第九章 多元函数微分法及其应用

9.1

1. 求下列函数的定义域，并指出其中的开区域与闭区域，连通集与非连通集，有界集与无界集.

(1) $z=\frac{1}{\sqrt{x+y}}+\frac{1}{\sqrt{x-y}}$；

解 函数的定义域为$D=\left\{(x,y)\middle|\, x+y>0, x-y>0\right\}$，是无界开区域.

(2) $z=\ln\left[x\ln(y-x)\right]$；

解 由$x\ln(y-x)>0$得

$$\begin{cases}x>0\\ y-x>1\end{cases}\quad 或\quad \begin{cases}x<0\\ 0<y-x<1\end{cases}$$

即

$$0<x<y-1\quad 或\quad \begin{cases}x<0\\ x<y<x+1\end{cases}$$

所以函数的定义域为$D=\left\{(x,y)\middle|\, 0<x<y-1\right\}\cup\left\{(x,y)\middle|\, x<0, x<y<x+1\right\}$，是无界非连通开集.

(3) $u=\arccos\frac{z}{\sqrt{x^2+y^2}}$.

解 由$\left|\frac{z}{\sqrt{x^2+y^2}}\right|\le 1$和$x^2+y^2\ne 0$得

$$x^2+y^2-z^2\ge 0\quad 和\quad x^2+y^2\ne 0$$

所以函数的定义域为$D=\left\{(x,y,z)\middle|\, x^2+y^2-z^2\ge 0, x^2+y^2\ne 0\right\}$，是无界非连通集.

2. 若$f\left(x+y,\frac{y}{x}\right)=x^2-y^2$，求$f(x,y)$.

解 令$s=x+y, t=\frac{y}{x}$，则$x=\frac{s}{1+t}, y=\frac{st}{1+t}$，所以

$$f(s,t)=\left(\frac{s}{1+t}\right)^2-\left(\frac{st}{1+t}\right)^2=\frac{s^2(1-t)}{1+t}$$

故

$$f(x,y)=\frac{x^2(1-y)}{1+y}$$

3. 求下列极限.

(1) $\lim\limits_{(x,y)\to(1,0)}\frac{\ln(x+\mathrm{e}^y)}{\sqrt{x^2+y^2}}$；

解 $\lim\limits_{(x,y)\to(1,0)}\frac{\ln(x+\mathrm{e}^y)}{\sqrt{x^2+y^2}}=\frac{\ln(1+\mathrm{e}^0)}{\sqrt{1^2+0^2}}=\ln 2$

(2) $\lim\limits_{(x,y)\to(0,0)}\frac{xy}{\sqrt{2-\mathrm{e}^{xy}}-1}$；

解

$$\begin{aligned}\lim_{(x,y)\to(0,0)}\frac{xy}{\sqrt{2-\mathrm{e}^{xy}}-1}&=\lim_{(x,y)\to(0,0)}\frac{xy\left(\sqrt{2-\mathrm{e}^{xy}}+1\right)}{1-\mathrm{e}^{xy}}\\&=\lim_{(x,y)\to(0,0)}\frac{xy}{1-\mathrm{e}^{xy}}\cdot\lim_{(x,y)\to(0,0)}\left(\sqrt{2-\mathrm{e}^{xy}}+1\right)\\&=(-1)\cdot 2=-2\end{aligned}$$

其中

$$\lim_{(x,y)\to(0,0)}\frac{xy}{1-\mathrm{e}^{xy}}\overset{u=xy}{=}\lim_{u\to0}\frac{u}{1-\mathrm{e}^u}=\lim_{u\to0}\frac{u}{-u}=-1$$

(3) $\lim\limits_{(x,y)\to(0,0)}(x+y)\ln(x^2+y^2)$.

解 令 $x=r\cos\theta, y=r\sin\theta$，则 $(x,y)\to(0,0)$ 等价于 $r\to0$，于是

$$\begin{aligned}\lim_{(x,y)\to(0,0)}(x+y)\ln(x^2+y^2)&=\lim_{r\to0}(r\cos\theta+r\sin\theta)\ln r^2\\&=\lim_{r\to0}2r\ln r(\cos\theta+\sin\theta)=0\end{aligned}$$

其中 $|\cos\theta+\sin\theta|\le\sqrt{2}$ 及

$$\lim_{r\to0}r\ln r=\lim_{r\to0}\frac{\ln r}{r}=\lim_{r\to0}\frac{\frac{1}{r}}{-\frac{1}{r^2}}=\lim_{r\to0}-r=0$$

4. 证明：极限 $\lim\limits_{(x,y)\to(0,0)}\frac{x^2y^2}{x^2y^2+(x-y)^2}$ 不存在.

证 因为

$$\lim_{\substack{(x,y)\to(0,0)\\ y=x}}\frac{x^2y^2}{x^2y^2+(x-y)^2}=\lim_{x\to0}\frac{x^4}{x^4}=1$$

$$\lim_{\substack{(x,y)\to(0,0)\\ y=-x}}\frac{x^2y^2}{x^2y^2+(x-y)^2}=\lim_{x\to0}\frac{x^4}{x^4+4x^2}=0$$

沿两种不同的路径的极限不同，所以极限 $\lim\limits_{(x,y)\to(0,0)}\frac{x^2y^2}{x^2y^2+(x-y)^2}$ 不存在.

5. 指出下列函数的间断点.

(1) $z=\frac{1}{x^2+y^2}$；

解 满足 $x^2+y^2=0$ 的点，即点 $(0,0)$ 为函数的间断点.

(2) $u=\frac{e^{\frac{1}{z}}}{x-y^2}$.

解 满足 $z=0$ 或 $x-y^2=0$ 的点，即平面 $z=0$ 与抛物柱面 $x-y^2=0$ 上的点为函数的间断点. 间断点集为 $\left\{(x,y,z)\,\middle|\, z=0,x-y^2=0\right\}$.

6. 讨论函数 $f(x,y)=\begin{cases}xy\frac{x^2-y^2}{x^2+y^2},(x,y)\neq(0,0)\\ 0,(x,y)=(0,0)\end{cases}$ 在点 $(0,0)$ 处的连续性.

解 当 $(x,y)\neq(0,0)$ 时，有

$$0\le|f(x,y)|=\left|xy\frac{x^2-y^2}{x^2+y^2}\right|\le|xy|\frac{x^2+y^2}{x^2+y^2}=|xy|$$

又 $\lim\limits_{(x,y)\to(0,0)}0=0, \lim\limits_{(x,y)\to(0,0)}|xy|=0$，所以 $\lim\limits_{(x,y)\to(0,0)}|f(x,y)|=0$，故

$$\lim_{(x,y)\to(0,0)}f(x,y)=0=f(0,0)$$

即 $f(x,y)$ 在点 $(0,0)$ 处连续.

9.2

1. 求下列函数的偏导数.

(1) $z=x^2y+\sin\frac{x}{y}$；

解 $\frac{\partial z}{\partial x}=2xy+\cos\frac{x}{y}\cdot\frac{1}{y}=2xy+\frac{1}{y}\cos\frac{x}{y}$

$$\frac{\partial z}{\partial y}=x^2+\cos\frac{x}{y}\cdot\left(-\frac{x}{y^2}\right)=x^2-\frac{x}{y^2}\cos\frac{x}{y}$$

(2) $z=(1+xy)^y$；

解 $\frac{\partial z}{\partial x}=y(1+xy)^{y-1}\cdot y=y^2(1+xy)^{y-1}$

$$\frac{\partial z}{\partial y}=(1+xy)^y\frac{\partial}{\partial y}\left[y\ln(1+xy)\right]$$

$$=(1+xy)^y\left[\ln(1+xy)+y\frac{x}{1+xy}\right]$$

$$=(1+xy)^y\left[\ln(1+xy)+\frac{xy}{1+xy}\right]$$

(3) $u=\arctan(x-y)^z$.

