曲线 C: $\begin{cases} x^2+y^2-z=0 \\ x+y+z=1 \end{cases}$ 求C上距离坐标原点最远和最近的点。 书P16 例15

考查函数 $u=x^2+y^2+z^2$ 在条件 $\begin{cases} x^2+y^2-z=0 \\ x+y+z=1 \end{cases}$ 下的最值

令 $F(x,y,z,\lambda,\mu)=x^2+y^2+z^2+\lambda(x^2+y^2-z)+\mu(x+y+z-1)$

由 $\begin{cases} \frac{\partial F}{\partial x}=0 \\ \frac{\partial F}{\partial y}=0 \\ \frac{\partial F}{\partial z}=0 \\ \frac{\partial F}{\partial \lambda}=0 \\ \frac{\partial F}{\partial \mu}=0 \end{cases} \Rightarrow \begin{cases} x=\frac{-1+\sqrt{3}}{2} \\ y=\frac{-1+\sqrt{3}}{2} \\ z=2-\sqrt{3} \\ \lambda \text{ 可以不求} \\ \mu \text{ 可以不求} \end{cases}$ 或 $\begin{cases} x=\frac{-1-\sqrt{3}}{2} \\ y=\frac{-1-\sqrt{3}}{2} \\ z=2+\sqrt{3} \\ \lambda \text{ 可以不求} \\ \mu \text{ 可以不求} \end{cases}$

易知u有最大值和最小值。

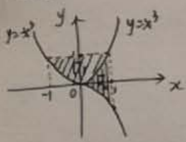
$u(\frac{-1+\sqrt{3}}{2}, \frac{-1+\sqrt{3}}{2}, 2-\sqrt{3}) = 9-5\sqrt{3}$

$u(\frac{-1-\sqrt{3}}{2}, \frac{-1-\sqrt{3}}{2}, 2+\sqrt{3}) = 9+5\sqrt{3}$

距离最大为 $\sqrt{9+5\sqrt{3}}$，最小为 $\sqrt{9-5\sqrt{3}}$。

5. 计算下列二重积分值

(1) 设区域D由曲线 $y=-x^3$，直线 $x=1$ 与 $y=1$ 围成，计算二重积分 $\iint\limits_D [2+xy\cos(x^2+y^2)]dxdy$



解：D如左图分割成 D_1 和 D_2。

D_1 关于y轴对称，$xy\cos(x^2+y^2)$ 为x的奇函数，$\iint\limits_{D_1} xy\cos(x^2+y^2)dxdy=0$

D_2 关于x轴对称，$xy\cos(x^2+y^2)$ 为y的奇函数，$\iint\limits_{D_2} xy\cos(x^2+y^2)dxdy=0$

故 $\iint\limits_D [2+xy\cos(x^2+y^2)]dxdy$

$=\iint\limits_D 2dxdy + \iint\limits_{D_1} xy\cos(x^2+y^2)dxdy + \iint\limits_{D_2} xy\cos(x^2+y^2)dxdy$

$=2D=2\times2\times1=4$

(2) 计算 $I=\iint\limits_D (x^3+y^3)dxdy$，$D: x^2+y^2\le 2x-2y$

坐标变换

解：$D: (x-1)^2+(y+1)^2\le 2$。

令 $u=x-1$，$v=y+1$，则 $dxdy=dudv$

$I=\iint\limits_{u^2+v^2\le 2} [(u+1)^3+(v-1)^3]dudv$

$=\iint\limits_{u^2+v^2\le 2} (\underline{u^3}+3u^2+\underline{3u}+1+\underline{v^3}-3v^2+\underline{3v}-1)dudv$

$\underline{\underline{\text{由奇函数性质}}}\ \iint\limits_{u^2+v^2\le 2}(3u^2-3v^2)dudv \ \underline{\underline{\text{对称性}}}\ 0$

对称性：① 积分区域关于 y=x 对称

$\Downarrow (x,y)\to(y,x)$

$\iint\limits_D f(x,y)dxdy=\iint\limits_D f(y,x)dxdy$

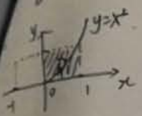
② 积分区域关于 y=-x 对称

$\Downarrow (x,y)\to(-y,-x)$

$\iint\limits_D f(x,y)dxdy=\iint\limits_D f(-y,-x)dxdy$

$\iint_D |x^2-y|\,dxdy$，$-1\le x\le 1$，$0\le y\le 1$.

