Homework of Linear mapping

1. Let $F: R^2 \to R^2$ be the map defined by $F\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2x \\ 3y \end{bmatrix}$ for any $\begin{bmatrix} x \\ y \end{bmatrix} \in R^2$.

   Describe the image by F of the points lying on the unit circle centered at 0, i.e. $\left\{\begin{bmatrix} x \\ y \end{bmatrix} \in \mathrm{R}^2 \mid \mathrm{x}^2 + y^2 = 1\right\}$.

   Solution: Let $\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 2x \\ 3y \end{bmatrix}$,we have $x = \frac{u}{2}, y = \frac{v}{3}$. Putting them into the unit circle equation, we have $\frac{u^2}{4} + \frac{v^2}{9} = 1$. Thus, the image is $\{(u,v) \mid \frac{u^2}{4} + \frac{v^2}{9} = 1\}$.

2. Let $F: R^2 \to R^2$ be the map defined by $F\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} xy \\ y \end{bmatrix}$ for any $\begin{bmatrix} x \\ y \end{bmatrix} \in R^2$.

   Describe the image by F of the line $\left\{\begin{bmatrix} x \\ y \end{bmatrix} \in \mathrm{R}^2 \mid x = 2\right\}$.

Solution: Let $\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} xy \\ y \end{bmatrix}$, since $x = 2$, we have $\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 2y \\ y \end{bmatrix}$, thus the image is $\{(u,v) \mid u = 2v\}$

3. Let V be a linear space of dimension n, and let $\{v_1, v_2, \cdots, v_n\}$ be a basis for V. Let F be a linear map from V into itself. Show that F is uniquely defined if one knows $F(v_j)$ for $j \in \{1, 2, \cdots, n\}$. Is it also true if F is an arbitrary map from om V into itself?

Solution: Let F' is another linear mapping and $F'(v_i)=F(v_i)$ or $j\in\{1,2,\cdots,n\}$. Because $\{v_1,v_2,\cdots,v_n\}$ is a basis for V. We have

$x=a_1v_1+a_2v_2+\cdots+a_nv_n$, thus

$F(x)=F(a_1v_1+\cdots+a_nv_n)=\mathrm{a}_1F(v_1)+\cdots+a_nF(v_n)$.

and

$F'(x)=F'(a_1v_1+\cdots+a_nv_n)=\mathrm{a}_1F'(v_1)+\cdots+a_nF'(v_n)$

Since $F'(v_i)=F(v_i)$ for $j\in\{1,2,\cdots,n\}$, we have $F(x)=F'(x)$

Thus, $F(x)$ is uniquely difined.

When the F is an arbitrary map, the answer is false. For example , Consider a nonlinear mapping $F(x)=(c_1x_1+c_2x_2)^2$ defined on R^2. Let $e_1=(1,0),e_2=(0,1)$ be a basis of R^2. If $F(e_1)=1,F(e_2)=1$. we have $c_1^2=1$ and $c_2^2=1$, thus , c_1 and c_2 may be 1 or -1. Thus, the mapping $F(x)=(c_1x_1+c_2x_2)^2$ cannot be defined when $F(e_1)=1,F(e_2)=1$ are known.

4. Let V,W be two linear space over the same field, and let $T:V\to W$ be a linear mapping. Show that the following set is a subspace of V.

$$\{x\in V\mid T(x)=0\}$$

Prove: For any $x,y\in\{x\in V\mid T(x)=0\}$, $T(x+y)=T(x)+T(y)=0$

$T(kx)=kT(x)=0$.

The set is closed under the addition and scalar multiplication. Thus it is a subspace.

5. Let $\mathrm{T}:\mathrm{R}^{n\times n}\to R^{n\times n}$ be the map defined for any $n\times n$ dimensional matrix $A\in R^{n\times n}$ by

$$T(A)=\frac{1}{2}(A+A^T)$$

where A^T denotes the transpose of matrix A.

1) Show that T is a linear mapping.
2) Show that the kernel of T consists in the linear space of all skew-symmetric matrices.
3) Show that the range of T consists in the linear space of all symmetric matrices.
4) What is the dimension of the linear space of all symmetric matrices, and the dimension of the linear space of all skew-symmetric matrices?

Answer: (1) For $A,B\in R^{n\times n}$, $k\in R$

$$T(A+B)=\frac{1}{2}\left[(A+B)^T+(A+B)\right]=\frac{1}{2}\left[A^T+B^T+A+B\right]=\frac{1}{2}\left[A^T+A+B^T+B\right]$$
$$=T(A)+T(B)$$

$$T(kA)=\frac{1}{2}\left[(kA)^T+kA\right]=k\frac{1}{2}(A^T+A)=kT(A)$$

Therefore T is a linear map.

(2)The kernel is $\{A\in R^{n\times n}\,|\,T(A)=0\}$, since $T(A)=\frac{1}{2}(A^T+A)=0$, we have $A^T=-A$. Thus the kernel consists in the linear space of all skew-symmetric matrices.

(3)The range of T is $\left\{T(A)\,|\,A\in R^{n\times n}\right\}$, since $T(A)=\frac{1}{2}(A^T+A)$ and $\left[T(A)\right]^T=\left[\frac{1}{2}(A^T+A)\right]^T=\frac{1}{2}(A+A^T)=T(A)$, thus the range of T consists in the linear space of all symmetric matrices.

(4) Since $\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, ……… $\begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$\begin{bmatrix} 0 & 1 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, 。。。。。。 $\begin{bmatrix} 0 & & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ 。。。。。 $\begin{bmatrix} 0 & & \cdots & 1 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 1 & 0 & & 0 \end{bmatrix}$ is a basis of the linear space of all symmetric matrices, , so its dimension is n+(n-1)+⋯+1=$\frac{n(n+1)}{2}$。Similarly, the dimension of the linear space of all skew-symmetric matrices is (n-1)+(n-2)+⋯+1=$\frac{n(n-1)}{2}$。

6. Consider the mapping $F\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right)=\begin{bmatrix} x \\ x-y \\ x-z \\ x-y-z \end{bmatrix}$

1) Determine the kernel of F .

To determine the kernel, we need to solve the linear equation

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \\ 1 & -1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

By the RREF form, we have $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, only $\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ can satisfy this equation. Thus, the kernel is $\left\{\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\right\}$

2) Determine the range of F .

The range of F is the set $\left\{ \begin{bmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \\ 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \middle| \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in R^3 \right\}$. The range is the space spanned by the column vectors of matrix $\begin{bmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \\ 1 & -1 & -1 \end{bmatrix}$. The range is the space spanned by $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -1 \\ -1 \end{bmatrix} \right\}$. Since these vectors are linear independent, the dimension of range is 3.

7. Let $\mathrm{T}: R^3 \to R^3$ be a linear mapping which associated matrix has the form $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ with respect to the canonical basis of R^3 ($e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$). What is the matrix associated with T in the basis generated by the three vectors $v_1 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}, v_3 = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$.

Solution:

According to the equivalence of matrices, we know that the matrix associated with T in the basis generated by the three vectors v_1, v_2, v_3 is B, which is equivalent to the matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ by the following equation

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & -1 \end{bmatrix} \text{B}$$

Therefore,

$$B = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & -1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & -1 \end{bmatrix} =$$

$$\begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & -1 \end{bmatrix}^{T} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1/\sqrt{2} & \sqrt{2} & 0 \\ -1/\sqrt{2} & \sqrt{2} & 0 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 3/2 & 1/2 & 0 \\ 1/2 & 3/2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$