解 $\frac{\partial u}{\partial x}=\frac{1}{1+\left[(x-y)^z\right]^2}\cdot z(x-y)^{z-1}=\frac{z(x-y)^{z-1}}{1+(x-y)^{2z}}$

$$\frac{\partial u}{\partial y}=\frac{1}{1+\left[(x-y)^z\right]^2}\cdot z(x-y)^{z-1}\cdot(-1)=\frac{-z(x-y)^{z-1}}{1+(x-y)^{2z}}$$

$$\frac{\partial u}{\partial z}=\frac{1}{1+\left[(x-y)^z\right]^2}\cdot(x-y)^z\ln(x-y)=\frac{(x-y)^z\ln(x-y)}{1+(x-y)^{2z}}$$

2. 设 $f(x,y)=x+(y-1)\arcsin\sqrt{\frac{x}{y}}$ ，求 $f'_x(x,1)$.

解 $f(x,1)=x$，所以

$$f'_x(x,1)=\frac{\mathrm{d}}{\mathrm{d}x}f(x,1)=\frac{\mathrm{d}x}{\mathrm{d}x}=1$$

3. 设 $f(x,y)=\begin{cases}\frac{1}{2xy}\sin(x^2y), xy\neq 0\\ 0，xy=0\end{cases}$ ，求 $f'_x(0,1)$ 及 $f'_y(0,1)$.

解 $f'_x(0,1)=\lim\limits_{x\to 0}\frac{f(x,1)-f(0,1)}{x}=\lim\limits_{x\to 0}\frac{\frac{1}{2x}\sin x^2-0}{x}=\frac{1}{2}\lim\limits_{x\to 0}\frac{\sin x^2}{x^2}=\frac{1}{2}$

$$f_y'(0,1)=\lim_{y\to 1}\frac{f(0,y)-f(0,1)}{y-1}=\lim_{y\to 1}\frac{0-0}{y-1}=0$$

4. 求函数 $z=x^2\mathrm{e}^{2y}$ 的二阶偏导数.

解 $\dfrac{\partial z}{\partial x}=2x\mathrm{e}^{2y}$，$\dfrac{\partial z}{\partial y}=2x^2\mathrm{e}^{2y}$

$$\frac{\partial^2 z}{\partial x^2}=2\mathrm{e}^{2y}，\quad \frac{\partial^2 z}{\partial x\partial y}=\frac{\partial^2 z}{\partial y\partial x}=4x\mathrm{e}^{2y}\quad，\quad \frac{\partial^2 z}{\partial y^2}=4x^2\mathrm{e}^{2y}$$

5. 设 $z=x\ln(xy)$，求 $\dfrac{\partial^3 z}{\partial x^2\partial y}$ 及 $\dfrac{\partial^3 z}{\partial x\partial y^2}$.

解 $\dfrac{\partial z}{\partial x}=\ln(xy)+x\cdot\dfrac{y}{xy}=\ln(xy)+1$

$$\frac{\partial^2 z}{\partial x^2}=\frac{y}{xy}=\frac{1}{x}，\quad \frac{\partial^3 z}{\partial x^2\partial y}=0，\quad \frac{\partial^2 z}{\partial x\partial y}=\frac{x}{xy}=\frac{1}{y}，\quad \frac{\partial^3 z}{\partial x\partial y^2}=-\frac{1}{y^2}$$

6. 设 $y=\mathrm{e}^{-kn^2t}\sin nx$，求证：$\dfrac{\partial y}{\partial t}=k\dfrac{\partial^2 y}{\partial x^2}$.

解 因为

$$\frac{\partial y}{\partial t}=-kn^2\mathrm{e}^{-kn^2t}\sin nx，\quad \frac{\partial y}{\partial x}=n\mathrm{e}^{-kn^2t}\cos nx，\quad \frac{\partial^2 y}{\partial x^2}=-n^2\mathrm{e}^{-kn^2t}\sin nx$$

所以

$$\frac{\partial y}{\partial t}=-kn^2\mathrm{e}^{-kn^2t}\sin nx=k\left(-n^2\mathrm{e}^{-kn^2t}\sin nx\right)=k\frac{\partial^2 y}{\partial x^2}$$

6. 设 $f(x,y)=\begin{cases}\dfrac{x^3y}{x^6+y^6},x^2+y^2\neq 0\\ 0，x^2+y^2=0\end{cases}$，试证：$f(x,y)$ 在点 $(0,0)$ 处不连续，但在点 $(0,0)$ 处两个偏导数都存在，且两个偏导数在点 $(0,0)$ 处不连续.

证 因为

$$\lim_{\substack{(x,y)\to(0,0)\\ y=kx^3}}f(x,y)=\lim_{x\to 0}f(x,kx^3)=\lim_{x\to 0}\frac{x^3(kx^3)}{x^6+(kx^3)^6}=\lim_{x\to 0}\frac{k}{1+k^6x^{12}}=k$$

对于不同 k 的极限值不同，所以 $\lim\limits_{(x,y)\to(0,0)}f(x,y)$ 不存在，故 $f(x,y)$ 在 $(0,0)$ 处不连续.

当 $x^2+y^2\neq 0$时，有

$$f'_x(x,y)=\frac{3x^2y^7-3x^8y}{\left(x^6+y^6\right)^2}$$

$$f'_y(x,y)=\frac{x^9-5x^3y^6}{\left(x^6+y^6\right)^2}$$

当 $x^2+y^2\neq 0$时，有

$$f'_x(0,0)=\lim_{\Delta x\to 0}\frac{f(\Delta x,0)-f(0,0)}{\Delta x}=\lim_{x\to 0}\frac{0-0}{\Delta x}=0$$

$$f'_y(0,0)=\lim_{\Delta y\to 0}\frac{f(0,\Delta y)-f(0,0)}{\Delta y}=\lim_{\Delta y\to 0}\frac{0-0}{\Delta y}=0$$

因为

$$\lim_{\substack{(x,y)\to(0,0)\\ y=kx^4}}f'_x(x,y)=\lim_{x\to 0}f'_x\left(x,kx^4\right)=\lim_{x\to 0}\frac{3x^2k^7x^{28}-3kx^{12}}{\left(x^6+k^6x^{24}\right)^2}=-3k$$

所以 $\lim\limits_{(x,y)\to(0,0)}f'_x(x,y)$不存在，故 $f'_x(x,y)$在 $(0,0)$点不连续.

因为

$$\lim_{\substack{(x,y)\to(0,0)\\ y=x}}f'_y(x,y)=\lim_{x\to 0}f'_y(x,x)=\lim_{x\to 0}\frac{x^9-5x^9}{\left(x^6+x^6\right)^2}=\lim_{x\to 0}\frac{-1}{x^3}=\infty$$

所以 $\lim\limits_{(x,y)\to(0,0)}f'_y(x,y)$不存在，故 $f'_y(x,y)$在 $(0,0)$点不连续.

9.3

1. 求下列函数的全微分.

(1) $z=\dfrac{y}{\sqrt{x^2+y^2}}$；

解 因为

$$\frac{\partial z}{\partial x}=\frac{-y}{x^2+y^2}\frac{x}{\sqrt{x^2+y^2}}=\frac{-xy}{\left(x^2+y^2\right)^{\frac{3}{2}}}$$

$$\frac{\partial z}{\partial y}=\frac{\sqrt{x^2+y^2}-y\dfrac{y}{\sqrt{x^2+y^2}}}{x^2+y^2}=\frac{x^2}{\left(x^2+y^2\right)^{\frac{3}{2}}}$$

连续，所以

$$\mathrm{d}z=\frac{\partial z}{\partial x}\mathrm{d}x+\frac{\partial z}{\partial y}\mathrm{d}y=-\frac{xy}{\left(x^2+y^2\right)^{\frac{3}{2}}}\mathrm{d}x+\frac{x^2}{\left(x^2+y^2\right)^{\frac{3}{2}}}\mathrm{d}y$$

(2) $u=x^{yz}$.