当$(x,y)\in D_1$时，



D如左图. D_1为D中$x\ge 0$的部分.

$|x^2-y|=\begin{cases} x^2-y, & y\le x^2 \\ y-x^2, & y>x^2 \end{cases}$

由于D关于y轴对称，$|x^2-y|$关于x是偶函数（关于x是偶函数）

$$\iint_D |x^2-y|\,dxdy = 2\iint_{D_1}|x^2-y|\,dxdy = 2\left(\int_0^1 dx\int_0^{x^2}(x^2-y)dy+\int_0^1 dx\int_{x^2}^1(y-x^2)dy\right)$$

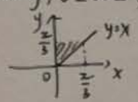
$$=2\left(\int_0^1 \frac{x^4}{2}dx+\int_0^1\left(\frac{1}{2}-x^2+\frac{x^4}{2}\right)dx\right)$$

$$=2\int_0^1\left(x^4-x^2+\frac{1}{2}\right)dx$$

$$=\frac{2}{5}-\frac{2}{3}+1=\frac{11}{15}$$

7. 计算 $\iint_D \frac{\cos y}{y}dxdy$，$0\le x\le\frac{\pi}{3}$，$x\le y\le\frac{\pi}{3}$.

解：D如下图



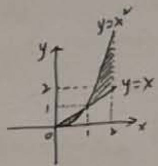
$\to x$

$\uparrow y$

$$\iint_D \frac{\cos y}{y}dxdy=\int_0^{\frac{\pi}{3}}dy\int_0^y\frac{\cos y}{y}dx=\int_0^{\frac{\pi}{3}}\cos y\,dy=\frac{\sqrt{3}}{2}$$

8. 交换累次积分的顺序.

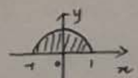
(1) $\int_0^2 dx\int_x^{x^2} f(x,y)dy$ (2) $\int_0^1 dy\int_{\sqrt{1-y^2}}^{-\sqrt{1-y^2}} f(x,y)dx$.



解(1) $\int_0^2 dx\int_x^{x^2} f(x,y)dy=\underline{\int_0^1 dx\int_x^{x^2} f(x,y)dy}+\int_1^2 dx\int_x^{x^2} f(x,y)dy$

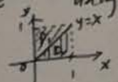
$=\underline{\int_0^1 dy\int_y^{\sqrt{y}}-f(x,y)dx}+\int_1^2 dy\int_{\sqrt{y}}^{y}f(x,y)dx+\int_2^4 dy\int_{\sqrt{y}}^{2}f(x,y)dx=\int_0^2 dy\int_{\sqrt{y}}^{y}f(x,y)dx+\int_2^4 dy\int_{\sqrt{y}}^{2}f(x,y)dx$

(2) $\int_0^1 dy\int_{\sqrt{1-y^2}}^{-\sqrt{1-y^2}} f(x,y)dx=-\iint_{\substack{0\le y\le 1\\ -\sqrt{1-y^2}\le x\le\sqrt{1-y^2}}} f(x,y)dxdy=-\int_{-1}^1 dx\int_0^{\sqrt{1-x^2}}f(x,y)dy$



9. 计算 $I=\iint_D e^{\max\{x^2,y^2\}}dxdy$，其中 $D: 0\le x\le 1, 0\le y\le 1$.

解：D如图



$$I=\iint_{D_1}e^{y^2}dxdy+\iint_{D_2}e^{x^2}dxdy=2\iint_{D_1}e^{y^2}dxdy=2\int_0^1 dy\int_0^y e^{y^2}dx=2\int_0^1 e^{y^2}\cdot y\,dy=e^{y^2}\Big|_0^1=e-1$$

$\iint_D f(x,y)dxdy=\iint_D f(r\cos\theta,r\sin\theta)r\,drd\theta$

10. 计算概率积分 $\int_0^{+\infty}e^{-x^2}dx$. (书P58例11. [illegible])

解：设 $I=\int_0^{+\infty}e^{-x^2}dx$，则 $I=\int_0^{+\infty}e^{-y^2}dy$.

