2021春季学期高等数学B期中试题答案

一、选择题（每小题1分，共5小题，满分5分）

1. (D)；　2. (D)；　3. (A)；　4. (C)；　5. (B).

二、(3分) 求微分方程 $y''+5y'+4y=(3-2x)\mathrm{e}^{-x}$ 的通解.

解 特征方程为

$$r^2+5r+4=0$$

解得 $r_1=-1, r_2=-4$，对应的齐次线性微分方程的通解为

$$Y=C_1\mathrm{e}^{-x}+C_2\mathrm{e}^{-4x}$$

其中 C_1, C_2 为任意常数.

设微分方程的特解为

$$y^*=x(ax+b)\mathrm{e}^{-x}=(ax^2+bx)\mathrm{e}^{-x}$$

代入方程得

$$(ax^2-4ax+bx+2a-2b)\mathrm{e}^{-x}+5(-ax^2+2ax-bx+b)\mathrm{e}^{-x}+4(ax^2+bx)\mathrm{e}^{-x}=(3-2x)\mathrm{e}^{-x}$$

即

$$(6ax+2a+3b)\mathrm{e}^{-x}=(3-2x)\mathrm{e}^{-x}$$

比较同类项的系数得

$$\begin{cases}6a=-2\\2a+3b=3\end{cases}$$

解得 $a=-\dfrac{1}{3}, b=\dfrac{11}{9}$，所以

$$y^*=\left(-\frac{1}{3}x^2+\frac{11}{9}x\right)\mathrm{e}^{-x}$$

方程的通解为

$$y=C_1\mathrm{e}^{-x}+C_2\mathrm{e}^{-4x}+\left(-\frac{1}{3}x^2+\frac{11}{9}x\right)\mathrm{e}^{-x}$$

三、(4 分) 设函数 $z=f(2x+y,x-y,x\sin y)$，其中 f 具有连续的二阶偏导数，求 $\mathrm{d}z$ 和 $\dfrac{\partial^2 z}{\partial x\partial y}$.

解 一阶偏导数为

$$\frac{\partial z}{\partial x}=2f_1'+f_2'+\sin yf_3',\quad \frac{\partial z}{\partial y}=f_1'-f_2'+x\cos yf_3'$$

所以全微分为

$$\mathrm{d}z=\left(2f_1'+f_2'+\sin yf'\right)\mathrm{d}x+\left(f_1'-f_2'+x\cos yf_3'\right)\mathrm{d}y$$

二阶混合偏导数为

$$\begin{aligned}\frac{\partial^2 z}{\partial x\partial y}&=\frac{\partial}{\partial y}\left(2f_1'+f_2'+\sin yf_3'\right)\\&=2\left(f_{11}''-f_{12}''+x\cos yf_{13}''\right)+\left(f_{21}''-f_{22}''+x\cos yf_{23}''\right)+\cos yf_3'+\sin y\left(f_{31}''-f_{32}''+x\cos yf_{33}''\right)\\&=2f_{11}''-f_{12}''+\left(2x\cos y+\sin y\right)f_{13}''-f_{22}''+\left(x\cos y-\sin y\right)f_{23}''+x\sin y\cos yf_{33}''+\cos yf_3'\end{aligned}$$

四、(3 分) 已知 $y=f(x,y,z),z=g(x,y,z)$，其中 f,g 具有连续的偏导数，求 $\dfrac{\mathrm{d}z}{\mathrm{d}x}$.

解 （解法一） 方程组对 x 求导得

$$\begin{cases}\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{\partial f}{\partial x}+\dfrac{\partial f}{\partial y}\dfrac{\mathrm{d}y}{\mathrm{d}x}+\dfrac{\partial f}{\partial z}\dfrac{\mathrm{d}z}{\mathrm{d}x}\\ \dfrac{\mathrm{d}z}{\mathrm{d}x}=\dfrac{\partial g}{\partial x}+\dfrac{\partial g}{\partial y}\dfrac{\mathrm{d}y}{\mathrm{d}x}+\dfrac{\partial g}{\partial z}\dfrac{\mathrm{d}z}{\mathrm{d}x}\end{cases}$$