解 因为

$$\frac{\partial u}{\partial x}=yzx^{yz-1}，\quad \frac{\partial u}{\partial y}=zx^{yz}\ln x，\quad \frac{\partial u}{\partial z}=yx^{yz}\ln x$$

连续，所以

$$\mathrm{d}u=\frac{\partial u}{\partial x}\mathrm{d}x+\frac{\partial u}{\partial y}\mathrm{d}y+\frac{\partial u}{\partial z}\mathrm{d}z=yzx^{yz-1}\mathrm{d}x+zx^{yz}\ln x\mathrm{d}y+yx^{yz}\ln x\mathrm{d}z$$

2. 求函数 $u=\cos(xy+xz)$ 在点 $\left(1,\frac{\pi}{6},\frac{\pi}{6}\right)$ 处的全微分.

解 因为

$$\frac{\partial u}{\partial x}=-\sin(xy+xz)\cdot(y+z)=-(y+z)\sin(xy+xz)$$

$$\frac{\partial u}{\partial y}=-x\sin(xy+xz)，\quad \frac{\partial u}{\partial z}=-x\sin(xy+xz)$$

连续，且在点 $\left(1,\frac{\pi}{6},\frac{\pi}{6}\right)$ 处

$$\left.\frac{\partial u}{\partial x}\right|_{\left(1,\frac{\pi}{6},\frac{\pi}{6}\right)}=\left.-(y+z)\sin(xy+xz)\right|_{\left(1,\frac{\pi}{6},\frac{\pi}{6}\right)}=-\frac{\sqrt{3}}{6}\pi$$

$$\left.\frac{\partial u}{\partial y}\right|_{\left(1,\frac{\pi}{6},\frac{\pi}{6}\right)}=\left.-x\sin(xy+xz)\right|_{\left(1,\frac{\pi}{6},\frac{\pi}{6}\right)}=-\frac{\sqrt{3}}{2}$$

$$\left.\frac{\partial u}{\partial z}\right|_{\left(1,\frac{\pi}{6},\frac{\pi}{6}\right)}=\left.-x\sin(xy+xz)\right|_{\left(1,\frac{\pi}{6},\frac{\pi}{6}\right)}=-\frac{\sqrt{3}}{2}$$

所以

$$\left.\mathrm{d}u\right|_{\left(1,\frac{\pi}{6},\frac{\pi}{6}\right)}=\left.\frac{\partial u}{\partial x}\right|_{\left(1,\frac{\pi}{6},\frac{\pi}{6}\right)}\mathrm{d}x+\left.\frac{\partial u}{\partial y}\right|_{\left(1,\frac{\pi}{6},\frac{\pi}{6}\right)}\mathrm{d}y+\left.\frac{\partial u}{\partial z}\right|_{\left(1,\frac{\pi}{6},\frac{\pi}{6}\right)}\mathrm{d}z$$

$$=-\frac{\sqrt{3}}{6}\pi\mathrm{d}x-\frac{\sqrt{3}}{2}\mathrm{d}y-\frac{\sqrt{3}}{2}\mathrm{d}z$$

3. 当 $x=2,y=1,\Delta x=0.1,\Delta y=-0.2$ 时，求函数 $z=\frac{y}{x}$ 的全增量和全微分.

解 设 $f(x,y)=\dfrac{y}{x}$，则全增量为

$$\Delta z=f(2+0.1,1-0.2)-f(2,1)=\frac{1-0.2}{2+0.1}-\frac{1}{2}=-0.119$$

又函数的全微分为

$$\mathrm{d}z=-\frac{y}{x^2}\Delta x+\frac{1}{x}\Delta y$$

所以当 $x=2,y=1,\Delta x=0.1,\Delta y=-0.2$ 时的全微分为

$$\mathrm{d}z=-\frac{1}{2^2}\times 0.1+\frac{1}{2}\cdot(-0.2)=-0.125$$

4. 证明：函数 $f(x,y)=\sqrt{|xy|}$ 在点 $(0,0)$ 处连续且偏导数存在，但不可微.

证 因为

$$\lim_{(x,y)\to(0,0)}f(x,y)=\lim_{(x,y)\to(0,0)}\sqrt{|xy|}=0=f(0,0)$$

所以 $f(x,y)$ 在点 $(0,0)$ 处连续.

因为

$$f'_x(0,0)=\lim_{x\to 0}\frac{f(x,0)-f(0,0)}{x}=\lim_{x\to 0}\frac{0-0}{x}=0$$

$$f'_y(0,0)=\lim_{y\to 0}\frac{f(0,y)-f(0,0)}{y}=\lim_{y\to 0}\frac{0-0}{y}=0$$

所以 $f(x,y)$ 在点 $(0,0)$ 处偏导数都存在.

因为

$$\lim_{\substack{(x,y)\to(0,0)\\ y=kx}}\frac{f(x,y)-f(0,0)-f'_x(0,0)x-f'_y(0,0)y}{\sqrt{x^2+y^2}}$$

$$=\lim_{\substack{(x,y)\to(0,0)\\ y=kx}}\frac{\sqrt{|xy|}-0}{\sqrt{x^2+y^2}}=\lim_{x\to 0}\frac{\sqrt{|k|}|x|}{\sqrt{1+k^2}|x|}=\frac{\sqrt{|k|}}{\sqrt{1+k^2}}$$

当 k 取不同值时得到的极限值不同，所以二重极限

$$\lim_{(x,y)\to(0,0)}\frac{f(x,y)-f(0,0)-f'_x(0,0)x-f'_y(0,0)y}{\sqrt{x^2+y^2}}$$

不存在，可知 $f(x,y)$ 在点 $(0,0)$ 处不可微.

5. 设函数 $z=f(x,y)$ 在凸区域 D 上，$\frac{\partial z}{\partial x}\equiv 0$ 的充要条件是什么？$\frac{\partial^2 z}{\partial x\partial y}\equiv 0$ 的充要条件是什么？$\mathrm{d}z\equiv 0$ 的充要条件是什么？（凸区域 D 是指 D 内任意两点间的直线段都位于 D 内的区域）

解 $\frac{\partial z}{\partial x}\equiv 0 \Leftrightarrow z=\varphi(y)$，$\varphi$ 为 y 的任一函数.

$$\frac{\partial^2 z}{\partial x\partial y}\equiv 0 \Leftrightarrow \frac{\partial z}{\partial x}=g(x) \Leftrightarrow z=\Phi(x)+\Psi(y)，\quad \Phi,\Psi \text{ 为任意两个可微函数.}$$

$$\mathrm{d}x\equiv 0 \Leftrightarrow \frac{\partial z}{\partial x}\equiv 0,\frac{\partial z}{\partial y}\equiv 0 \Leftrightarrow z=C，\quad C \text{ 为任意常数}$$

9.4

1. 设 $z=u^2\ln v$，而 $u=\frac{x}{y},v=3x-2y$，求 $\frac{\partial z}{\partial x},\frac{\partial z}{\partial y}$.

解 $$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial x}=2u\cdot\ln v+\frac{u^2}{v}\cdot 3=\frac{2x}{y^2}\ln(3x-2y)+\frac{3x^2}{(3x-2y)y^2}$$

$$\frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial y}=2u\cdot\ln v\cdot\left(-\frac{x}{y^2}\right)+\frac{u^2}{v}\cdot(-2)=-\frac{2x^2}{y^3}\ln(3x-2y)+\frac{2x^2}{(3x-2y)y^2}$$

2. 设 $z=\tan(3t+2x^2-y)$，而 $x=\frac{1}{t},y=\sqrt{t}$，求 $\frac{\mathrm{d}z}{\mathrm{d}t}$.