$$I^2=\int_0^{+\infty}e^{-x^2}dx\cdot\int_0^{+\infty}e^{-y^2}dy=\iint_{\substack{0\le x<+\infty\\0\le y<+\infty}}e^{-(x^2+y^2)}dxdy=\int_0^{\frac{\pi}{2}}d\theta\int_0^{+\infty}e^{-r^2}r\,dr=\frac{\pi}{2}\left(-\frac{e^{-r^2}}{2}\right)\Big|_0^{+\infty}=\frac{\pi}{4}\Rightarrow I=\frac{\sqrt{\pi}}{2}$$

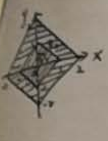
$f(x,y)=\begin{cases} x^2, & |x|+|y|<1 \\ \frac{1}{\sqrt{x^2+y^2}}, & 1\le |x|+|y|\le 2 \end{cases}$ 计算二重积分 $\iint\limits_D f(x,y)dxdy$，其中 $D=\{(x,y)\mid |x|+|y|\le 2\}$

记 $D_1=\{(x,y)\mid |x|+|y|<1\}$ $D_2=\{(x,y)\mid 1\le|x|+|y|\le 2\}$

$\iint\limits_D f(x,y)dxdy=\boxed{\iint\limits_{D_1}x^2dxdy}+\iint\limits_{D_2}\frac{1}{\sqrt{x^2+y^2}}dxdy$

$\iint\limits_{D_1} x^2dxdy = 4\int_0^1dx\int_0^{1-x}x^2dy = 4\int_0^1(x^2-x)dx = \frac{4}{3}-1=\frac{1}{3}$

$=2\iint\limits_{\substack{x+y\le1\\x\ge0\\y\ge0}}(x^2+y^2)dxdy+4\iint\limits_{\substack{1\le x+y\le2\\x\ge0\\y\ge0}}\frac{1}{\sqrt{x^2+y^2}}dxdy$



$x+y\le1$
$r\cos\theta+r\sin\theta=r(\cos\theta+\sin\theta)\le1$
$r\le\frac{1}{\cos\theta+\sin\theta}$
$1\le x+y\le2$
$\frac{1}{\cos\theta+\sin\theta}\le r\le\frac{2}{\cos\theta+\sin\theta}$

$=2\int_0^{\frac{\pi}{2}}d\theta\int_0^{\frac{1}{\cos\theta+\sin\theta}}r^3dr+4\int_0^{\frac{\pi}{2}}d\theta\int_{\frac{1}{\cos\theta+\sin\theta}}^{\frac{2}{\cos\theta+\sin\theta}}\frac{1}{r}\cdot rdr$ $(dxdy=rdrd\theta)$

$=\frac{1}{2}\int_0^{\frac{\pi}{2}}\left(\frac{1}{\cos\theta+\sin\theta}\right)^4d\theta+4\int_0^{\frac{\pi}{2}}\frac{1}{\cos\theta+\sin\theta}d\theta$

$=\frac{1}{8}\int_0^{\frac{\pi}{2}}\csc^4(\theta+\frac{\pi}{4})d\theta+2\sqrt{2}\int_0^{\frac{\pi}{2}}\csc(\theta+\frac{\pi}{4})d\theta$

$=[-\frac{1}{8}\int_0^{\frac{\pi}{2}}[1+\cot^2(\theta+\frac{\pi}{4})]d\cot(\theta+\frac{\pi}{4})+2\sqrt{2}\ln(\csc(\theta+\frac{\pi}{4})-\cot(\theta+\frac{\pi}{4}))]\Big|_0^{\frac{\pi}{2}}$

$=[-\frac{1}{8}(\cot(\theta+\frac{\pi}{4})+\frac{1}{3}\cot^3(\theta+\frac{\pi}{4}))+2\sqrt{2}\ln(\csc(\theta+\frac{\pi}{4})-\cot(\theta+\frac{\pi}{4}))]\Big|_0^{\frac{\pi}{2}}$

$=\boxed{\frac{1}{3}}+4\sqrt{2}\ln(\sqrt{2}+1)$

$\cos\theta+\sin\theta=\sqrt{2}\sin(\theta+\frac{\pi}{4})$

$\frac{1}{\csc\theta}=\sin\theta$

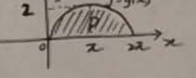
$\frac{1}{\cot\theta}=\tan\theta$

$(\cot\theta)'=-\csc^2\theta$

$1+\cot^2\theta=\csc^2\theta$

$\int\csc x dx=\int\frac{\csc x(\csc x-\cot x)}{\csc x-\cot x}dx$
$=\int\frac{\csc^2x-\cot x\csc x}{\csc x-\cot x}dx$
$=\ln|\csc x-\cot x|+C$ 不定积分

