^2+4y^2+9)\mathrm{d}\sigma$，其中 $D=\{(x,y)\mid x^2+y^2\le 4\}$.

解 函数 $f(x,y)=x^2+4y^2+9$ 在 D 上的最大值为 25, 最小值为 9, 所以在 D 上

$$9 \le x^2+4y^2+9 \le 25$$

且不恒等, 又 D 的面积等于 4π，由二重积分性质得

$$9\cdot(4\pi) < \iint_D (x^2+4y^2+9)\mathrm{d}\sigma < 25\cdot(4\pi)$$

即

$$36\pi < \iint_D (x^2+4y^2+9)\mathrm{d}\sigma < 100\pi$$

10. 2

1. 计算下列二重积分.

(1) $\iint_D (x+y)\,\mathrm{d}x\mathrm{d}y$ 其中 D 是以 $(0,0),(1,0),(1,1)$ 为顶点的三角形闭区域；

解

$$\iint_D (x+y)\,\mathrm{d}x\mathrm{d}y = \int_0^1 \mathrm{d}x \int_0^x (x+y)\mathrm{d}y$$

$$= \int_0^1 \left(xy+\frac{y^2}{2}\right)\Bigg|_{y=0}^{y=x} \mathrm{d}x = \int_0^1 \frac{3}{2}x^2\mathrm{d}x = \frac{1}{2}x^3\Big|_0^1 = \frac{1}{2}$$

(2) $\iint_D (x^3+3x^2y+y^3)\mathrm{d}x\mathrm{d}y$，其中 $D=\{(x,y)\mid 0\le x\le 1, 0\le y\le 1\}$；

解

$$\iint\limits_D \left(x^3+3x^2y+y^3\right)\mathrm{d}x\mathrm{d}y=\int_0^1\mathrm{d}y\int_0^1\left(x^3+3x^2y+y^3\right)\mathrm{d}x$$

$$=\int_0^1\left(\frac{x^4}{4}+x^3y+xy^3\right)\Big|_{x=0}^{x=1}\mathrm{d}y$$

$$=\int_0^1\left(\frac{1}{4}+y+y^3\right)\mathrm{d}y=\left(\frac{1}{4}y+\frac{y^2}{2}+\frac{y^4}{4}\right)\Big|_0^1=1$$

(3) $\iint\limits_D x\sqrt{y}\,\mathrm{d}x\mathrm{d}y$，其中 D 是由两条抛物线 $y=\sqrt{x}, y=x^2$ 所围成的闭区域；

解

$$\iint\limits_D x\sqrt{y}\,\mathrm{d}x\mathrm{d}y=\int_0^1\mathrm{d}x\int_{x^2}^{\sqrt{x}}x\sqrt{y}\,\mathrm{d}y=\int_0^1\frac{2}{3}xy^{\frac{3}{2}}\Big|_{y=x^2}^{y=\sqrt{x}}\mathrm{d}x$$

$$=\frac{2}{3}\int_0^1\left(x^{\frac{7}{4}}-x^4\right)\mathrm{d}x=\frac{2}{3}\left(\frac{4}{11}x^{\frac{11}{4}}-\frac{1}{5}x^5\right)\Big|_0^1=\frac{6}{55}$$

(4) $\iint\limits_D \sqrt{1-\sin^2(x+y)}\,\mathrm{d}x\mathrm{d}y$，其中 $D=\left\{(x,y)\mid 0\le x\le\pi, 0\le y\le\pi\right\}$；

解

$$\iint\limits_D \sqrt{1-\sin^2(x+y)}\,\mathrm{d}x\mathrm{d}y=\iint\limits_D\left|\cos(x+y)\right|\mathrm{d}x\mathrm{d}y$$

$$=\int_0^{\frac{\pi}{2}}\mathrm{d}x\int_0^{\frac{\pi}{2}-x}\cos(x+y)\mathrm{d}y-\int_0^{\frac{\pi}{2}}\mathrm{d}x\int_{\frac{\pi}{2}-x}^{\pi}\cos(x+y)\mathrm{d}y$$

$$-\int_{\frac{\pi}{2}}^{\pi}\mathrm{d}x\int_0^{\frac{3\pi}{2}-x}\cos(x+y)\mathrm{d}y+\int_{\frac{\pi}{2}}^{\pi}\mathrm{d}x\int_{\frac{3\pi}{2}-x}^{\pi}\cos(x+y)\mathrm{d}y=2\pi$$

(5) $\iint\limits_D\left[x^2y+\sin\left(xy^2\right)\right]\mathrm{d}x\mathrm{d}y$，其中 D 是由曲线 $x^2-y^2=1$ 与直线 $y=0, y=1$ 所围成的闭区域.

解

$$\iint\limits_D\left[x^2y+\sin\left(xy^2\right)\right]\mathrm{d}x\mathrm{d}y=\iint\limits_D x^2y\,\mathrm{d}x\mathrm{d}y+\iint\limits_D\sin\left(xy^2\right)\mathrm{d}x\mathrm{d}y$$

$$=2\iint\limits_{D_1}x^2y\,\mathrm{d}x\mathrm{d}y+0=2\int_0^1\mathrm{d}y\int_0^{\sqrt{1+y^2}}x^2y\,\mathrm{d}x=2\int_0^1\frac{x^3y}{3}\Big|_{x=0}^{x=\sqrt{1+y^2}}\mathrm{d}y$$

$$=\frac{2}{3}\int_0^1 y\left(\sqrt{1+y^2}\right)^3\mathrm{d}y=\frac{1}{3}\int_0^1\left(1+y^2\right)^{\frac{3}{2}}\mathrm{d}\left(1+y^2\right)$$

$$=\frac{2}{15}\left(1+y^2\right)^{\frac{5}{2}}\Big|_0^1=\frac{2}{15}\left(4\sqrt{2}-1\right)$$

2．交换下列二次积分的积分次序.

(1) $\int_1^{e} dx \int_0^{\ln x} f(x,y) dy$；

解　积分区域为 $D=\left\{(x,y)\,\middle|\,1\le x\le e, 0\le y\le \ln x\right\}=\left\{(x,y)\,\middle|\,0\le y\le 1, e^y\le x\le e\right\}$，所以

$$\int_1^{e} dx \int_0^{\ln x} f(x,y) dy = \int_0^1 dy \int_{e^y}^{e} f(x,y) dx$$

(2) $\int_0^1 dx \int_x^{2x} f(x,y) dy$；

解　积分区域为

$$D=\left\{(x,y)\,\middle|\,0\le x\le 1, x\le y\le 2x\right\}$$
$$=\left\{(x,y)\,\middle|\,0\le y\le 1, \frac{y}{2}\le x\le y\right\}\cup\left\{(x,y)\,\middle|\,1\le y\le 2, y\le x\le 1\right\}$$

所以

$$\int_0^1 dx \int_x^{2x} f(x,y) dy = \int_0^1 dy \int_{\frac{y}{2}}^{y} f(x,y) dx + \int_1^2 dy \int_{\frac{y}{2}}^{1} f(x,y) dx$$

(3) $\int_0^2 dy \int_{y^2}^{2y} f(x,y) dx$；

解　积分区域 $D=\left\{(x,y)\,\middle|\,0\le y\le 2, y^2\le x\le 2y\right\}=\left\{(x,y)\,\middle|\,0\le x\le 4, \frac{x}{2}\le y\le \sqrt{x}\right\}$，

所以

$$\int_0^2 dy \int_{y^2}^{2y} f(x,y) dx = \int_0^4 dx \int_{\frac{x}{2}}^{\sqrt{x}} f(x,y) dy$$

(4) $\int_0^{\frac{a}{2}} dy \int_{\sqrt{a^2-2ay}}^{\sqrt{a^2-y^2}} f(x,y) dx + \int_{\frac{a}{2}}^{a} dy \int_0^{\sqrt{a^2-y^2}} f(x,y) dx \ (a>0)$.

