Homework 1

April 11, 2023

1. Let V be a vector space and let $\mathbf{x}, \mathbf{y} \in V$. Show that

   (a) $\beta\mathbf{0} = \mathbf{0}$ for each scalar β.
   Proof:
   Let x is a vector in V.
   $\beta\mathbf{0} = \beta(x - x) = \beta x - \beta x = \mathbf{0}$.

   (b) $\mathbf{x} + \mathbf{y} = \mathbf{0}$ implies that $\mathbf{y} = -\mathbf{x}$.
   Proof:
   $\because x + y = \mathbf{0}$ and $\beta\mathbf{0} = \mathbf{0}$ for each scalar β,
   $\therefore y = y + (-1) * (x + y) = -x$.

   (c) $(-1)\mathbf{x} = -\mathbf{x}$.
   Proof:
   $(-1)x + x = (1 - 1)x = 0x$
   $\because x = 1x = (1 + 0)x = x + 0x$, $\therefore 0x = \mathbf{0}$
   $\therefore (-1)x + x = \mathbf{0}$.
   $\because x + (-x) = \mathbf{0}$, $\therefore (-1)x = -x$

2. Let V be the set of all ordered pairs of real numbers with addition defined by

$$(x_1, x_2) + (y_1, y_2) = (x_1 + y_1, x_2 + y_2)$$

   and scalar multiplication defined by

$$\alpha \circ (x_1, x_2) = (\alpha x_1, x_2)$$

   Scalar multipliacation for this system is defined in an unusual way, and consequently we use the symbol o to avoid confusion with the ordinary scalar multiplication of row vectors. Is V a vector space with these operations? Justify your answer.

   Proof:

   Suppose V is a vector space, and let $x_1 \in R, x_2 \neq 0$.

For addition, $(x_1, x_2) + (x_1, x_2) = (2x_1, 2x_2)$;

For scalar multipliacation,

$(x_1, x_2) + (x_1, x_2) = (1 + 1)(x_1, x_2) = (2x_1, x_2)$;

However, $\because x_2 \neq 0$, $\therefore (2x_1, 2x_2) \neq (2x_1, x_2)$,

$\therefore (x_1, x_2) + (x_1, x_2) \neq (x_1, x_2) + (x_1, x_2)$, which is contradicory.

All in all, the suppostion is incorrected. V is not a vector space.

3. Let R^+ denote the set of positive real numbers. Define the operation of scalar multiplication, denoted o, by

$$\alpha \circ x = x^\alpha$$

for each $x \in R^+$ and for any real number α. Define the operation of addition, denoted $\oplus$, by

$$x \oplus y = x \cdot y \quad \text{for all} \quad x, y \in R^+$$

Thus, for this system, the scalar product of -3 times $\frac{1}{2}$ is given by

$$-3 \circ \frac{1}{2} = \left(\frac{1}{2}\right)^{-3} = 8$$

and the sum of 2 and 5 is given by

$$2 \oplus 5 = 2 \cdot 5 = 10$$

Is R^+ a vector space with these operations? Prove your answer.

Solve:

R^+ is a vector space.

Proof:

Let $x, y, z \in R^+$, $\alpha, \beta \in R$

$\because x \oplus y = xy > 0$, $\therefore x \oplus y \in V$

$\because \alpha \circ x = x^\alpha > 0$, $\therefore \alpha \circ x \in V$

Then, proof V satisfies the eight properties:

(a) $x \oplus y = x * y = y * x = y \oplus x$.
(b) $(x \oplus y) \oplus z = (xy)z = x(yz) = x \oplus (y \oplus z)$.
(c) Let $\mathbf{0} = 1$. Then, $x \oplus \mathbf{0} = x * 1 = x$
(d) For each x, there is a $-x = \frac{1}{x}$,
letting $x \oplus -x = x * \frac{1}{x} = 1 = \mathbf{0}$
(e) $(\alpha * \beta) \circ x = x^\alpha\beta = (x^\beta)^\alpha = \alpha \circ (\beta \circ x)$

(f) $(\alpha + \beta) \circ x = x^{\alpha\beta} = x^{\alpha} \oplus x^{\beta} = \alpha \circ x \oplus \beta \circ x$

(g) $\alpha \circ (x \oplus y) = (xy)^{\alpha} = x^{\alpha} * y^{\alpha} = \alpha \circ x \oplus \alpha \circ y$

(h) $1 \circ x = x^1 = x$

As a result, R^+ is a vector space.

4. Suppose $\mathbf{u}_1, \ldots, \mathbf{u}_p$ and $\mathbf{v}_1, \ldots, \mathbf{v}_q$ are vectors in a vector space V, and let $H = \text{Span}(\mathbf{u}_1, \ldots, \mathbf{u}_p)$ and $K = \text{Span}(\mathbf{v}_1, \ldots, \mathbf{v}_q)$.

   (a) Show that $H \bigcap K$ is a subspace of V.
   Proof:
   $\because \forall u \in H$, there's always $u = a_1 u_1 + \ldots + a_p u_p$, $a_1, \ldots a_p \in R$
   $\therefore u \in V$
   And for the same issue, $\forall v \in K$, there's always $v \in V$
   $\therefore H, K \subseteq V$
   $\because H \bigcap K \subseteq H \subseteq V$, $\therefore H \bigcap K$ is a subset of V
   $\forall x, y \in H \bigcap K$, $\because H \bigcap K \subseteq H$, $\therefore x, y \in H$
   Define $\alpha \in R$.
   $\because H$ is a vector space.
   $\therefore x + y, \alpha x \in H$, for the same, $x + y, \alpha x \in K$
   $\therefore x + y, \alpha x \in H \bigcap K$, and $H \bigcap K$ is a subspace.