解

$$\begin{aligned}
\frac{\mathrm{d}z}{\mathrm{d}t}&=\frac{\partial z}{\partial t}+\frac{\partial u}{\partial x}\cdot\frac{\mathrm{d}x}{\mathrm{d}t}+\frac{\partial u}{\partial y}\cdot\frac{\mathrm{d}y}{\mathrm{d}t}\\
&=\sec^2(3t+2x^2-y)\cdot 3+\sec^2(3t+2x^2-y)-4x\cdot\left(-\frac{1}{t^2}\right)\\
&+\sec^2(3t+2x^2-y)\cdot(-1)\cdot\frac{1}{2\sqrt{t}}\\
&=\left(3-\frac{4}{t^3}-\frac{1}{2\sqrt{t}}\right)\sec^2\left(3t+\frac{2}{t^2}-\sqrt{t}\right)
\end{aligned}$$

3. 设 $u=\frac{\mathrm{e}^{ax}(y-z)}{a^2+1}$，而 $y=a\sin x,z=\cos x$，求 $\frac{\mathrm{d}u}{\mathrm{d}x}$.

解

$$\frac{\mathrm{d}z}{\mathrm{d}x}=\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}\cdot\frac{\mathrm{d}y}{\mathrm{d}x}+\frac{\partial u}{\partial z}\cdot\frac{\mathrm{d}z}{\mathrm{d}x}$$

$$=\frac{a\mathrm{e}^{ax}\left(y-z\right)}{a^2+1}+\frac{\mathrm{e}^{ax}}{a^2+1}\cdot a\cos x+\frac{\mathrm{e}^{ax}}{a^2+1}\cdot\left(-1\right)\cdot\left(-\sin x\right)=\mathrm{e}^{ax}\sin x$$

4. 求下列函数的一阶偏导数（其中 f 具有一阶连续偏导数）.

(1) $z=f\left(x+y,x^2+y^2\right)$；

解 $\frac{\partial z}{\partial x}=f_1'\cdot 1+f_2'\cdot\left(2x\right)=f_1'+2xf_2'$

$\frac{\partial z}{\partial y}=f_1'\cdot 1+f_2'\cdot\left(2y\right)=f_1'+2yf_2'$

(2) $u=f\left(\frac{x}{y},\frac{y}{z}\right)$.

解 $\frac{\partial u}{\partial x}=f_1'\cdot\frac{1}{y}=\frac{1}{y}f_1'$

$\frac{\partial u}{\partial y}=f_1'\cdot\left(-\frac{x}{y^2}\right)+f_2'\cdot\frac{1}{z}=-\frac{x}{y^2}f_1'+\frac{1}{z}f_2'$

$\frac{\partial u}{\partial z}=f_2'\cdot\left(-\frac{y}{z^2}\right)=-\frac{y}{z^2}f_2'$

5. 设 $z=xy+x\varphi\left(\frac{y}{x}\right)$，其中 φ 可导，证明：$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=z+xy$.

证 $\frac{\partial z}{\partial x}=y+\varphi+x\varphi'\cdot\left(-\frac{y}{x^2}\right)=y+\varphi-x\varphi'$

$\frac{\partial z}{\partial y}=x+x\varphi'\cdot\frac{1}{x}=x+\varphi'$

所以

$$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=x\left(y+\varphi-\frac{y}{x}\varphi'\right)+y\left(x+\varphi'\right)=xy+x\varphi+xy=z+xy$$

6. 求下列函数的二阶偏导数（其中 f 具有二阶连续偏导数）.

(1) $z=f\left(xy,y\right)$；

解 $\frac{\partial z}{\partial x}=f_1'\cdot y=yf_1'$，$\frac{\partial z}{\partial y}=f_1'\cdot x+f_2'=xf_1'+f_2'$

$$\frac{\partial^2 z}{\partial x^2}=\frac{\partial}{\partial x}(yf_1')=yf_{11}''\cdot y=y^2 f_{11}''$$

$$\frac{\partial^2 z}{\partial x\partial y}=\frac{\partial}{\partial y}(yf_1')=f_1'+y(f_{11}''\cdot x+f_{12}'')=f_1'+xyf_{11}''+yf_{12}''$$

$$\frac{\partial^2 z}{\partial y^2}=\frac{\partial}{\partial y}(xf_1'+f_2')=x(f_{11}''\cdot x+f_{12}'')+(f_{21}''\cdot x+f_{22}'')=xf_{11}''+2xf_{12}''+f_{22}''$$

(2) $z=f\left(xe^x,x,y\right)$.

解 $\frac{\partial z}{\partial x}=f_1'\cdot(x+1)e^x+f_2'$，$\frac{\partial z}{\partial y}=f_3'$

$$\frac{\partial^2 z}{\partial x^2}=\frac{\partial}{\partial x}\left[(x+1)e^x f_1'+f_2'\right]$$
$$=(x+2)e^x f_1'+(x+1)e^x\left[f_{11}''\cdot(x+1)e^x+f_{12}''\right]+\left[f_{21}''\cdot(x+1)e^x+f_{22}''\right]$$

$$\frac{\partial^2 z}{\partial x\partial y}=\frac{\partial}{\partial y}\left[(x+1)e^x f_1'+f_2'\right]=(x+1)e^x f_{13}''+f_{23}''$$

$$\frac{\partial^2 z}{\partial y^2}=\frac{\partial}{\partial y}(f_3')=f_{33}''$$

7. 已知函数 $z=f(x,y)$ 具有二阶连续偏导数，且满足方程 $a^2\frac{\partial^2 z}{\partial x^2}-\frac{\partial^2 z}{\partial y^2}=0$，作变换，令 $u=x+ay,v=x-ay\,(a\neq 0)$，试求 z 作为 u,v 的函数所应满足的方程.

解 $\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\cdot 1+\frac{\partial z}{\partial v}=\frac{\partial z}{\partial u}+\frac{\partial z}{\partial v}$

$$\frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\cdot a+\frac{\partial z}{\partial v}\cdot(-a)=a\left(\frac{\partial z}{\partial u}-\frac{\partial z}{\partial v}\right)$$

$$\frac{\partial^2 z}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}+\frac{\partial z}{\partial v}\right)=\left(\frac{\partial^2 z}{\partial u^2}\cdot 1+\frac{\partial^2 z}{\partial u\partial v}\cdot 1\right)+\left(\frac{\partial^2 z}{\partial v\partial u}\cdot 1+\frac{\partial^2 z}{\partial v^2}\cdot 1\right)$$
$$=\frac{\partial^2 z}{\partial u^2}+2\frac{\partial^2 z}{\partial u\partial v}+\frac{\partial^2 z}{\partial v^2}$$

$$\frac{\partial^2 z}{\partial y^2}=\frac{\partial}{\partial y}\left[a\left(\frac{\partial z}{\partial u}-\frac{\partial z}{\partial v}\right)\right]=a\left[\frac{\partial^2 z}{\partial u^2}\cdot a+\frac{\partial^2 z}{\partial u\partial v}\cdot(-a)\right]-a\left[\frac{\partial^2 z}{\partial v\partial u}\cdot a+\frac{\partial^2 z}{\partial v^2}\cdot(-a)\right]$$

$$=a^2\left(\frac{\partial^2 z}{\partial u^2}-2\frac{\partial^2 z}{\partial u\partial v}+\frac{\partial^2 z}{\partial v^2}\right)$$

所以

$$a^2\frac{\partial^2 z}{\partial x^2}-\frac{\partial^2 z}{\partial y^2}=a^2\left(\frac{\partial^2 z}{\partial u^2}+2\frac{\partial^2 z}{\partial u\partial v}+\frac{\partial^2 z}{\partial v^2}\right)-a^2\left(\frac{\partial^2 z}{\partial u^2}-2\frac{\partial^2 z}{\partial u\partial v}+\frac{\partial^2 z}{\partial v^2}\right)$$

$$=4a^2\frac{\partial^2 z}{\partial u\partial v}=0$$

故所应满足的方程为

$$\frac{\partial^2 z}{\partial u\partial v}=0$$

8. 如果函数 $s=f(x,y,z)$ 满足关系 $f(tx,ty,tz)=t^k f(x,y,z), t>0$，则称此函数为 k 次齐次函数. 证明：当 f 可微时，k 次齐次函数满足方程 $x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}+z\frac{\partial f}{\partial z}=kf(x,y,z)$；反之，满足该方程的函数必为 k 次齐次函数.