解　积分区域为

$$D=\left\{(x,y)\,\middle|\,0\le y\le \frac{a}{2}, \sqrt{a^2-2ay}\le x\le \sqrt{a^2-y^2}\right\}\cup\left\{(x,y)\,\middle|\,\frac{a}{2}\le y\le a, 0\le x\le \sqrt{a^2-y^2}\right\}$$
$$=\left\{(x,y)\,\middle|\,0\le x\le a, \frac{a^2-x^2}{2a}\le y\le \sqrt{a^2-x^2}\right\}$$

所以

$$\int_0^{\frac{a}{2}} \mathrm{d}y \int_{\sqrt{a^2-2ay}}^{\sqrt{a^2-y^2}} f(x,y)\mathrm{d}x + \int_{\frac{a}{2}}^{a} \mathrm{d}y \int_0^{\sqrt{a^2-y^2}} f(x,y)\mathrm{d}x$$

$$= \int_0^a \mathrm{d}x \int_{\frac{a^2-x^2}{2a}}^{\sqrt{a^2-x^2}} f(x,y)\mathrm{d}y$$

3. 计算二次积分 $\int_0^{\frac{\pi}{6}} \mathrm{d}y \int_y^{\frac{\pi}{6}} \frac{\cos x}{x} \mathrm{d}x$.

解

$$\int_0^{\frac{\pi}{6}} \mathrm{d}y \int_y^{\frac{\pi}{6}} \frac{\cos x}{x} \mathrm{d}x = \int_0^{\frac{\pi}{6}} \mathrm{d}x \int_0^{x} \frac{\cos x}{x} \mathrm{d}y$$

$$= \int_0^{\frac{\pi}{6}} \frac{\cos x}{x} \cdot y\Big|_{y=0}^{y=x} \mathrm{d}x = \int_0^{\frac{\pi}{6}} \cos x \,\mathrm{d}x = \sin x\Big|_0^{\frac{\pi}{6}} = \frac{1}{2}$$

4. 设平面薄片所占的闭区域 D 由直线 $x+y=2, y=x$ 和 x 轴所围成，它的面密度 $\mu(x,y)=x^2+y^2$，求该薄片的质量.

解 所求薄片的质量为

$$m = \iint_D \mu(x,y)\mathrm{d}\sigma = \iint_D (x^2+y^2)\mathrm{d}x\mathrm{d}y = \int_0^1 \mathrm{d}y \int_y^{2-y} (x^2+y^2)\mathrm{d}x$$

$$= \int_0^1 \left(\frac{x^3}{3} + xy^2\right)\Big|_{x=y}^{x=2-y} \mathrm{d}y = \int_0^1 \left[\frac{1}{3}(2-y)^3 + 2y^2 - \frac{7}{3}y^3\right]\mathrm{d}y = \frac{4}{3}$$

5. 求由平面 $x=0, y=0, x+y=1$ 所围成的柱体被平面 $z=0$ 及抛物面 $x^2+y^2=6-z$ 截得的立体的体积.

解 记 $D=\left\{(x,y)\,\middle|\, 0\le x\le 1, 0\le y\le 1-x\right\}$, 则立体体积为

$$\iint_D \left[6-(x^2+y^2)\right]\mathrm{d}x\mathrm{d}y = \int_0^1 \mathrm{d}x \int_0^{1-x} (6-x^2-y^2)\mathrm{d}y$$

$$= \int_0^1 \left(6y - x^2y - \frac{y^3}{3}\right)\Big|_{y=x}^{y=1-x} \mathrm{d}x = \int_0^1 \left[6(1-x) - x^2 + x^3 - \frac{1}{3}(1-x)^3\right]\mathrm{d}x = \frac{17}{6}$$

6. 利用极坐标计算下列二重积分.

(1) $\iint_D \sqrt{x^2+y^2}\,\mathrm{d}x\mathrm{d}y$，其中 D 是圆环闭区域 $\left\{(x,y)\,\middle|\, a^2\le x^2+y^2\le b^2\right\}$;

解 在极坐标系下, 积分区域 $D=\left\{(r,\theta)\,\middle|\, 0\le\theta\le 2\pi, a\le r\le b\right\}$，所以

$$\iint_D \sqrt{x^2+y^2}\,\mathrm{d}x\mathrm{d}y = \iint_D r\cdot r\mathrm{d}r\mathrm{d}\theta = \int_0^{2\pi}\,\mathrm{d}\theta\int_a^b r^2\mathrm{d}r$$

$$=(2\pi)\cdot\frac{1}{3}\left(b^3-a^3\right)=\frac{2}{3}\pi\left(b^3-a^3\right)$$

(2) $\iint_D \left(x^2+y^2\right)\mathrm{d}x\mathrm{d}y$，其中 $D=\left\{(x,y)\,\middle|\, x^2+y^2\ge 2x, x^2+y^2\le 4x\right\}$；

解 在极坐标系下，积分区域为 $D=\left\{(r,\theta)\,\middle|\, -\frac{\pi}{2}\le\theta\le\frac{\pi}{2}, 2\cos\theta\le r\le 4\cos\theta\right\}$，所以

$$\iint_D \left(x^2+y^2\right)\mathrm{d}x\mathrm{d}y = \iint_D r^2\cdot r\mathrm{d}r\mathrm{d}\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\mathrm{d}\theta\int_{2\cos\theta}^{4\cos\theta} r^3\mathrm{d}r = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{r^4}{4}\Big|_{r=2\cos\theta}^{r=4\cos\theta}\mathrm{d}\theta$$

$$=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 60\cos^4\theta\,\mathrm{d}\theta = 120\int_0^{\frac{\pi}{2}}\cos^4\theta\,\mathrm{d}\theta = 120\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}=\frac{45\pi}{2}$$

(3) $\iint_D \left(x^2+y^2\right)^{\frac{3}{2}}\mathrm{d}x\mathrm{d}y$，其中 $D=\left\{(x,y)\,\middle|\, x^2+y^2\le 1, x^2+y^2\le 2x\right\}$.

解 记 $D_1=\left\{(x,y)\mid (x,y)\in D, y\ge 0\right\}$, 则

$$\iint_D \left(x^2+y^2\right)^{\frac{3}{2}}\mathrm{d}x\mathrm{d}y = 2\iint_{D_1}\left(x^2+y^2\right)^{\frac{3}{2}}\mathrm{d}x\mathrm{d}y$$

在极坐标系下，

$$D_1=\left\{(r,\theta)\,\middle|\, 0\le\theta\le\frac{\pi}{3}, 0\le r\le 1\right\}\cup\left\{(r,\theta)\,\middle|\,\frac{\pi}{3}\le\theta\le\frac{\pi}{2}, 0\le r\le 2\cos x\right\}$$

所以

$$\iint_D \left(x^2+y^2\right)^{\frac{3}{2}}\mathrm{d}x\mathrm{d}y = 2\iint_{D_1} r^3\cdot r\mathrm{d}r\mathrm{d}\theta = 2\left[\int_0^{\frac{\pi}{3}}\mathrm{d}\theta\int_0^1 r^4\mathrm{d}r + \int_{\frac{\pi}{3}}^{\frac{\pi}{2}}\mathrm{d}\theta\int_0^{2\cos\theta} r^4\mathrm{d}r\right]$$

$$=2\left[\frac{\pi}{3}\cdot\frac{1}{5}+\frac{1}{5}\int_{\frac{\pi}{3}}^{\frac{\pi}{2}}\cdot 2^5\cos^5\theta\,\mathrm{d}\theta\right]=\frac{2\pi}{15}+\frac{64}{5}\int_{\frac{\pi}{3}}^{\frac{\pi}{2}}\left(1-\sin^2\theta\right)^2\mathrm{d}\sin\theta$$

$$=\frac{2\pi}{15}+\frac{64}{5}\left(\sin\theta-\frac{2}{3}\sin^2\theta+\frac{1}{5}\sin^5\theta\right)\Bigg|_{\frac{\pi}{3}}^{\frac{\pi}{2}}=\frac{2}{15}\left(\pi+\frac{256-147\sqrt{3}}{5}\right)$$

7. 化下列二次积分为极坐标形式的二次积分.