解得

$$\frac{\mathrm{d}z}{\mathrm{d}x}=\frac{\dfrac{\partial g}{\partial x}-\dfrac{\partial f}{\partial y}\dfrac{\partial g}{\partial x}+\dfrac{\partial f}{\partial x}\dfrac{\partial g}{\partial y}}{1-\dfrac{\partial f}{\partial y}-\dfrac{\partial g}{\partial z}+\dfrac{\partial f}{\partial y}\dfrac{\partial g}{\partial z}-\dfrac{\partial f}{\partial z}\dfrac{\partial g}{\partial y}}$$

（解法二） 设 $F(x,y,z)=f(x,y,z)-y,G(x,y,z)=g(x,y,z)-z$，则

$$\frac{\mathrm{d}z}{\mathrm{d}x}=-\frac{\dfrac{\partial(F,G)}{\partial(y,x)}}{\dfrac{\partial(F,G)}{\partial(y,z)}}=-\frac{\begin{vmatrix}\dfrac{\partial f}{\partial y}-1 & \dfrac{\partial f}{\partial x}\\ \dfrac{\partial g}{\partial y} & \dfrac{\partial g}{\partial x}\end{vmatrix}}{\begin{vmatrix}\dfrac{\partial f}{\partial y}-1 & \dfrac{\partial f}{\partial z}\\ \dfrac{\partial g}{\partial y} & \dfrac{\partial g}{\partial z}-1\end{vmatrix}}=\frac{\dfrac{\partial g}{\partial x}-\dfrac{\partial f}{\partial y}\dfrac{\partial g}{\partial x}+\dfrac{\partial f}{\partial x}\dfrac{\partial g}{\partial y}}{1-\dfrac{\partial f}{\partial y}-\dfrac{\partial g}{\partial z}+\dfrac{\partial f}{\partial y}\dfrac{\partial g}{\partial z}-\dfrac{\partial f}{\partial z}\dfrac{\partial g}{\partial y}}$$

（解法三）对方程组取全微分得

$$\begin{cases} \mathrm{d}y = \dfrac{\partial f}{\partial x}\mathrm{d}x + \dfrac{\partial f}{\partial y}\mathrm{d}y + \dfrac{\partial f}{\partial z}\mathrm{d}z \\ \mathrm{d}z = \dfrac{\partial g}{\partial x}\mathrm{d}x + \dfrac{\partial g}{\partial y}\mathrm{d}y + \dfrac{\partial g}{\partial z}\mathrm{d}z \end{cases}$$

解出 $\mathrm{d}z$ 得

$$\mathrm{d}z = \frac{\dfrac{\partial g}{\partial x} - \dfrac{\partial f}{\partial y}\dfrac{\partial g}{\partial x} + \dfrac{\partial f}{\partial x}\dfrac{\partial g}{\partial y}}{1 - \dfrac{\partial f}{\partial y} - \dfrac{\partial g}{\partial z} + \dfrac{\partial f}{\partial y}\dfrac{\partial g}{\partial z} - \dfrac{\partial f}{\partial z}\dfrac{\partial g}{\partial y}}\mathrm{d}x$$

所以

$$\frac{\mathrm{d}z}{\mathrm{d}x} = \frac{\dfrac{\partial g}{\partial x} - \dfrac{\partial f}{\partial y}\dfrac{\partial g}{\partial x} + \dfrac{\partial f}{\partial x}\dfrac{\partial g}{\partial y}}{1 - \dfrac{\partial f}{\partial y} - \dfrac{\partial g}{\partial z} + \dfrac{\partial f}{\partial y}\dfrac{\partial g}{\partial z} - \dfrac{\partial f}{\partial z}\dfrac{\partial g}{\partial y}}$$

五、(3 分) 设山坡的高度为 $z = 5 - x^2 - 2y^2$，一个登山者在山坡上点 $\left(-\frac{3}{2}, -1, \frac{3}{4}\right)$ 处，在下列情形下该向什么方向 $\vec{l} = a\vec{i} + b\vec{j}$ 移动？（1）爬的最快 (即高度 z 增加的最快);（2）在同一水平线上;（3）以斜率 1 爬坡(即以倾角 45° 爬坡).