证

(1) 对 $f(tx,ty,tz)=t^k f(x,y,z)$ 关于 t 求导得

$$xf_1'(tx,ty,tz)+yf_2'(tx,ty,tz)+zf_3'(tx,ty,tz)=kt^{k-1}f(x,y,z)$$

两边乘 t 得

$$txf_1'(tx,ty,tz)+tyf_2'(tx,ty,tz)+tzf_3'(tx,ty,tz)=kt^k f(x,y,z)=kf(tx,ty,tz)$$

所以

$$uf_1'(u,v,w)+vf_2'(u,v,w)+wf_3'(u,v,w)=kf(u,v,w)$$

故

$$xf_1'(x,y,z)+yf_2'(x,y,z)+zf_3'(x,y,z)=kf(x,y,z)$$

即

$$x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}+z\frac{\partial f}{\partial z}=kf(x,y,z)$$

(2) 已知

$$xf_1'(x,y,z)+yf_2'(x,y,z)+zf_3'(x,y,z)=kf(x,y,z)$$

所以

$$txf_1'(tx,ty,tz)+tyf_2'(tx,ty,tz)+ztf_3'(tx,ty,tz)=kf(tx,ty,tz)$$

令 $\varphi(t)=f(tx,ty,tz)$，则

$$\begin{aligned}\varphi'(t)&=xf_1'(tx,ty,tz)+yf_2'(tx,ty,tz)+zf_3'(tx,ty,tz)\\&=\frac{1}{t}\left[txf_1'(tx,ty,tz)+tyf_2'(tx,ty,tz)+tzf_3'(tx,ty,tz)\right]\\&=\frac{1}{t}\cdot kf(tx,ty,tz)=\frac{k\varphi(t)}{t}\end{aligned}$$

分离变量得

$$\frac{\mathrm{d}\varphi(t)}{\varphi(t)}=\frac{k}{t}\mathrm{d}t$$

积分得

$$\ln\varphi(t)=k\ln t+\ln C\Rightarrow\varphi(t)=Ct^k$$

令 $t=1$ 得 $C=\varphi(1)=f(x,y,z)$，所以

$$f(tx,ty,tz)=t^kf(x,y,z)$$

满足方程 $x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}+z\frac{\partial f}{\partial z}=kf(x,y,z)$ 的函数必为 k 次齐次函数.

9. 利用微分运算法则，求函数 $z=f\left(xy,\frac{x}{y}\right)$ 的全微分和偏导数.

解 全微分为

$$\begin{aligned}\mathrm{d}z&=f_1'\cdot\mathrm{d}(xy)+f_2'\cdot\mathrm{d}\left(\frac{x}{y}\right)=f_1'(y\mathrm{d}x+x\mathrm{d}y)+f_2'\cdot\frac{y\mathrm{d}x-x\mathrm{d}y}{y^2}\\&=\left(yf_1'+\frac{1}{y}f_2'\right)\mathrm{d}x+\left(xf_1'-\frac{x}{y^2}f_2'\right)\mathrm{d}y\end{aligned}$$

偏导数为

$$\frac{\partial z}{\partial x}=yf_1'+\frac{1}{y}f_2',\qquad\frac{\partial z}{\partial y}=xf_1'-\frac{x}{y^2}f_2'$$

9.5

1. 求由方程 $\frac{x}{z}=\ln\frac{z}{y}$ 所确定的隐函数 $z=z(x,y)$ 的一阶及二阶偏导数.

解 方程两边对 x 求偏导得

$$\frac{1\cdot z-x\dfrac{\partial z}{\partial x}}{z^2}=\frac{1}{\dfrac{z}{y}}\cdot\frac{1}{y}\cdot\frac{\partial z}{\partial x}$$

解得

$$\frac{\partial z}{\partial x}=\frac{z}{x+z}$$

方程两边对 y 求偏导得

$$-\frac{x}{z^2}\cdot\frac{\partial z}{\partial y}=\frac{1}{\dfrac{z}{y}}\cdot\frac{y\dfrac{\partial z}{\partial y}-z}{y^2}$$

解得

$$\frac{\partial z}{\partial y}=\frac{z^2}{y(x+z)}$$

2. 利用微分运算法则，求由方程 $z-y-x+x\mathrm{e}^{z-y-x}=0$ 所确定的隐函数 $z=z(x,y)$ 的全微分和偏导数.

解 方程两边取全微分得

$$\mathrm{d}z-\mathrm{d}y-\mathrm{d}x+\mathrm{e}^{z-y-x}\mathrm{d}x+x\mathrm{e}^{z-y-x}(\mathrm{d}z-\mathrm{d}y-\mathrm{d}x)=0$$

整理得

$$\left(1+x\mathrm{e}^{z-y-x}\right)\mathrm{d}z-\left(1+x\mathrm{e}^{z-y-x}\right)\mathrm{d}y+\left(\mathrm{e}^{z-y-x}-1-x\mathrm{e}^{z-y-x}\right)\mathrm{d}x=0$$

从而得到全微分

$$\mathrm{d}z=\frac{x\mathrm{e}^{z-y-x}-\mathrm{e}^{z-y-x}+1}{1+x\mathrm{e}^{z-y-x}}\mathrm{d}x+\mathrm{d}y$$

并由此得到偏导数

$$\frac{\partial z}{\partial x}=\frac{x\mathrm{e}^{z-y-x}-\mathrm{e}^{z-y-x}+1}{1+x\mathrm{e}^{z-y-x}},\quad \frac{\partial z}{\partial y}=1$$

3. 设函数 $z=z(x,y)$ 由方程 $F\left(x+\dfrac{z}{y},y+\dfrac{z}{x}\right)=0$ 所确定，其中 F 具有连续偏导数，

证明： $x\dfrac{\partial z}{\partial x}+y\dfrac{\partial z}{\partial y}=z-xy$.

证 方程两边对 x, y 求偏导得

$$F_1' \cdot \left(1 + \frac{1}{y}\frac{\partial z}{\partial x}\right) + F_2' \cdot \frac{x\frac{\partial z}{\partial x} - z \cdot 1}{x^2} = 0$$

$$F_1' \cdot \frac{y\frac{\partial z}{\partial y} - z \cdot 1}{y^2} + F_2' \cdot \left(1 + \frac{1}{x}\frac{\partial z}{\partial y}\right) = 0$$

解得

$$\frac{\partial z}{\partial x} = \frac{-yx^2F_1' + yzF_2'}{x(xF_1' + yF_2')} \quad , \quad \frac{\partial z}{\partial y} = \frac{xzF_1' - xy^2F_2'}{y(xF_1' + yF_2')}$$

所以

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = \frac{-yx^2F_1' + yzF_2'}{xF_1' + yF_2'} + \frac{xzF_1' - xy^2F_2'}{xF_1' + yF_2'} = z - xy$$

4．设函数 $z = z(x, y)$ 由方程 $F(x + y, y - z) = 0$ 所确定，其中 F 具有二阶连续偏导数，求 $\frac{\partial^2 z}{\partial x \partial y}$.