(1) $\int_0^1\mathrm{d}x\int_0^1 f(x,y)\mathrm{d}y$；

解 积分区域为

$$D=\left\{(x,y)\,\middle|\,0\le x\le 1,0\le y\le 1\right\}$$

$$=\left\{(r,\theta)\,\middle|\,0\le\theta\le\frac{\pi}{4},0\le r\le\frac{1}{\cos\theta}\right\}\cup\left\{(r,\theta)\,\middle|\,\frac{\pi}{4}\le\theta\le\frac{\pi}{2},0\le r\le\frac{1}{\sin\theta}\right\}$$

所以

$$\int_0^1 \mathrm{d}x\int_0^1 f(x,y)\mathrm{d}y=\int_0^{\frac{\pi}{4}}\mathrm{d}\theta\int_0^{\frac{1}{\cos\theta}} f(r\cos\theta,r\sin\theta)r\mathrm{d}r+\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\mathrm{d}\theta\int_0^{\frac{1}{\sin\theta}} f(r\cos\theta,r\sin\theta)r\mathrm{d}r$$

（2） $\int_0^1 \mathrm{d}x\int_{1-x}^{\sqrt{1-x^2}} f(x,y)\mathrm{d}y$.

解 积分区域为

$$D=\left\{(x,y)\,\middle|\,0\le x\le 1,1-x\le y\le\sqrt{1-x^2}\right\}=\left\{(r,\theta)\,\middle|\,0\le\theta\le\frac{\pi}{2},\frac{1}{\cos\theta+\sin\theta}\le r\le 1\right\}$$

所以

$$\int_0^1 \mathrm{d}x\int_{1-x}^{\sqrt{1-x^2}} f(x,y)\mathrm{d}y=\int_0^{\frac{\pi}{2}}\mathrm{d}\theta\int_{\frac{1}{\cos\theta+\sin\theta}}^1 f(r\cos\theta,r\sin\theta)r\mathrm{d}r$$

8. 把下列积分化成极坐标形式,并计算积分.

（1） $\int_0^1 \mathrm{d}x\int_{x^2}^{x}\frac{1}{\sqrt{x^2+y^2}}\mathrm{d}y$；

解 积分区域为

$$D=\left\{(x,y)\,\middle|\,0\le x\le 1,x^2\le y\le x\right\}=\left\{(r,\theta)\,\middle|\,0\le\theta\le\frac{\pi}{4},0\le r\le\frac{\sin\theta}{\cos^2\theta}\right\}$$

所以

$$\int_0^1 \mathrm{d}x\int_{x^2}^{x}\frac{1}{\sqrt{x^2+y^2}}\mathrm{d}y=\iint_D\frac{1}{\sqrt{x^2+y^2}}\mathrm{d}x\mathrm{d}y=\iint_D\frac{1}{r}\cdot r\mathrm{d}r\mathrm{d}\theta$$

$$=\int_0^{\frac{\pi}{4}}\mathrm{d}\theta\int_0^{\frac{\sin\theta}{\cos^2\theta}}\mathrm{d}r=\int_0^{\frac{\pi}{4}}\frac{\sin\theta}{\cos^2\theta}\mathrm{d}\theta=\frac{1}{\cos\theta}\bigg|_0^{\frac{\pi}{4}}=\sqrt{2}-1$$

（2） $\int_0^a \mathrm{d}x\int_{-x}^{-a+\sqrt{a^2-x^2}}\frac{1}{\sqrt{x^2+y^2}\cdot\sqrt{4a^2-x^2-y^2}}\mathrm{d}y\,(a>0)$.

解 积分区域为

$$D=\left\{(x,y)\,\middle|\,0\le x\le a,-x\le y\le -a+\sqrt{a^2-x^2}\right\}=\left\{(r,\theta)\,\middle|\,-\frac{\pi}{4}\le\theta\le 0,0\le r\le -2a\sin\theta\right\}$$

所以

$$\int_0^a \mathrm{d}x\int_{-x}^{-a+\sqrt{a^2-x^2}} \frac{1}{\sqrt{x^2+y^2}\sqrt{4a^2-x^2-y^2}}\mathrm{d}y = \iint_D \frac{1}{\sqrt{x^2+y^2}\sqrt{4a^2-x^2-y^2}}\mathrm{d}x\mathrm{d}y$$

$$= \iint_D \frac{1}{r\sqrt{4a^2-r^2}}\cdot r\mathrm{d}r\mathrm{d}\theta = \int_{-\frac{\pi}{4}}^{0}\mathrm{d}\theta\int_0^{-2a\sin\theta}\frac{1}{\sqrt{4a^2-r^2}}\mathrm{d}r$$

$$= \int_{-\frac{\pi}{4}}^{0}\arcsin\frac{r}{2a}\Big|_{r=0}^{r=-2a\sin\theta}\mathrm{d}\theta = \int_{-\frac{\pi}{4}}^{0}-\theta\,\mathrm{d}\theta = \frac{\pi^2}{32}$$

9. 计算以 xOy 平面上的圆周 $x^2+y^2=ax$ 围成的闭区域为底，而以曲面 $z=x^2+y^2$ 为顶的曲顶柱体的体积.

解 记 $D=\left\{(x,y)\middle|x^2+y^2\le ax\right\}$，则曲顶柱体的体积为

$$V=\iint_D\left(x^2+y^2\right)\mathrm{d}x\mathrm{d}y=\iint_D r^2\cdot r\mathrm{d}r\mathrm{d}\theta=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\mathrm{d}\theta\int_0^{a\cos\theta}r^3\mathrm{d}r=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{r^4}{4}\Big|_{r=0}^{r=a\cos\theta}d\theta$$

$$=\frac{a^4}{4}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos^4\theta\,\mathrm{d}\theta=\frac{a^4}{2}\int_0^{\frac{\pi}{2}}\cos^4\theta\,\mathrm{d}\theta=\frac{a^4}{2}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}=\frac{3}{32}\pi a^4$$

10. 3

1. 计算下列三重积分.

⑴ $\iiint_\Omega xyz\,\mathrm{d}x\mathrm{d}y\mathrm{d}z$，其中 Ω 为球面 $x^2+y^2+z^2=1$ 及三个坐标面所围成的在第一卦限内的闭区域；

解 积分区域为

$$\Omega=\left\{(x,y,z)\middle|\,0\le x\le 1,0\le y\le\sqrt{1-x^2},0\le z\le\sqrt{1-x^2-y^2}\right\}$$

所以

$$\iiint_\Omega xyz\,\mathrm{d}x\mathrm{d}y\mathrm{d}z=\int_0^1\mathrm{d}x\int_0^{\sqrt{1-x^2}}\mathrm{d}y\int_0^{\sqrt{1-x^2-y^2}}xyz\,\mathrm{d}z=\int_0^1\mathrm{d}x\int_0^{\sqrt{1-x^2}}\frac{xyz^2}{2}\Big|_{z=0}^{z=\sqrt{1-x^2-y^2}}\mathrm{d}y$$

$$=\int_0^1\mathrm{d}x\int_0^{\sqrt{1-x^2}}\frac{xy}{2}\left(1-x^2-y^2\right)\mathrm{d}y=\int_0^1\frac{x}{2}\left[\frac{y^2}{2}\left(1-x^2\right)-\frac{y^4}{4}\right]\Big|_{y=0}^{y=\sqrt{1-x^2}}\mathrm{d}x$$

$$=\int_0^1\frac{1}{8}\left(1-x^2\right)^2\mathrm{d}x=\frac{1}{48}$$

⑵ $\iiint_\Omega y\cos(x+z)\mathrm{d}x\mathrm{d}y\mathrm{d}z$，其中 Ω 是由柱面 $y=\sqrt{x}$ 和平面 $y=0,z=0,x+z=\frac{\pi}{2}$ 所围成的闭区域；

解 积分区域为

$$\Omega=\left\{(x,y,z)\,\middle|\,0\le x\le\frac{\pi}{2},0\le y\le\sqrt{x},0\le z\le\frac{\pi}{2}-x\right\}$$