解 （1）$\mathbf{grad}\, z = \dfrac{\partial z}{\partial x}\vec{i} + \dfrac{\partial z}{\partial y}\vec{j} = -2x\vec{i} - 4y\vec{j}$

$$\mathbf{grad}\, z\Big|_{\left(-\frac{3}{2}, -1\right)} = \left(-2x\vec{i} - 4y\vec{j}\right)\Big|_{\left(-\frac{3}{2}, -1\right)} = 3\vec{i} + 4\vec{j}$$

所以爬的最快时的移动方向为 $\vec{l}_1 = 3\vec{i} + 4\vec{j}$.

（2）若在同一水平线上，则移动方向与梯度 $\mathbf{grad}\, z\Big|_{\left(-\frac{3}{2}, -1\right)} = 3\vec{i} + 4\vec{j}$ 方向垂直，该方向为

$$\vec{l}_2 = 4\vec{i} - 3\vec{j}, \quad \vec{l}_3 = -4\vec{i} + 3\vec{j}$$

（3）（解法一） $\vec{l}^{\,0} = \dfrac{\vec{l}}{|\vec{l}|} = \dfrac{a\vec{i} + b\vec{j}}{\sqrt{a^2 + b^2}}$

若以斜率 1 爬坡，则

$$\left.\frac{\partial z}{\partial \vec{l}}\right|_{\left(-\frac{3}{2},-1\right)}=\left.\mathbf{grad}\, z\right|_{\left(-\frac{3}{2},-1\right)}\cdot \vec{l}^{\,0}=\frac{3a+4b}{\sqrt{a^2+b^2}}=1$$

整理得

$$a^2+3ab+\frac{15}{8}b^2=0$$

令 $b=1$，则上式化为

$$a^2+3a+\frac{15}{8}=0$$

解得 $a=-\frac{3}{2}+\frac{\sqrt{6}}{4}$， $a=-\frac{3}{2}-\frac{\sqrt{6}}{4}$ （舍），令 $b=-1$，则上式化为

$$a^2-3a+\frac{15}{8}=0$$

解得 $a=\frac{3}{2}+\frac{\sqrt{6}}{4}$， $a=\frac{3}{2}-\frac{\sqrt{6}}{4}$ （舍），所以以斜率1爬坡时的移动方向为

$$\vec{l}_4=\left(-\frac{3}{2}+\frac{\sqrt{6}}{4}\right)\vec{i}+\vec{j},\quad \vec{l}_5=\left(\frac{3}{2}+\frac{\sqrt{6}}{4}\right)\vec{i}-\vec{j}$$

（解法二） $\vec{l}^{\,0}=a\vec{i}+b\vec{j}, a^2+b^2=1$

若以斜率1爬坡，则

$$\left.\frac{\partial z}{\partial \vec{l}}\right|_{\left(-\frac{3}{2},-1\right)}=\left.\mathbf{grad}\, z\right|_{\left(-\frac{3}{2},-1\right)}\cdot \vec{l}^{\,0}=3a+4b=1$$

联立方程

$$\begin{cases} a^2+b^2=1 \\ 3a+4b=1 \end{cases}$$

解得 $a=\frac{3\pm 8\sqrt{6}}{25}, b=\frac{4\mp 6\sqrt{6}}{25}$，所以以斜率1爬坡时的移动方向为

$$\vec{l}_4=\frac{3+8\sqrt{6}}{25}\vec{i}+\frac{4-6\sqrt{6}}{25}\vec{j},\quad \vec{l}_5=\frac{3-8\sqrt{6}}{25}\vec{i}+\frac{4+6\sqrt{6}}{25}\vec{j}$$

六、(3 分) 求函数 $f(x,y,z)=xy+2yz$ 在约束条件 $x^2+y^2+z^2=10$ 下的最大值和最小值.