   (b) Show that H and K are subspaces of $H + K$.
   Proof:
   $\because H = \text{Span}(\mathbf{u}_1, \ldots, \mathbf{u}_2)$, H is a vector space.
   The same, K is a vector space.
   $\because H + K$ contains the linear conbination of H and K,
   $\therefore H$, K is the subspace of $H + K$.

   (c) Show that $H + K = \text{Span}(\mathbf{u}_1, \ldots, \mathbf{u}_p, \mathbf{v}_1, \ldots, \mathbf{v}_q)$.
   Proof:
   Define $x \in H$, $x = a_1\mathbf{u}_1 + \ldots + a_p\mathbf{u}_p$, $a_1, \ldots, a_p \in R$,
   $y \in K$, $y = b_1\mathbf{v}_1 + \ldots + b_q\mathbf{v}_q$, $b_1, \ldots, b_q \in R$.
   $\therefore \forall z \in H + K, z = \alpha x + \beta y$, $\alpha, \beta \in R$,
   $z = \alpha a_1\mathbf{u}_1 + \ldots + \alpha a_p\mathbf{u}_p + \beta b_1\mathbf{v}_1 + \ldots + \beta b_q\mathbf{v}_q$.
   $\therefore z \in \text{Span}(\mathbf{u}_1, \ldots, \mathbf{u}_p, \mathbf{v}_1, \ldots, \mathbf{v}_q)$,
   $H + K \subseteq \text{Span}(\mathbf{u}_1, \ldots, \mathbf{u}_p, \mathbf{v}_1, \ldots, \mathbf{v}_q)$.
   $\forall w \in \text{Span}(\mathbf{u}_1, \ldots, \mathbf{u}_p, \mathbf{v}_1, \ldots, \mathbf{v}_q)$,
   $w = m_1\mathbf{u}_1 + \ldots + m_p\mathbf{u}_p + n_1\mathbf{v}_1 + \ldots + n_q\mathbf{v}_q$,
   $m_1, \ldots m_p, n_1, \ldots, n_q \in R$.
   $\because m_1\mathbf{u}_1 + \ldots + m_p\mathbf{u}_p \in H$, $n_1\mathbf{v}_1 + \ldots + n_q\mathbf{v}_q \in K$.
   $\therefore w \in H + K$, $\text{Span}(\mathbf{u}_1, \ldots, \mathbf{u}_p, \mathbf{v}_1, \ldots, \mathbf{v}_q) \subseteq H + K$.
   $\therefore H + K = \text{Span}(\mathbf{u}_1, \ldots, \mathbf{u}_p, \mathbf{v}_1, \ldots, \mathbf{v}_q)$.

5. Let $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k\}$ be a spanning set for a vector space V.

(a) If we add another vector, $\mathbf{x}_{k+1}$, to the set, will we still have a spanning set? Explain.

Solve:

Yes, we still have a spanning set.

$\because V = \text{Span}(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k)$

$\therefore \forall \mathbf{v} \in V, \mathbf{v} = \mathbf{a}_1\mathbf{x}_1 + \ldots + \mathbf{a}_k\mathbf{x}_k,\ \mathbf{a}_1, \ldots, \mathbf{a}_k \in R$

$\therefore \mathbf{v} = \mathbf{a}_1\mathbf{x}_1 + \ldots + \mathbf{a}_k\mathbf{x}_k + 0 * \mathbf{x}_{k+1}$

$\therefore V = \text{Span}(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k, \mathbf{x}_{k+1})$

(b) If we delete one of the vectors,say, $\mathbf{x}_k$, from the set, will we still have a spanning set? Explain.

Solve:

That has many possibilities:

$\because V = \text{Span}(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k)$

$\therefore k \geq \text{Dim}V$

If $k = \text{Dim}V$

$\because$The number of $\mathbf{x}_1, \ldots, \mathbf{x}_{k-1}$ is less than Dim$V$,

$\therefore$ It is not a spanning set of V after deleting.

If $k \geq \text{Dim}V$

(i) If $\mathbf{x}_k$ is the liner combination of $\mathbf{x}_1, \ldots, \mathbf{x}_{k-1}$

Suppose $\forall \mathbf{x}_k = \mathbf{b}_1\mathbf{x}_1 + \ldots + \mathbf{b}_{k-1}\mathbf{x}_{k-1},\ \mathbf{b}_1, \ldots, \mathbf{b}_{k-1} \in R$

$\forall \mathbf{v} \in V, \mathbf{v} = \mathbf{a}_1\mathbf{x}_1 + \ldots + \mathbf{a}_k\mathbf{x}_k,\ \mathbf{a}_1, \ldots, \mathbf{a}_k \in R$

$\mathbf{v} = \mathbf{a}_1\mathbf{x}_1 + \ldots + \mathbf{a}_{k-1}\mathbf{x}_{k-1} + \mathbf{b}_1\mathbf{x}_1 + \ldots + \mathbf{b}_{k-1}\mathbf{x}_{k-1}$

$= (\mathbf{a}_1\mathbf{b}_1)\mathbf{x}_1 + \ldots + (\mathbf{a}_{k-1}\mathbf{b}_{k-1})\mathbf{x}_{k-1}$

$\therefore$ It is a spanning set of V after deleting.

(ii) If $\mathbf{x}_k$ cann't be written as a liner combination of $\mathbf{x}_1, \ldots, \mathbf{x}_{k-1}$

Suppose It is a spanning set of V after deleting

$\exists \mathbf{v} \in V$ while $\mathbf{v} = \mathbf{x}_k$, whitch cann't be written as a linear combination of $\mathbf{x}_1, \ldots, \mathbf{x}_{k-1}$, bringing contradiction.

$\therefore$ It is not a spanning set of V after deleting.