所以

$$\iiint\limits_{\Omega} y\cos(x+z)\mathrm{d}x\mathrm{d}y\mathrm{d}z=\int_0^{\frac{\pi}{2}}\mathrm{d}x\int_0^{\sqrt{x}}\mathrm{d}y\int_0^{\frac{\pi}{2}-x}y\cos(x+z)\mathrm{d}z$$

$$=\int_0^{\frac{\pi}{2}}\mathrm{d}x\int_0^{\sqrt{x}}y\sin(x+z)\Big|_{z=0}^{z=\frac{\pi}{2}-x}\mathrm{d}y=\int_0^{\frac{\pi}{2}}\mathrm{d}x\int_0^{\sqrt{x}}y(1-\sin x)\mathrm{d}y$$

$$=\int_0^{\frac{\pi}{2}}\frac{y^2}{2}(1-\sin x)\Big|_{y=0}^{y=\sqrt{x}}\mathrm{d}x=\frac{1}{2}\int_0^{\frac{\pi}{2}}(x-x\sin x)\mathrm{d}x$$

$$=\frac{1}{2}\left(\frac{x^2}{2}+x\cos x-\sin x\right)\Big|_0^{\frac{\pi}{2}}=\frac{1}{2}\left(\frac{\pi^2}{8}-1\right)$$

(3) $\iiint\limits_{\Omega}\left(y^2+x^3y^4z^5\right)\mathrm{d}x\mathrm{d}y\mathrm{d}z$，其中 $\Omega=\left\{(x,y,z)\,\middle|\,\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\le 1\right\}$；

解 平面 $y=y\,(-b\le y\le b)$ 截 Ω 所得区域为

$$D_y=\left\{(x,z)\,\middle|\,\frac{x^2}{a^2}+\frac{z^2}{c^2}\le 1-\frac{y^2}{b^2}\right\}$$

所以

$$\iiint\limits_{\Omega}\left(y^2+x^3y^4z^5\right)\mathrm{d}x\mathrm{d}y\mathrm{d}z=\iiint\limits_{\Omega}y^2\mathrm{d}x\mathrm{d}y\mathrm{d}z+0$$

$$=\int_{-b}^{b}y^2\mathrm{d}y\iint\limits_{D_y}\mathrm{d}x\mathrm{d}z=\int_{-b}^{b}y^2\cdot\pi\left(a\sqrt{1-\frac{y^2}{b^2}}\right)\left(c\sqrt{1-\frac{y^2}{b^2}}\right)\mathrm{d}y$$

$$=\frac{\pi ac}{b^2}\int_{-b}^{b}\left(b^2y^2-y^4\right)\mathrm{d}y=\frac{2\pi ac}{b^2}\int_0^{b}\left(b^2y^2-y^4\right)\mathrm{d}y$$

$$=\frac{2\pi ac}{b^2}\left(\frac{b^3}{3}-\frac{b^5}{5}\right)=\frac{4}{15}\pi ab^3c$$

(4) $\iiint\limits_{\Omega}y[1+xf(z)]\mathrm{d}V$，其中 Ω 是由不等式组 $-1\le x\le 1,x^3\le y\le 1,0\le z\le x^2+y^2$ 所限定的闭区域，$f(z)$ 为任一连续函数.

解 用柱面 $y=-x^3$ 将 Ω 分成 Ω_1 和 Ω_2 两部分：

$$\Omega_1=\left\{(x,y,z)\,\middle|\,0\le y\le 1,-\sqrt[3]{y}\le x\le\sqrt[3]{y},0\le z\le x^2+y^2\right\}$$

$$\Omega_2=\left\{(x,y,z)\,\middle|\,-1\le x\le 0,x^3\le y\le -x^3,0\le z\le x^2+y^2\right\}$$

由对称性知

$$\iiint_{\Omega} xyf(z)\mathrm{d}x\mathrm{d}y\mathrm{d}z = \iiint_{\Omega_1} xyf(z)\mathrm{d}x\mathrm{d}y\mathrm{d}z + \iiint_{\Omega_2} xyf(z)\mathrm{d}x\mathrm{d}y\mathrm{d}z = 0$$

于是

$$\iiint_{\Omega} y[1+xf(z)]\mathrm{d}x\mathrm{d}y\mathrm{d}z = \iiint_{\Omega} y\,\mathrm{d}x\mathrm{d}y\mathrm{d}z + \iiint_{\Omega} xyf(z)\mathrm{d}x\mathrm{d}y\mathrm{d}z$$

$$= \iiint_{\Omega} y\,\mathrm{d}x\mathrm{d}y\mathrm{d}z = \int_{-1}^{1}\mathrm{d}x\int_{x^2}^{1}\mathrm{d}y\int_{0}^{x^2+y^2} y\,\mathrm{d}z = \int_{-1}^{1}\mathrm{d}x\int_{x^2}^{1}\left(yx^2+y^3\right)\mathrm{d}y$$

$$= \int_{-1}^{1}\left(\frac{1}{2}+x^2-x^8-\frac{1}{2}x^{12}\right)\mathrm{d}x = \frac{80}{117}$$

2. 利用柱坐标计算下列三重积分.

(1) $\iiint_{\Omega}\frac{1}{1+x^2+y^2}\mathrm{d}x\mathrm{d}y\mathrm{d}z$，其中$\Omega$是由锥面$x^2+y^2=z^2$及平面$z=1$所围成的闭区域；

解 在柱坐标系下，积分区域为

$$\Omega = \left\{(r,\theta,z)\,\middle|\,0\le\theta\le 2\pi, 0\le r\le 1,\ r\le z\le 1\right\}$$

所以

$$\iiint_{\Omega}\frac{1}{1+x^2+y^2}\mathrm{d}x\mathrm{d}y\mathrm{d}z = \iiint_{\Omega}\frac{1}{1+r^2}\cdot r\,\mathrm{d}r\mathrm{d}\theta\mathrm{d}z = \int_0^{2\pi}\mathrm{d}\theta\int_0^1\mathrm{d}r\int_r^1\frac{r}{1+r^2}\mathrm{d}z$$

$$= 2\pi\int_0^1\frac{r(1-r)}{1+r^2}\mathrm{d}r = 2\pi\left[\int_0^1\frac{r}{1+r^2}\mathrm{d}r - \int_0^1\left(1-\frac{1}{1+r^2}\right)\mathrm{d}r\right] = \pi\left(\ln 2 - 2 + \frac{\pi}{2}\right)$$

(2) $\iiint_{\Omega}\left(x^2+y^2\right)\mathrm{d}x\mathrm{d}y\mathrm{d}z$，其中$\Omega$是旋转抛物面$2z=x^2+y^2$与平面$z=2, z=8$所围成的闭区域.

解 在柱坐标系下，积分区域为

$$\Omega = \left\{(r,\theta,z)\,\middle|\,2\le z\le 8, 0\le\theta\le 2\pi, 0\le r\le\sqrt{2z}\right\}$$

所以

$$\iiint_{\Omega}\left(x^2+y^2\right)\mathrm{d}x\mathrm{d}y\mathrm{d}z = \iiint_{\Omega} r^2\cdot r\mathrm{d}r\mathrm{d}\theta\mathrm{d}z = \int_2^8\mathrm{d}z\int_0^{2\pi}\mathrm{d}\theta\int_0^{\sqrt{2z}} r^3\mathrm{d}r$$

$$= 2\pi\int_2^8\frac{1}{4}\left(\sqrt{2z}\right)^4\mathrm{d}z = 2\pi\int_2^8 z^2\mathrm{d}z = 2\pi\left.\frac{z^3}{3}\right|_2^8 = 2\pi\cdot\frac{1}{3}\left(8^3-2^3\right) = 336\pi$$

3. 利用球坐标计算下列三重积分.