解 方程两边对 x, y 求偏导得

$$F_1' \cdot 1 + F_2' \cdot \left(-\frac{\partial z}{\partial x}\right) = 0$$

$$F_1' \cdot 1 + F_2' \cdot \left(1 - \frac{\partial z}{\partial y}\right) = 0$$

解得

$$\frac{\partial z}{\partial x} = \frac{F_1'}{F_2'}, \ \frac{\partial z}{\partial y} = 1 + \frac{F_1'}{F_2'}$$

求二阶偏导数得

$$\frac{\partial^2 z}{\partial x\partial y}=\frac{\partial}{\partial y}\left(\frac{F_1'}{F_2'}\right)$$

$$=\frac{\left[F_{11}''\cdot 1+F_{12}''\cdot\left(1-\frac{\partial z}{\partial y}\right)\right]F_2'-F_1'\left[F_{21}''\cdot 1+F_{22}''\cdot\left(1-\frac{\partial z}{\partial y}\right)\right]}{\left(F_2'\right)^2}$$

$$=\frac{\left[F_{11}''+F_{12}''\cdot\left(-\frac{F_1'}{F_2'}\right)\right]F_2'-F_1'\left[F_{21}''+F_{22}''\cdot\left(-\frac{F_1'}{F_2'}\right)\right]}{\left(F_2'\right)^2}$$

$$=\frac{\left(F_2'\right)^2F_{11}''-2F_1'F_2'F_{12}''+\left(F_1'\right)^2F_{22}''}{\left(F_2'\right)^3}$$

5. 求下列方程组所确定的隐函数的导数或偏导数.

(1) 设 $\begin{cases}z=x^2+y^2\\x^2+2y^2+3z^2=20\end{cases}$，求 $\frac{\mathrm{d}y}{\mathrm{d}x},\frac{\mathrm{d}z}{\mathrm{d}x}$；

解 方程组对 x 求导得

$$\begin{cases}\frac{\mathrm{d}z}{\mathrm{d}x}=2x+2y\frac{\mathrm{d}y}{\mathrm{d}x}\\2x+4y\frac{\mathrm{d}y}{\mathrm{d}x}+6z\frac{\mathrm{d}z}{\mathrm{d}x}=0\end{cases}$$

解得

$$\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{x(1+6z)}{2y(1+3z)},\quad \frac{\mathrm{d}z}{\mathrm{d}x}=\frac{x}{1+3z}$$

(2) 设 $\begin{cases}x=\mathrm{e}^u+u\sin v\\y=\mathrm{e}^u-u\cos v\end{cases}$，求 $\frac{\partial u}{\partial x},\frac{\partial u}{\partial y},\frac{\partial v}{\partial x},\frac{\partial v}{\partial y}$；

解 方程组对 x 求偏导得

$$\begin{cases}1=\mathrm{e}^u\frac{\partial u}{\partial x}+\frac{\partial u}{\partial x}\cdot\sin v+u\cdot\cos v\frac{\partial v}{\partial x}\\0=\mathrm{e}^u\frac{\partial u}{\partial x}-\frac{\partial u}{\partial x}\cdot\cos v-u\cdot(-\sin v)\frac{\partial u}{\partial x}\end{cases}$$

化简得

$$\begin{cases}\left(e^u+\sin v\right)\frac{\partial u}{\partial x}+u\cos v\cdot\frac{\partial v}{\partial x}=1\\\left(e^u-\sin v\right)\frac{\partial u}{\partial x}+u\sin v\frac{\partial v}{\partial x}=0\end{cases}$$

解得

$$\frac{\partial u}{\partial x}=\frac{\sin v}{e^u(\sin v-\cos v)+1},\quad \frac{\partial v}{\partial x}=\frac{-e^u+\cos v}{u\left[e^u(\sin v-\cos v)-1\right]}$$

方程组对 y 求偏导得

$$\begin{cases}0=e^u\dfrac{\partial u}{\partial y}+\dfrac{\partial u}{\partial y}\cdot\sin v+u\cdot\cos v\dfrac{\partial v}{\partial y}\\ 1=e^u\dfrac{\partial u}{\partial y}-\dfrac{\partial u}{\partial x}\cdot\cos v-u\cdot(-\sin v)\dfrac{\partial u}{\partial y}\end{cases}$$

化简得

$$\begin{cases}(e^u+\sin v)\dfrac{\partial u}{\partial y}+u\cos v\dfrac{\partial v}{\partial y}=0\\ (e^u-\sin v)\dfrac{\partial u}{\partial y}+u\sin v\dfrac{\partial v}{\partial y}=1\end{cases}$$

解得

$$\frac{\partial u}{\partial y}=-\frac{\cos v}{e^u(\sin v-\cos v)+1},\quad \frac{\partial v}{\partial y}=\frac{e^u+\sin v}{u\left[e^u(\sin v-\cos v)+1\right]}$$

6. 设 $y=f(x,t)$，而 t 是由方程 $F(x,y,t)=0$ 所确定的 x,y 的函数，其中 f,F 均有一阶连续偏导数，求 $\dfrac{dy}{dx}$.

解 对两个方程关于 x 求导得

$$\begin{cases}\dfrac{dy}{dx}=\dfrac{\partial f}{\partial x}+\dfrac{\partial f}{\partial t}\dfrac{dt}{dx}\\ \dfrac{\partial F}{\partial x}+\dfrac{\partial F}{\partial y}\dfrac{dy}{dx}+\dfrac{\partial F}{\partial t}\dfrac{dt}{dx}\end{cases}$$

解得

$$\frac{dy}{dx}=\frac{\dfrac{\partial f}{\partial x}\dfrac{\partial F}{\partial t}-\dfrac{\partial f}{\partial t}\dfrac{\partial F}{\partial x}}{\dfrac{\partial F}{\partial t}+\dfrac{\partial f}{\partial t}\dfrac{\partial F}{\partial y}}$$

9.6

1. 求曲线 $x=\dfrac{t}{1+t},y=\dfrac{1+t}{t},z=t^2$ 在对应于 $t=1$ 的点处的切线及法平面方程.

解 $t=1$ 对应点为 $\left(\dfrac{1}{2},2,1\right)$，该点的切向量为

$$\bar{T}=\left\{\frac{\mathrm{d}x}{\mathrm{d}t},\frac{\mathrm{d}y}{\mathrm{d}t},\frac{\mathrm{d}z}{\mathrm{d}t}\right\}\Big|_{t=1}=\left\{\frac{1}{(1+t)^2},-\frac{1}{t^2},2t\right\}\Big|_{t=1}=\left\{\frac{1}{4},-1,2\right\}$$

所以切线方程为

$$\frac{x-\frac{1}{2}}{\frac{1}{4}}=\frac{y-2}{-1}=\frac{z-1}{2}$$

即

$$\frac{x-\frac{1}{2}}{1}=\frac{y-2}{-4}=\frac{z-1}{8}$$

法平面方程为

$$\frac{1}{4}\cdot\left(x-\frac{1}{2}\right)-1\cdot(y-2)+2\cdot(z-1)=0$$

即

$$2x-8y+16z-1=0$$

2. 求曲线$\begin{cases}x^2+y^2+z^2-3x=0\\2x-3y+5z-4=0\end{cases}$在点$(1,1,1)$处的切线及法平面方程.