解 设拉格朗日函数

$$L(x,y,z,\lambda)=xy+2yz+\lambda\left(x^2+y^2+z^2-10\right)$$

令

$$\begin{cases} \dfrac{\partial L}{\partial x} = y + 2\lambda x = 0 \\ \dfrac{\partial L}{\partial y} = x + 2z + 2\lambda y = 0 \\ \dfrac{\partial L}{\partial z} = 2y + 2\lambda z = 0 \\ \dfrac{\partial L}{\partial \lambda} = x^2 + y^2 + z^2 - 10 = 0 \end{cases}$$

由前三个方程解得 $y = \pm\sqrt{5}x, z = 2x$ 或 $y = 0, x = -2z$，将 $y = \pm\sqrt{5}x, z = 2x$ 代入第四个方程得

$$x^2 + 5x^2 + 4x^2 - 10 = 0$$

解得 $x = \pm 1$，所以 $y = \mp\sqrt{5}, z = \pm 2$，极值嫌疑点为 $(\pm 1, \pm\sqrt{5}, \pm 2), (\pm 1, \mp\sqrt{5}, \pm 2)$，且

$$f(\pm 1, \pm\sqrt{5}, \pm 2) = 5\sqrt{5},\ f(\pm 1, \mp\sqrt{5}, \pm 2) = -5\sqrt{5}$$

将 $y = 0, x = -2z$ 代入第四个方程得

$$4z^2 + z^2 - 10 = 0$$

解得 $z = \pm\sqrt{2}$，所以 $x = \mp 2\sqrt{2}, y = 0$，极值嫌疑点为 $(\mp 2\sqrt{2}, 0, \pm\sqrt{2})$，且

$$f(\mp 2\sqrt{2}, 0, \pm\sqrt{2}) = 0$$

比较之，最大值为 $f(\pm 1, \pm\sqrt{5}, \pm 2) = 5\sqrt{5}$，最小值为 $f(\pm 1, \mp\sqrt{5}, \pm 2) = -5\sqrt{5}$.

七、(3 分) 计算积分 $\int_1^2 \mathrm{d}x \int_{\sqrt{x}}^{x} \sin\frac{\pi x}{2y} \mathrm{d}y + \int_2^4 \mathrm{d}x \int_{\sqrt{x}}^{2} \sin\frac{\pi x}{2y} \mathrm{d}y$.

解 $\int_1^2 \mathrm{d}x \int_{\sqrt{x}}^{x} \sin\frac{\pi x}{2y} \mathrm{d}y + \int_2^4 \mathrm{d}x \int_{\sqrt{x}}^{2} \sin\frac{\pi x}{2y} \mathrm{d}y = \int_1^2 \mathrm{d}y \int_y^{y^2} \sin\frac{\pi x}{2y} \mathrm{d}x$

$$= \int_1^2 \mathrm{d}y \int_y^{y^2} \frac{2y}{\pi} \sin\frac{\pi x}{2y} \mathrm{d}\left(\frac{\pi x}{2y}\right) = \int_1^2 -\frac{2y}{\pi} \cos\frac{\pi x}{2y}\Big|_{x=y}^{x=y^2} \mathrm{d}y = \int_1^2 -\frac{2y}{\pi} \cos\frac{\pi y}{2} \mathrm{d}y$$

$$= -\frac{4}{\pi^2} \int_1^2 y \mathrm{d}\left(\sin\frac{\pi x}{2}\right) = -\frac{4}{\pi^2}\left[y \sin\frac{\pi x}{2}\Big|_1^2 - \int_1^2 \sin\frac{\pi y}{2} \mathrm{d}y \right] = -\frac{4}{\pi^2}\left(-1 + \frac{2}{\pi} \cos\frac{\pi y}{2}\Big|_1^2\right)$$

$$= -\frac{4}{\pi^2}\left(-1 - \frac{2}{\pi}\right) = \frac{4}{\pi^2} + \frac{8}{\pi^3}$$

八、(3 分) 求曲面 $S_1: z = x^2 + y^2 + 1$ 在点 $(1,-1,3)$ 处的切平面方程，并求该切平面与曲面 $S_2: z = x^2 + y^2$ 围成立体的体积.