(1) $\iiint_{\Omega}(x+z)\mathrm{d}x\mathrm{d}y\mathrm{d}z$，其中$\Omega$是由锥面$z=\sqrt{x^2+y^2}$及球面$z=\sqrt{1-x^2-y^2}$所围成的闭区域；

解 在球坐标系下，积分区域为

$$\Omega=\left\{(\rho,\varphi,\theta)\middle|0\le\theta\le 2\pi,0\le\varphi\le\frac{\pi}{4},0\le\rho\le 1\right\}$$

所以

$$\iiint_\Omega (x+z)\mathrm{d}x\mathrm{d}y\mathrm{d}z=\iiint_\Omega x\,\mathrm{d}x\mathrm{d}y\mathrm{d}z+\iiint_\Omega z\,\mathrm{d}x\mathrm{d}y\mathrm{d}z$$

$$=0+\iiint_\Omega \rho\cos\varphi\cdot\rho^2\sin\varphi\, d\rho\mathrm{d}\varphi\mathrm{d}\theta=\int_0^{2\pi}\mathrm{d}\theta\int_0^{\frac{\pi}{4}}\mathrm{d}\varphi\int_0^1\rho^3\cos\varphi\sin\varphi\,\mathrm{d}\rho$$

$$=2\pi\int_0^{\frac{\pi}{4}}\frac{\rho^4}{4}\cos\varphi\sin\varphi\Big|_{\rho=0}^{\rho=1}\mathrm{d}\varphi=\frac{\pi}{2}\int_0^{\frac{\pi}{4}}\cos\varphi\sin\varphi\,\mathrm{d}\varphi=\frac{\pi}{4}\sin^2\varphi\Big|_0^{\frac{\pi}{4}}=\frac{\pi}{8}$$

(2) $\iiint_\Omega \frac{x^2+y^2}{z^2}\mathrm{d}V$，其中Ω是由不等式组 $x^2+y^2+z^2\ge 1$，$x^2+y^2+(z-1)^2\le 1$所确定的闭区域；

解 在球坐标系下，积分区域为

$$\Omega=\left\{(\rho,\varphi,\theta)\middle|0\le\theta\le 2\pi,0\le\varphi\le\frac{\pi}{3},1\le\rho\le 2\cos\varphi\right\}$$

所以

$$\iiint_\Omega\frac{x^2+y^2}{z^2}\mathrm{d}V=\iiint_\Omega\frac{\rho^2\sin^2\varphi}{\rho^2\cos^2\varphi}\cdot\rho^2\sin\varphi\,\mathrm{d}\rho\mathrm{d}\varphi\mathrm{d}\theta$$

$$=\int_0^{2\pi}\mathrm{d}\theta\int_0^{\frac{\pi}{3}}\mathrm{d}\varphi\int_1^{2\cos\varphi}\frac{\sin^3\varphi}{\cos^2\varphi}\rho^2\mathrm{d}\rho=2\pi\int_0^{\frac{\pi}{3}}\frac{\sin^3\varphi}{\cos^2\varphi}\frac{1}{3}\left[(2\cos\varphi)^2-1\right]\mathrm{d}\varphi$$

$$=\frac{2\pi}{3}\left[\int_0^{\frac{\pi}{3}}8\sin^3\varphi\cos\varphi\,\mathrm{d}\varphi-\int_0^{\frac{\pi}{3}}\frac{1-\cos^2\varphi}{\cos^2\varphi}\sin\varphi\,\mathrm{d}\varphi\right]$$

$$=\frac{2\pi}{3}\left[2\sin^4\varphi\Big|_0^{\frac{\pi}{3}}+\left(-\frac{1}{\cos\varphi}-\cos\varphi\right)\Big|_0^{\frac{\pi}{3}}=\frac{2\pi}{3}\left[\frac{9}{8}+\left(-2-\frac{1}{2}+2\right)\right]=\frac{5}{12}\pi\right.$$

(3) $\iiint_\Omega\left(x^3y-3xy^2+3xy\right)\mathrm{d}x\mathrm{d}y\mathrm{d}z$，其中Ω是球体 $(x-1)^2+(y-1)^2+(z-2)^2\le 1$.

解 令 $X=x-1,Y=y-1,Z=z-2$，则 $\Omega=\left\{(X,Y,Z)\middle|X^2+Y^2+Z^2\le 1\right\}$ ，所以

$$\iiint_{\Omega}\left(x^3y-3xy^2+3xy\right)\mathrm{d}x\mathrm{d}y\mathrm{d}z$$
$$=\iiint_{\Omega}\left[(X+1)^3(Y+1)-3(X+1)(Y+1)^2+3(X+1)(Y+1)\right]\mathrm{d}X\mathrm{d}Y\mathrm{d}Z$$
$$=\iiint_{\Omega}\left[(X+1)^3Y+X^3+3X-3X(Y+1)^2-6Y+3(XY+X+Y)\right]\mathrm{d}X\mathrm{d}Y\mathrm{d}Z$$
$$+\iiint_{\Omega}\left[3\left(X^2-Y^2\right)+1\right]\mathrm{d}X\mathrm{d}Y\mathrm{d}Z=0+\iiint_{\Omega}\mathrm{d}X\mathrm{d}Y\mathrm{d}Z=\frac{4\pi}{3}$$

4. 把积分 $\int_{-1}^{1}\mathrm{d}x\int_{0}^{\sqrt{1-x^2}}\mathrm{d}y\int_{1}^{1+\sqrt{1-x^2-y^2}}\frac{1}{\sqrt{x^2+y^2+z^2}}\mathrm{d}z$ 化成球坐标形式，并计算积分值.

解　积分区域为

$$\Omega=\left\{(x,y,z)\middle|-1\le x\le 1,0\le y\le\sqrt{1-x^2},1\le z\le 1+\sqrt{1-x^2-y^2}\right\}$$
$$=\left\{(\rho,\varphi,\theta)\middle|0\le\theta\le\pi,0\le\varphi\le\frac{\pi}{4},\frac{1}{\cos\varphi}\le\rho\le 2\cos\varphi\right\}$$

所以

$$\int_{-1}^{1}\mathrm{d}x\int_{0}^{\sqrt{1-x^2}}\mathrm{d}y\int_{1}^{1+\sqrt{1-x^2-y^2}}\frac{1}{\sqrt{x^2+y^2+z^2}}\mathrm{d}z=\iiint_{\Omega}\frac{1}{\sqrt{x^2+y^2+z^2}}\mathrm{d}x\mathrm{d}y\mathrm{d}z$$
$$=\iiint_{\Omega}\frac{1}{\rho}\cdot\rho^2\sin\varphi\mathrm{d}\rho\mathrm{d}\varphi d\theta=\int_0^{\pi}\mathrm{d}\theta\int_0^{\frac{\pi}{4}}\mathrm{d}\varphi\int_{\frac{1}{\cos\varphi}}^{2\cos\varphi}\rho\sin\varphi\,\mathrm{d}\rho$$
$$=\pi\int_0^{\frac{\pi}{4}}\frac{\rho^2}{2}\sin\varphi\Bigg|_{\rho=\frac{1}{\cos\varphi}}^{\rho=2\cos\varphi}\mathrm{d}\varphi=\frac{\pi}{2}\int_0^{\frac{\pi}{4}}\left(4\cos^2\varphi-\frac{1}{\cos^2\varphi}\right)\sin\varphi\,\mathrm{d}\varphi$$
$$=\frac{\pi}{2}\left(-\frac{1}{\cos\varphi}-\frac{4}{3}\cos^3\varphi\right)\Bigg|_0^{\frac{\pi}{4}}=\frac{\pi}{2}\left[\left(1+\frac{4}{3}\right)-\left(\sqrt{2}+\frac{\sqrt{2}}{3}\right)\right]=\frac{\pi}{3}\left(\frac{7}{2}-2\sqrt{2}\right)$$

5. 设有一物体，占有空间闭区域 $\Omega=\left\{(x,y,z)\middle|x^2+y^2\le 2x,0\le z\le 1\right\}$，在点 (x,y,z) 处的密度为 $\rho(x,y,z)=x^2+y^2+z^2$，计算该物体的质量.

解　该物体的质量为

$$m=\iiint_{\Omega}\left(x^2+y^2+z^2\right)\mathrm{d}x\mathrm{d}y\mathrm{d}z=\iiint_{\Omega}\left(r^2+z^2\right)\cdot r\mathrm{d}r\mathrm{d}\theta\mathrm{d}z$$

$$=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\mathrm{d}\theta\int_0^{2\cos\theta}\mathrm{d}r\int_0^1\left(r^2+z^2\right)r\mathrm{d}z=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\mathrm{d}\theta\int_0^{2\cos\theta}\left(r^3+\frac{1}{3}r\right)\mathrm{d}r$$

$$=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left(4\cos^4\theta+\frac{2}{3}\cos^2\theta\right)\mathrm{d}\theta=2\int_0^{\frac{\pi}{2}}\left(4\cos^4\theta+\frac{2}{3}\cos^2\theta\right)\mathrm{d}\theta$$

$$=8\int_0^{\frac{\pi}{2}}\cos^4\theta\,\mathrm{d}\theta+\frac{4}{3}\int_0^{\frac{\pi}{2}}\cos^2\theta\,\mathrm{d}\theta=8\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}+\frac{4}{3}\cdot\frac{1}{2}\cdot\frac{\pi}{2}=\frac{11}{6}\pi$$

10.4

1. 求锥面 $z=\sqrt{x^2+y^2}$ 被柱面 $z^2=2x$ 所割下部分的曲面面积.