解　设$F(x,y,z)=x^2+y^2+z^2-3x,G(x,y,z)=2x-3y+5z-4$，则

$$\frac{\partial(F,G)}{\partial(y,z)}\Big|_{(1,1,1)}=\begin{vmatrix}2y & 2z\\-3 & 5\end{vmatrix}_{(1,1,1)}=\begin{vmatrix}2 & 2\\-3 & 5\end{vmatrix}=16$$

$$\frac{\partial(F,G)}{\partial(z,x)}\Big|_{(1,1,1)}=\begin{vmatrix}2z & 2x-3\\5 & 2\end{vmatrix}_{(1,1,1)}=\begin{vmatrix}2 & -1\\5 & 2\end{vmatrix}=9$$

$$\frac{\partial(F,G)}{\partial(x,y)}\Big|_{(1,1,1)}=\begin{vmatrix}2x-3 & 2y\\2 & -3\end{vmatrix}_{(1,1,1)}=\begin{vmatrix}-1 & 2\\2 & -3\end{vmatrix}=-1$$

所以切线方程为

$$\frac{x-1}{16}=\frac{y-1}{9}=\frac{z-1}{-1}$$

法平面方程为

$$16(x-1)+9(y-1)-(z-1)=0$$

即

$$16x+9y-z-24=0$$

3. 求曲面$z=\sqrt{x^2+y^2}$在点$(3,4,5)$处的切平面及法线方程.

解 法向量为

$$\vec{n}=\left\{\frac{\partial z}{\partial x},\frac{\partial z}{\partial y},-1\right\}\Bigg|_{\substack{x=3\\y=4}}=\left\{\frac{3}{5},\frac{4}{5},-1\right\}$$

所以切平面方程为

$$\frac{3}{5}(x-3)+\frac{4}{5}(y-4)-(z-5)=0$$

即

$$3x+4y-5z=0$$

法线方程为

$$\frac{x-3}{\frac{3}{5}}=\frac{y-4}{\frac{4}{5}}=\frac{z-5}{-1}$$

即

$$\frac{x-3}{3}=\frac{y-4}{4}=\frac{z-5}{-5}$$

4. 求曲面 $x^3+y^3+z^3+xyz-6=0$ 在点 $(1,2,-1)$ 处的切平面及法线方程.

解 设 $F(x,y,z)=x^3+y^3+z^3+xyz-6$，则法向量为

$$\vec{n}=\left\{\frac{\partial F}{\partial x},\frac{\partial F}{\partial y},\frac{\partial F}{\partial z}\right\}\Bigg|_{(1,2,-1)}=\left\{3x^2+yz,3y^2+xz,3z^2+xy\right\}\Big|_{(1,2,-1)}=\{1,11,5\}$$

所以切平面方程为

$$1\cdot(x-1)+11\cdot(y-2)-5\cdot(z+1)=0$$

即

$$x+11y+5z-18=0$$

法线方程为

$$\frac{x-1}{1}=\frac{y-2}{11}=\frac{z+1}{5}$$

5. 设 $f(u,v)$ 可微，证明：曲面 $f(ax-bz,ay-cz)=0$ 上任一点的切平面都与某一定直线平行，其中 a,b,c 是不同时为零的常数.

证 曲面上任一点 (x,y,z) 处的法向量为

$$\vec{n}=\{af_1',af_2',-bf_1'-cf_2'\}$$

又 $\vec{l}=\{b,c,a\}$ 为某一定直线的方向向量，且

$$\vec{n}\cdot\vec{l}=(af_1')\cdot b+(af_2')\cdot c+(-bf_1'-cf_2')\cdot a=0$$

所以 $\bar{n}$ 垂直于向量 $\bar{l}$，即以 $\bar{n}$ 为法向量的平面平行于以 $\bar{l}$ 为方向向量的直线，亦即曲面 $f(ax-bz,ay-cz)=0$ 上任一点的切平面都与以 $\{b,c,a\}$ 为方向向量的定直线平行.

9.7

1. 求函数 $z=\ln(x+y)$ 在抛物线 $y^2=4x$ 上点 $(1,2)$ 处，沿着这条抛物线在该点处偏向 x 轴正向的切线方向的方向导数.

解 对 $y^2=4x$ 关于 x 求导得 $2y\dfrac{\mathrm{d}y}{\mathrm{d}x}=4$，即 $\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{2}{y}$，所以切向量为

$$\bar{l}=\left\{1,\frac{\mathrm{d}y}{\mathrm{d}x}\right\}\Big|_{(1,2)}=\left\{1,\frac{y}{2}\right\}\Big|_{(1,2)}=\{1,1\}$$

其单位向量为

$$\bar{l}^{o}=\frac{\bar{l}}{|\bar{l}|}=\left\{\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right\}$$

所以

$$\cos\alpha=\frac{1}{\sqrt{2}},\cos\beta=\frac{1}{\sqrt{2}}$$

又函数 $z=\ln(x+y)$ 的偏导数连续，且

$$\frac{\partial z}{\partial x}\Big|_{(1,2)}=\frac{1}{x+y}\Big|_{(1,2)}=\frac{1}{3}\quad,\quad\frac{\partial z}{\partial y}\Big|_{(1,2)}=\frac{1}{x+y}\Big|_{(1,2)}=\frac{1}{3}$$

所以

$$\frac{\partial z}{\partial \bar{l}}\Big|_{(1,2)}=\frac{\partial z}{\partial x}\Big|_{(1,2)}\cos\alpha+\frac{\partial z}{\partial y}\Big|_{(1,2)}\cos\beta=\frac{1}{3}\times\frac{1}{\sqrt{2}}+\frac{1}{3}\times\frac{1}{\sqrt{2}}=\frac{\sqrt{2}}{3}$$

3. 求函数 $u=x^2+y^2-2z^2+3xy+xyz-2z-3y$ 在点 $(1,2,3)$ 处沿从点 $(1,2,3)$ 到点 $(2,1,3)$ 的方向的方向导数.

解 从点 $(1,2,3)$ 到点 $(2,1,3)$ 的方向向量为 $\bar{l}=\{1,-1,0\}$，其单位向量为

$$\bar{l}^{o}=\frac{\bar{l}}{|\bar{l}|}=\left\{\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}},0\right\}$$

所以

$$\cos\alpha=\frac{1}{\sqrt{2}},\cos\beta=-\frac{1}{\sqrt{2}},\cos\gamma=0$$

又函数的偏导数连续，且

$$\frac{\partial u}{\partial x}\Big|_{(1,2,3)}=(2x+3y+yz)\Big|_{(1,2,3)}=14$$

$$\frac{\partial u}{\partial y}\Big|_{(1,2,3)}=(2y+3x+xz)\Big|_{(1,2,3)}=7$$

$$\frac{\partial u}{\partial z}\Big|_{(1,2,3)}=(-4z+xy-2)\Big|_{(1,2,3)}=-12$$

所以

$$\frac{\partial z}{\partial \vec{l}}\Big|_{(1,2,3)}=\frac{\partial u}{\partial x}\Big|_{(1,2,3)}\cos\alpha+\frac{\partial u}{\partial y}\Big|_{(1,2,3)}\cos\beta+\frac{\partial u}{\partial z}\Big|_{(1,2,3)}\cos\gamma$$

$$=14\times\frac{1}{\sqrt{2}}+7\times\frac{1}{\sqrt{2}}+(-12)\times 0=\frac{7}{\sqrt{2}}$$

3. 设 $f(x,y)$ 在点 $(0,0)$ 处可微，沿 $\vec{i}+\sqrt{3}\vec{j}$ 方向的方向导数为 1，沿 $\sqrt{3}\vec{i}+\vec{j}$ 方向的方向导数为 $\sqrt{3}$，求 $f(x,y)$ 在点 $(0,0)$ 处变化最快的方向和这个最大的变化率.