解 曲面 S_1 在点 $(1,-1,3)$ 处的法向量为

$$\vec{n} = \left\{\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}, -1\right\}\Big|_{(1,-1,3)} = \{2x, 2y, -1\}\Big|_{(1,-1,3)} = \{2,-2,-1\}$$

所以切平面方程为

$$2(x-1) - 2(y+1) - (z-3) = 0$$

即

$$z = 2x - 2y - 1$$

记

$$D = \left\{(x,y) \middle| x^2 + y^2 \le 2x - 2y - 1\right\} = \left\{(x,y) \middle| (x-1)^2 + (y+1)^2 \le 1\right\}$$

则切平面与曲面 S_2 围成立体的体积为

$$V = \iint_D \left(2x - 2y - 1 - x^2 - y^2\right)\mathrm{d}x\mathrm{d}y = \iint_D \left[1 - (x-1)^2 - (y+1)^2\right]\mathrm{d}x\mathrm{d}y$$

令 $\begin{cases} x - 1 = r\cos\theta \\ y + 1 = r\sin\theta \end{cases}$，则

$$D = \left\{(r,\theta) \middle| 0 \le \theta \le 2\pi, 0 \le r \le 1\right\}$$

所以

$$V = \iint_D \left(1 - r^2\right) r\mathrm{d}r\mathrm{d}\theta = \int_0^{2\pi} \mathrm{d}\theta \int_0^1 \left(1 - r^2\right) r\mathrm{d}r = 2\pi \cdot \frac{1}{4} = \frac{\pi}{2}$$

九、(3 分) 设函数 $f(x,y) = |x + 2y|\varphi(x,y)$，其中 $\varphi(x,y)$ 在点 $(0,0)$ 处连续，且 $\varphi(0,0) = a$（a 为常数），讨论函数 $f(x,y)$ 在点 $(0,0)$ 处偏导数的存在性以及函数 $f(x,y)$ 在点 $(0,0)$ 处的可微性.

解 当 $a \ne 0$ 时，极限

$$\lim_{x\to 0} \frac{f(x,0) - f(0,0)}{x} = \lim_{x\to 0} \frac{|x|\varphi(x,0) - 0}{x} = a\lim_{x\to 0} \frac{|x|}{x}$$

$$\lim_{y\to 0} \frac{f(0,y) - f(0,0)}{y} = \lim_{y\to 0} \frac{2|y|\varphi(0,y) - 0}{y} = 2a\lim_{y\to 0} \frac{|y|}{y}$$

都不存在，所以函数 $f(x,y)$ 在点 $(0,0)$ 处的偏导数不存在，且微分不存在.

当$a=0$时，有

$$f_x'(0,0)=\lim_{x\to 0}\frac{f(x,0)-f(0,0)}{x}=\lim_{x\to 0}\frac{|x|\varphi(x,0)-0}{x}=0$$

$$f_y'(0,0)=\lim_{y\to 0}\frac{f(0,y)-f(0,0)}{y}=\lim_{y\to 0}\frac{2|y|\varphi(0,y)-0}{y}=0$$

所以函数$f(x,y)$在点$(0,0)$处的偏导数存在，且$f_x'(0,0)=0, f_y'(0,0)=0$.

又

$$\lim_{(x,y)\to(0,0)}\frac{f(x,y)-f(0,0)-0\cdot(x-0)-0\cdot(y-0)}{\sqrt{x^2+y^2}}=\lim_{(x,y)\to(0,0)}\frac{|x+2y|\varphi(x,y)}{\sqrt{x^2+y^2}}$$

$$\overset{x=r\cos\theta,y=r\sin\theta}{=}\lim_{r\to 0}\frac{|r\cos\theta+2r\sin\theta|\varphi(r\cos\theta,r\sin\theta)}{r}=\lim_{r\to 0}|\cos\theta+2\sin\theta|\varphi(r\cos\theta,r\sin\theta)=0$$

所以

$$f(x,y)-f(0,0)=0\cdot(x-0)-0\cdot(y-0)+o\left(\sqrt{x^2+y^2}\right)$$

故函数$f(x,y)$在点$(0,0)$处的可微，且$\mathrm{d}f\big|_{(0,0)}=0$.