解　曲面在 xOy 平面上的投影区域为

$$D=\left\{(x,y)\,\middle|\,x^2+y^2\le 2x\right\}$$

所求曲面的面积为

$$A=\iint_D\sqrt{1+\left(-\frac{x}{\sqrt{x^2+y^2}}\right)^2+\left(-\frac{y}{\sqrt{x^2+y^2}}\right)^2}\,\mathrm{d}x\mathrm{d}y$$

$$=\iint_D\sqrt{2}\,\mathrm{d}x\mathrm{d}y=\sqrt{2}\cdot\left(\pi\cdot 1^2\right)=\sqrt{2}\pi$$

2. 设平面薄片所占的闭区域 D 由抛物线 $y=x^2$ 及直线 $y=x$ 所围成，它在点处 (x,y) 的面密度 $\mu(x,y)=x^2y$，求该薄片的质心.

解　设薄片的质心坐标为 $(\bar{x},\bar{y})$，则

$$\bar{x}=\frac{\iint_D x\cdot x^2y\,\mathrm{d}\sigma}{\iint_D x^2y\,\mathrm{d}\sigma}=\frac{\iint_D x^3y\,\mathrm{d}\sigma}{\iint_D x^2y\,\mathrm{d}\sigma},\bar{y}=\frac{\iint_D y\cdot x^2y\,\mathrm{d}\sigma}{\iint_D x^2y\,\mathrm{d}\sigma}=\frac{\iint_D x^2y^2\,\mathrm{d}\sigma}{\iint_D x^2y\,\mathrm{d}\sigma}$$

其中

$$\iint_D x^2y\,\mathrm{d}\sigma=\int_0^1x^2\mathrm{d}x\int_{x^2}^x y\,\mathrm{d}y=\int_0^1\frac{1}{2}\left(x^4-x^6\right)\mathrm{d}x=\frac{1}{35}$$

$$\iint_D x^3y\mathrm{d}\sigma=\int_0^1x^3\mathrm{d}x\int_{x^2}^x y\mathrm{d}y=\int_0^1\frac{1}{2}\left(x^5-x^7\right)\mathrm{d}x=\frac{1}{48}$$

$$\iint_D x^2y^2\mathrm{d}\sigma=\int_0^1x^2\mathrm{d}x\int_{x^2}^x y^2\mathrm{d}y=\int_0^1\frac{1}{3}\left(x^5-x^8\right)\mathrm{d}x=\frac{1}{54}$$

所以

$$\bar{x}=\frac{\frac{1}{48}}{\frac{1}{35}}=\frac{35}{48},\bar{y}=\frac{\frac{1}{54}}{\frac{1}{35}}=\frac{35}{54}$$

故薄片的质心坐标为$\left(\frac{35}{48},\frac{35}{54}\right)$.

3. 设均匀薄片所占的闭区域D界于两个圆$r=a\cos\theta,r=b\cos\theta\left(0<a<b\right)$之间，求该薄片的质心.

解 设薄片的质心坐标为$\left(\bar{x},\bar{y}\right)$，由对称性知$\bar{y}=0$，又

$$\bar{x}=\frac{\iint\limits_{D}x\mathrm{d}\sigma}{\iint\limits_{D}\mathrm{d}\sigma}$$

其中

$$\iint\limits_{D}x\mathrm{d}\sigma=\iint\limits_{D}r\cos\theta\cdot r\mathrm{d}r\mathrm{d}\theta=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos\theta\,\mathrm{d}\theta\int_{a\cos\theta}^{b\cos\theta}r^{2}\mathrm{d}r$$

$$=\frac{2}{3}\left(b^{3}-a^{3}\right)\int_{0}^{\frac{\pi}{2}}\cos^{4}\theta\,\mathrm{d}\theta=\frac{2}{3}\left(b^{3}-a^{3}\right)\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}=\frac{\pi}{8}\left(b^{3}-a^{3}\right)$$

所以

$$\bar{x}=\frac{\frac{\pi}{8}\left(b^{3}-a^{3}\right)}{\frac{\pi}{4}\left(b^{2}-a^{2}\right)}=\frac{a^{2}+ab+b^{2}}{2\left(a+b\right)}$$

故薄片质心坐标为$\left(\frac{a^{2}+ab+b^{2}}{2\left(a+b\right)},0\right)$.

4. 设均匀物体所占的闭区域Ω由抛物面$y=\sqrt{x}$，$y=2\sqrt{x}$和平面$z=0,x+z=6$所围成，求该物体的质心.

解 设该物体的质心坐标为$\left(\bar{x},\bar{y},\bar{z}\right)$，则

$$\bar{x}=\frac{\iiint\limits_{\Omega}x\,\mathrm{d}x\mathrm{d}y\mathrm{d}z}{\iiint\limits_{\Omega}\mathrm{d}x\mathrm{d}y\mathrm{d}z},\bar{y}=\frac{\iiint\limits_{\Omega}y\,\mathrm{d}x\mathrm{d}y\mathrm{d}z}{\iiint\limits_{\Omega}\mathrm{d}x\mathrm{d}y\mathrm{d}z},\bar{z}=\frac{\iiint\limits_{\Omega}z\,\mathrm{d}x\mathrm{d}y\mathrm{d}z}{\iiint\limits_{\Omega}\mathrm{d}x\mathrm{d}y\mathrm{d}z}$$

其中

$$\iiint\limits_{\Omega} \mathrm{d}x\mathrm{d}y\mathrm{d}z = \int_0^6 \mathrm{d}x\int_{\sqrt{x}}^{2\sqrt{x}} \mathrm{d}y\int_0^{6-x} \mathrm{d}z = \int_0^6 (6-x)\sqrt{x}\,\mathrm{d}x = \frac{48}{5}\sqrt{6}$$

$$\iiint\limits_{\Omega} x\,\mathrm{d}x\mathrm{d}y\mathrm{d}z = \int_0^6 \mathrm{d}x\int_{\sqrt{x}}^{2\sqrt{x}} \mathrm{d}y\int_0^{6-x} x\,\mathrm{d}z = \int_0^6 x(6-x)\sqrt{x}\mathrm{d}x = \frac{864}{35}\sqrt{6}$$

$$\iiint\limits_{\Omega} y\,\mathrm{d}x\mathrm{d}y\mathrm{d}z = \int_0^6 \mathrm{d}x\int_{\sqrt{x}}^{2\sqrt{x}} \mathrm{d}y\int_0^{6-x} y\,\mathrm{d}z = \int_0^6 \frac{3x}{2}(6-x)\sqrt{x}\,\mathrm{d}x = 54$$

$$\iiint\limits_{\Omega} z\,\mathrm{d}x\mathrm{d}y\mathrm{d}z = \int_0^6 \mathrm{d}x\int_{\sqrt{x}}^{2\sqrt{x}} \mathrm{d}y\int_0^{6-x} z\,\mathrm{d}z = \int_0^6 \frac{1}{2}(6-x)^2\sqrt{x}\,\mathrm{d}x = \frac{576}{35}\sqrt{6}$$

所以

$$\bar{x} = \frac{\frac{864}{35}\sqrt{6}}{\frac{48}{5}\sqrt{6}} = \frac{18}{7}, \bar{y} = \frac{54}{\frac{48}{5}\sqrt{6}} = \frac{15}{16}\sqrt{6}, \bar{z} = \frac{\frac{576}{35}\sqrt{6}}{\frac{48}{5}\sqrt{6}} = \frac{12}{7}$$

故物体的质心坐标为$\left(\frac{18}{7},\frac{15}{16}\sqrt{6},\frac{12}{7}\right)$.

5．设一球占有闭区域$\Omega = \left\{(x,y,z)\middle| x^2+y^2+z^2 \le 2Rz\right\}$，它在内部各点处的密度的大小等于该点到坐标原点的距离的平方，试求该球的质心.