12. 设有界闭区域 G 是由曲线 $\begin{cases}x=t-\sin t\\y=1-\cos t\end{cases}$ $(0\le t\le2\pi)$ 与 $y=0$ 围成，计算二重积分 $\iint\limits_G y^2dxdy$ 摆线方程形式



$\cos2\alpha=1-2\sin^2\alpha$
$1-\cos2\alpha=2\sin^2\alpha$

解：$\iint\limits_G y^2dxdy=\int_0^{2\pi}dx\int_0^{y(x)}y^2dy=\int_0^{2\pi}\frac{1}{3}y^3(x)dx$

$=\int_0^{2\pi}\frac{1}{3}(1-\cos t)^3d(t-\sin t)$

$=\int_0^{2\pi}\frac{1}{3}(1-\cos t)^4dt=\frac{16}{3}\int_0^{2\pi}\sin^8\frac{t}{2}dt=\frac{32}{3}\int_0^{\pi}\sin^8udu=\frac{32}{3}\times\frac{7}{8}\times\frac{5}{6}\times\frac{3}{4}\times\frac{1}{2}\times\pi=\frac{35}{12}\pi$

13. 已知平面区域 $D=\{(r,\theta)\mid 2\le r\le2(1+\cos\theta),-\frac{\pi}{2}\le\theta\le\frac{\pi}{2}\}$，计算二重积分 $\iint\limits_D xdxdy$

解：$\iint\limits_D xdxdy=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}d\theta\int_2^{2(1+\cos\theta)}\underbrace{r\cos\theta\cdot rdr}_{\text{关于}\theta\text{是偶函数}}\overset{\text{偶}}{=}2\int_0^{\frac{\pi}{2}}d\theta\int_2^{2(1+\cos\theta)}r^2\cos\theta dr$

$(a+b)^3$
$=a^3+3a^2b+3ab^2+b^3$

$=\frac{2}{3}\int_0^{\frac{\pi}{2}}([2(1+\cos\theta)]^3-8)\cos\theta d\theta=\frac{16}{3}\int_0^{\frac{\pi}{2}}[(1+\cos\theta)^3-1]\cos\theta d\theta$

$=\frac{16}{3}\int_0^{\frac{\pi}{2}}(3\cos^2\theta+3\cos^3\theta+\cos^4\theta)d\theta$

$=\frac{16}{3}(3K_2+3K_3+K_4)$

$=\frac{16}{3}(3\cdot\frac{1}{2}\cdot\frac{\pi}{2}+3\cdot\frac{2}{3}\cdot1+\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2})$

$=16\cdot(\frac{\pi}{4}+\frac{2}{3}+\frac{\pi}{16})$

$=5\pi+\frac{32}{3}$.

$I_0=\int_0^{\pi}\sin^0udu=\pi$.

$I_n=\int\sin^nxdx$
$=-\int\sin^{n-1}xd\cos x=-\cos x\sin^{n-1}x+(n-1)\int\sin^{n-2}x\underbrace{\cos^2x}_{1-\sin^2x}dx=-\cos x\sin^{n-1}x+(n-1)I_{n-2}+(1-n)I_n$

$K_0=\int_0^{\frac{\pi}{2}}\cos^0\theta d\theta=\frac{\pi}{2}$
$K_1=\int_0^{\frac{\pi}{2}}\cos\theta d\theta=\sin\theta\Big|_0^{\frac{\pi}{2}}=1-0=1$

$\Rightarrow I_n=-\frac{\cos x\sin^{n-1}x}{n}+\frac{n-1}{n}I_{n-2}$

$K_n=\int\cos^nxdx$
$=\int\cos^{n-1}xd\sin x=\sin x\cos^{n-1}x+(n-1)\int\cos^{n-2}x\cdot\sin^2xdx=\sin x\cos^{n-1}x+(n-1)K_{n-2}-(n-1)K_n$

$\Rightarrow K_n=\frac{\sin x\cos^{n-1}x}{n}+\frac{n-1}{n}K_{n-2}$