解　设球体的质心坐标为$(\bar{x},\bar{y},\bar{z})$，密度函数为$\rho(x,y,z) = x^2+y^2+z^2$，由对称性知$\bar{x} = \bar{y} = 0$，又

$$\bar{z} = \frac{\iiint\limits_{\Omega} z(x^2+y^2+z^2)\mathrm{d}V}{\iiint\limits_{\Omega}(x^2+y^2+z^2)\mathrm{d}V}$$

其中

$$\iiint\limits_{\Omega}(x^2+y^2+z^2)\mathrm{d}V = \iiint\limits_{\Omega}\rho^2\cdot\rho^2\sin\varphi\,\mathrm{d}\rho\mathrm{d}\varphi d\theta$$

$$= \int_0^{2\pi}\mathrm{d}\theta\int_0^{\frac{\pi}{2}}\mathrm{d}\varphi\int_0^{2R\cos\varphi}\rho^4\sin\varphi\,\mathrm{d}\rho = 2\pi\int_0^{\frac{\pi}{2}}\frac{32}{5}R^5\cos^5\varphi\sin\varphi\,\mathrm{d}\varphi = \frac{32}{15}\pi R^5$$

$$\iiint\limits_{\Omega} z(x^2+y^2+z^2)\mathrm{d}V = \iiint\limits_{\Omega}\rho\cos\varphi\cdot\rho^2\cdot\rho^2\sin\varphi\,\mathrm{d}\rho\mathrm{d}\varphi\mathrm{d}\theta$$

$$= \int_0^{2\pi}\mathrm{d}\theta\int_0^{\frac{\pi}{2}}\mathrm{d}\varphi\int_0^{2R\cos\varphi}\rho^5\sin\varphi\cos\varphi\,\mathrm{d}\rho = \frac{8}{3}\pi R^6$$

所以

$$\bar{z}=\frac{\frac{8}{3}\pi R^6}{\frac{32}{15}\pi R^5}=\frac{5}{4}R$$

故物体的质心坐标为$\left(0,0,\frac{5}{4}R\right)$.

5．设均匀薄片（面密度为常数1）占有闭区域$D=\left\{(x,y)\left|\frac{x^2}{a^2}+\frac{y^2}{b^2}\le 1\right.\right\}$，求该薄片关于$y$轴的转动惯量.

解 薄片关于y轴的转动惯量为

$$I_y=\iint_D x^2\,\mathrm{d}x\mathrm{d}y=\int_{-a}^{a}x^2\mathrm{d}x\int_{-\frac{b}{a}\sqrt{1-x^2}}^{\frac{b}{a}\sqrt{1-x^2}}\mathrm{d}y=\frac{2b}{a}\int_{-a}^{a}x^2\sqrt{a^2-x^2}\,\mathrm{d}x$$

$$=\frac{4b}{a}\int_0^a x^2\sqrt{a^2-x^2}\,\mathrm{d}x\overset{x=a\sin t}{=}\frac{4b}{a}\int_0^{\frac{\pi}{2}}(a\sin t)^2(a\cos t)\cdot(a\cos t)\mathrm{d}t$$

$$=4a^3b\left[\int_0^{\frac{\pi}{2}}\sin^2 t\,\mathrm{d}t-\int_0^{\frac{\pi}{2}}\sin^4 t\,\mathrm{d}t\right]=4a^3b\left[\frac{1}{2}\cdot\frac{\pi}{2}-\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}\right]=\frac{1}{4}\pi a^3b$$

7．设一均匀物体（密度ρ为常数）占有的闭区域Ω由曲面$z=x^2+y^2$和平面$z=0,|x|=a,|y|=a$所围成.

(1) 求物体的体积；
(2) 求物体的质心；
(3) 求物体关于z轴的转动惯量.

解 (1) 物体的体积为

$$V=\iiint_\Omega \mathrm{d}V=4\int_0^a\mathrm{d}x\int_0^a\mathrm{d}y\int_0^{x^2+y^2}\mathrm{d}z=4\int_0^a\mathrm{d}x\int_0^a\left(x^2+y^2\right)\mathrm{d}y$$

$$=4\int_0^a\left(ax^2+\frac{a^3}{3}\right)\mathrm{d}x=\frac{8}{3}a^4$$

(2) 设物体的质心坐标为$(\bar{x},\bar{y},\bar{z})$，则由对称性知$\bar{x}=\bar{y}=0$，又

$$\bar{z}=\frac{\iiint_\Omega \rho z\,\mathrm{d}V}{\iiint_\Omega \rho\,\mathrm{d}V}=\frac{\iiint_\Omega z\,\mathrm{d}V}{\iiint_\Omega \mathrm{d}V}$$

其中

$$\iiint\limits_{\Omega} z\,\mathrm{d}V = 4\int_0^a \mathrm{d}x\int_0^a \mathrm{d}y\int_0^{x^2+y^2} z\,\mathrm{d}z = 4\int_0^a \mathrm{d}x\int_0^a \frac{1}{2}\left(x^4+2x^2y^2+y^4\right)\mathrm{d}y$$

$$= 2\int_0^a \left(ax^4+\frac{2}{3}a^3x^2+\frac{1}{5}a^5\right)dx = \frac{56}{45}a^6$$

所以

$$\bar{z} = \frac{\frac{56}{45}a^6}{\frac{8}{3}a^4} = \frac{7}{15}a^2$$

故物体的质心坐标为$\left(0,0,\frac{7}{15}a^2\right)$.

(3) 物体关于z轴的转动惯量为

$$I_z = \iiint\limits_{\Omega} \rho\left(x^2+y^2\right)\mathrm{d}V = 4\rho\int_0^a \mathrm{d}x\int_0^a \mathrm{d}y\int_0^{x^2+y^2}\left(x^2+y^2\right)\mathrm{d}z$$

$$= 4\rho\int_0^a \mathrm{d}x\int_0^a \left(x^4+2x^2y^2+y^4\right)\mathrm{d}y = \frac{112}{45}\rho a^4$$

总习题十

1. 设有空间闭区域 $\Omega_1 = \left\{(x,y,z)\,\middle|\, x^2+y^2+z^2 \le R^2,\ z \ge 0\right\}$，$\Omega_2 = \left\{(x,y,z)\,\middle|\, x^2+y^2+z^2 \le R^2,\ x \ge 0, y \ge 0, z \ge 0\right\}$，则有（　）

（A）$\iiint\limits_{\Omega_1} x\,\mathrm{d}V = 4\iiint\limits_{\Omega_2} x\,\mathrm{d}V$　　（B）$\iiint\limits_{\Omega_1} y\,\mathrm{d}V = 4\iiint\limits_{\Omega_2} y\,\mathrm{d}V$

（C）$\iiint\limits_{\Omega_1} z\,\mathrm{d}V = 4\iiint\limits_{\Omega_2} z\,\mathrm{d}V$　　（D）$\iiint\limits_{\Omega_1} xyz\,\mathrm{d}V = 4\iiint\limits_{\Omega_2} xyz\,\mathrm{d}V$

解 由对称性知

$$\iiint\limits_{\Omega_1} x\,\mathrm{d}V = \iiint\limits_{\Omega_1} y\,\mathrm{d}V = \iiint\limits_{\Omega_1} xyz\,\mathrm{d}V = 0$$

$$\iiint\limits_{\Omega_1} z\,\mathrm{d}V = 4\iiint\limits_{\Omega_2} z\,\mathrm{d}V$$

所以选(C).

2. 设$f(x)$为连续函数，$F(t) = \int_1^t \mathrm{d}y\int_y^t f(x)\mathrm{d}x$，则$F'(2) =$（　）

（A）$2f(2)$　　（B）$f(2)$　　（C）$-f(2)$　　（D）0

解 $F(t)=\int_1^t \mathrm{d}y\int_y^t f(x)\mathrm{d}x=\int_1^t \mathrm{d}x\int_1^x f(x)\mathrm{d}y=\int_1^t f(x)(x-1)\mathrm{d}x$

所以

$$F'(t)=f(t)(t-1),F'(2)=f(2)(2-1)=f(2)$$

故选(B).

3. 设 $f(x,y)$ 在闭区域 $D=\left\{(x,y)\,\middle|\,x^2+y^2\le y,x\ge 0\right\}$ 上连续，且 $f(x,y)=\sqrt{1-x^2-y^2}-\dfrac{8}{\pi}\iint\limits_D f(x,y)\mathrm{d}x\mathrm{d}y$. 求 $f(x,y)$.

解 设 $A=\iint\limits_D f(x,y)\mathrm{d}x\mathrm{d}y$，则 $f(x,y)=\sqrt{1-x^2-y^2}-\dfrac{8}{\pi}A$，所以

$$A=\iint\limits_D\left(\sqrt{1-x^2-y^2}-\frac{8}{\pi}A\right)\mathrm{d}x\mathrm{d}y=\iint\limits_D\sqrt{1-x^2-y^2}\mathrm{d}x\mathrm{d}y-\frac{8}{\pi}A\cdot\frac{\pi}{8}$$

于是

$$\begin{aligned}A&=\frac{1}{2}\iint\limits_D\sqrt{1-x^2-y^2}\mathrm{d}x\mathrm{d}y=\frac{1}{2}\iint\limits_D\sqrt{1-r^2}\cdot r\mathrm{d}r\mathrm{d}\theta\\&=\frac{1}{2}\int_0^{\frac{\pi}{2}}\mathrm{d}\theta\int_0^{\sin\theta}\sqrt{1-r^2}r\mathrm{d}r=\frac{1}{2}\int_0^{\frac{\pi}{2}}-\frac{1}{3}\left(1-r^2\right)^{\frac{3}{2}}\Big|_{r=0}^{r=\sin\theta}\mathrm{d}\theta\\&=\frac{1}{6}\int_0^{\frac{\pi}{2}}\left(1-\cos^2\theta\right)\mathrm{d}\theta=\frac{1}{6}\cdot\left(\frac{\pi}{2}-\frac{2}{3}\right)=\frac{\pi}{12}-\frac{1}{9}\end{aligned}$$

故

$$f(x,y)=\sqrt{1-x^2-y^2}+\frac{8}{9\pi}-\frac{2}{3}$$

4. 设 $f(x)$ 连续，且 $F(t)=\iiint\limits_\Omega\left[z^2+f\left(x^2+y^2\right)\right]\mathrm{d}V$，其中 Ω 由不等式组 $0\le z\le h,x^2+y^2\le t^2$ 所确定，求 $\dfrac{\mathrm{d}F}{\mathrm{d}t}$.

解

$$\begin{aligned}F(t)&=\iiint\limits_\Omega\left[z^2+f\left(x^2+y^2\right)\right]\mathrm{d}x\mathrm{d}y\mathrm{d}z=\iiint\limits_\Omega\left[z^2+f\left(r^2\right)\right]r\mathrm{d}r\mathrm{d}\theta\mathrm{d}z\\&=\int_0^{2\pi}\mathrm{d}\theta\int_0^t\mathrm{d}r\int_0^h\left[z^2+f\left(r^2\right)\right]r\mathrm{d}z=2\pi\int_0^t\left[\frac{z^3}{3}+f\left(r^2\right)z\right]r\Big|_{z=0}^{z=h}\mathrm{d}r\\&=2\pi h\int_0^t\left(\frac{h^3}{3}+f\left(r^2\right)\right)rdr\end{aligned}$$

求导得

$$F'(t)=2\pi ht\left(\frac{h^3}{3}+f\left(t^2\right)\right)$$

5. 有一融化过程中的雪堆，高 $h=h(t)$（t 为时间），侧面方程为 $z=h(t)-\frac{2\left(x^2+y^2\right)}{h(t)}$（长度单位为cm，时间单位为h）. 已知体积减小的速率与侧面积成正比（比例系数为0.9）. 问原高 $h(0)=130\text{cm}$ 的这个雪堆全部融化需要多少小时？

解 雪堆的体积为

$$V=\iiint_{\Omega}\mathrm{d}V=\int_0^{h(t)}\mathrm{d}z\iint_{x^2+y^2\le\frac{1}{2}\left[h^2(t)-h(t)z\right]}\mathrm{d}x\mathrm{d}y=\frac{\pi}{2}\int_0^{h(t)}\left[h^2(t)-h(t)z\right]\mathrm{d}z=\frac{\pi}{4}h^3(t)$$

雪堆的侧面积为

$$A=\iint_{x^2+y^2\le\frac{h^2(t)}{2}}\sqrt{1+\left(-\frac{4x}{h(t)}\right)^2+\left(-\frac{4y}{h(t)}\right)^2}\mathrm{d}x\mathrm{d}y=\iint_{x^2+y^2\le\frac{h^2(t)}{2}}\sqrt{1+\frac{16\left(x^2+y^2\right)}{h^2(t)}}\mathrm{d}x\mathrm{d}y$$

$$=\frac{1}{h(t)}\int_0^{2\pi}\mathrm{d}\theta\int_0^{\frac{h(t)}{\sqrt{2}}}\sqrt{h^2(t)+16r^2}\,r\mathrm{d}r=\frac{\pi}{24h(t)}\left[h^2(t)+16r^2\right]^{\frac{3}{2}}\Bigg|_{r=0}^{r=\frac{h(t)}{\sqrt{2}}}=\frac{13}{12}\pi h^2(t)$$

又由于 $\frac{\mathrm{d}V}{\mathrm{d}t}=-0.9A$，所以

$$\frac{\pi}{4}\cdot 3h^2(t)\frac{\mathrm{d}h}{\mathrm{d}t}=(0.9)\left(\frac{13}{12}\pi h^2(t)\right)$$

即

$$\frac{\mathrm{d}h}{\mathrm{d}t}=-\frac{13}{10}$$

解得 $h(t)=-\frac{13}{10}t+C$，由 $h(0)=130$ 得 $C=130$，所以

$$h(t)=130-\frac{13}{10}t$$

令 $h(t)=0$ 得 $t=100$ 小时.

6. 在均匀的半径为 R 的半圆形薄片的直径上，要接上一个一边与直径等长的同样材料的均匀矩形薄片，为了使整个均匀薄片的质心恰好落在圆心上，问接上去的均匀矩形薄片另一边的长度应是多少？

解 选取直角坐标系使整个均匀薄片占有区域

$$D=\left\{(x,y)\,\Big|\,-R\le x\le R,-l\le y\le\sqrt{R^2-x^2}\right\}$$

其中 l 是矩形薄片所求边长，设整个薄片的质心坐标为 $(\bar{x},\bar{y})$，由对称性知 $\bar{x}=0$，又

$$\bar{y}=\frac{\iint\limits_D y\,\mathrm{d}\sigma}{\iint\limits_D \mathrm{d}\sigma}$$

其中

$$\iint\limits_D y\,\mathrm{d}\sigma=\int_{-R}^{R}\mathrm{d}x\int_{-l}^{\sqrt{R^2-x^2}}y\,\mathrm{d}y=\frac{1}{2}\int_{-R}^{R}\left(R^2-x^2-l^2\right)\mathrm{d}x=\frac{2}{3}R^3-l^2R$$

令 $\bar{y}=0$ 得 $\frac{2}{3}R^3-l^2R=0$，解得 $l=\sqrt{\frac{2}{3}}R$.

7. 求由抛物线 $y=x^2$ 及直线 $y=1$ 所围成的均匀薄片（面密度为常数 μ）对于直线 $y=-1$ 的转动惯量.

解　薄片所占区域为

$$D=\left\{(x,y)\,\middle|\,-\sqrt{y}\le x\le\sqrt{y},0\le y\le 1\right\}$$

薄片对于直线 $y=-1$ 的转动惯量为

$$\begin{aligned}I_{y=-1}&=\iint\limits_D\mu(y+1)^2\,\mathrm{d}\sigma=\mu\int_0^1(y+1)^2\,\mathrm{d}y\int_{-\sqrt{y}}^{\sqrt{y}}\mathrm{d}x\\&=2\mu\int_0^1\sqrt{y}(y+1)^2\,\mathrm{d}y=2\mu\int_0^1\left(y^{\frac{5}{2}}+2y^{\frac{3}{2}}+y^{\frac{1}{2}}\right)\mathrm{d}y=\frac{368}{105}\mu\end{aligned}$$