解 将方向向量单位化

$$\frac{\vec{i}+\sqrt{3}\vec{j}}{\left|\vec{i}+\sqrt{3}\vec{j}\right|}=\frac{1}{2}\vec{i}+\frac{\sqrt{3}}{2}\vec{j},\quad \frac{\sqrt{3}\vec{i}+\vec{j}}{\sqrt{3}\left|\vec{i}+\vec{j}\right|}=\frac{\sqrt{3}}{2}\vec{i}+\frac{1}{2}\vec{j}$$

由题设可知

$$\begin{cases} f'_x(0,0)\cdot\dfrac{1}{2}+f'_y(0,0)\cdot\dfrac{\sqrt{3}}{2}=1 \\ f'_x(0,0)\cdot\dfrac{\sqrt{3}}{2}+f'_y(0,0)\cdot\dfrac{1}{2}=\sqrt{3} \end{cases}$$

解得 $f'_x(0,0)=2$，$f'_y(0,0)=0$，所以 $f(x,y)$ 在点 $(0,0)$ 处变化最快的方向是

$$\mathbf{grad}\, f\Big|_{(0,0)}=\{f'_x(0,0),f'_y(0,0)\}=\{2,0\}$$

最大的变化率为

$$\left|\mathbf{grad}\, f\Big|_{(0,0)}\right|=\sqrt{2^2+0^2}=2$$

4. 设 $u=\dfrac{z^2}{c^2}-\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}$，问 u 在点 (a,b,c) 处沿哪个方向增大最快？沿哪个方向减小最快？沿哪个方向变化率为零？

解　u 增大最快的方向为

$$\mathbf{grad}\, u\Big|_{(a,b,c)}=\left\{\frac{\partial u}{\partial x},\frac{\partial u}{\partial y},\frac{\partial u}{\partial z}\right\}\Big|_{(a,b,c)}=\left\{-\frac{2x}{a^2},-\frac{2y}{b^2},\frac{2z}{c^2}\right\}\Big|_{(a,b,c)}=\left\{-\frac{2}{a},-\frac{2}{b},\frac{2}{c}\right\}$$

u 减少最快的方向为

$$-\mathbf{grad}\, u\Big|_{(a,b,c)}=\left\{\frac{2}{a},\frac{2}{b},-\frac{2}{c}\right\}$$

沿与梯度 $\mathbf{grad}\, u\Big|_{(a,b,c)}=\left\{-\frac{2}{a},-\frac{2}{b},\frac{2}{c}\right\}$ 正交的方向 u 的变化率为零.

9.8

1. 求下列函数的极值.

(1)　$z=3axy-x^3-y^3\ (a>0)$；

解　求偏导数

$$\frac{\partial z}{\partial x}=3ay-3x^2,\ \frac{\partial z}{\partial y}=3ax-3y^2$$

令 $\frac{\partial z}{\partial x}=0,\frac{\partial z}{\partial y}=0$ 得

$$\begin{cases}3ay-3x^2=0\\3ax-3y^2=0\end{cases}\Rightarrow\begin{cases}x=0\\y=0\end{cases},\begin{cases}x=a\\y=a\end{cases}$$

驻点为 $(0,0)$，(a,a).

又

$$\frac{\partial^2 z}{\partial x^2}=-6x,\ \frac{\partial^2 z}{\partial x\partial y}=3a,\ \frac{\partial^2 z}{\partial y^2}=-6y$$

在点 $(0,0)$ 处，因为 $AC-B^2=0-(3a)^2=-9a^2<0$，所以无极值.

在点 (a,a) 处，因为 $AC-B^2=(-6a)(-6a)-(3a)^2=27a^2>0$，且 $A=-6a<0$，所以 (a,a) 是极大值点，极大值为 $z\big|_{(a,a)}=a^3$.

(2)　$z=\mathrm{e}^{2x}\left(x+2y+y^2\right)$.

解　求偏导数

$$\frac{\partial z}{\partial x}=\mathrm{e}^{2x}\left(1+2x+4y+2y^2\right),\ \frac{\partial z}{\partial y}=2\mathrm{e}^{2x}(1+y)$$

令 $\dfrac{\partial z}{\partial x}=0,\dfrac{\partial z}{\partial y}=0$ 得

$$\begin{cases}1+2x+4y+2y^2=0\\1+y=0\end{cases}\Rightarrow\begin{cases}x=\dfrac{1}{2}\\y=-1\end{cases}$$

驻点为 $\left(\dfrac{1}{2},-1\right)$.

又

$$\frac{\partial^2 z}{\partial x^2}=\mathrm{e}^{2x}\left(4+4x+8y+4y^2\right)\ ,\ \frac{\partial^2 z}{\partial x\partial y}=4\mathrm{e}^{2x}(1+y)，\ \frac{\partial^2 z}{\partial y^2}=2\mathrm{e}^{2x}$$

在点 $\left(\dfrac{1}{2},-1\right)$，因为 $AC-B^2=(2\mathrm{e})\times(2\mathrm{e})-0^2=4\mathrm{e}^2>0$，且 $A=2\mathrm{e}>0$，所以 $\left(\dfrac{1}{2},-1\right)$ 是极小值点，极小值为 $z\Big|_{\left(\frac{1}{2},-1\right)}=\mathrm{e}^{2x}\left(x+2y+y^2\right)\Big|_{\left(-\frac{1}{2},-1\right)}=-\dfrac{\mathrm{e}}{2}$.

2. 求下列函数在指定的约束条件下的极值.

(1) $z=x^2+y^2$，条件为 $x^6+y^6=1$；

解 设拉格朗日函数

$$L(x,y,\lambda)=x^2+y^2+\lambda\left(x^6+y^6-1\right)$$

令

$$\begin{cases}\dfrac{\partial L}{\partial x}=2x+6\lambda x^5=0\\\dfrac{\partial L}{\partial y}=2y+6\lambda y^5=0\\\dfrac{\partial L}{\partial \lambda}=x^6+y^6-1=0\end{cases}$$

当 $x\neq 0$ 且 $y\neq 0$ 时，在第一、二个方程中消去 λ 得 $y=\pm x$，与第三个方程联立解得

$$\begin{cases}x=\pm\dfrac{1}{\sqrt[6]{2}}\\y=\pm\dfrac{1}{\sqrt[6]{2}}\end{cases},\quad\begin{cases}x=\pm\dfrac{1}{\sqrt[6]{2}}\\y=\mp\dfrac{1}{\sqrt[6]{2}}\end{cases}$$

当 $x=0$ 时，第三个方程化为 $y^6-1=0$，解得 $y=\pm1$，当 $y=0$ 时，第三个方程化为 $x^6-1=0$，解得 $x=\pm1$. 综上，极值嫌疑点为 $(0,1)$，$(0,-1)$，$(1,0)$，$(-1,0)$，$\left(\frac{1}{\sqrt[6]{2}},\frac{1}{\sqrt[6]{2}}\right)$，$\left(\frac{1}{\sqrt[6]{2}},-\frac{1}{\sqrt[6]{2}}\right)$，$\left(-\frac{1}{\sqrt[6]{2}},\frac{1}{\sqrt[6]{2}}\right)$，$\left(-\frac{1}{\sqrt[6]{2}},-\frac{1}{\sqrt[6]{2}}\right)$，且

$$z\Big|_{(0,\pm1)}=z\Big|_{(\pm1,0)}=1,\quad z\Big|_{\left(\pm\frac{1}{\sqrt[6]{2}},\frac{1}{\sqrt[6]{2}}\right)}=z\Big|_{\left(\pm\frac{1}{\sqrt[6]{2}},-\frac{1}{\sqrt[6]{2}}\right)}=\sqrt[3]{4}$$

所以函数在约束条件下的极大值为 $\sqrt[3]{4}$（也为最大值），极小值为 1（也为最大值）.

(2) $u=xyz$，条件为 $x^2+y^2+z^2=1$， $x+y+z=0$.

解 设拉格朗日函数

$$L(x,y,z,\lambda,\mu)=xyz+\lambda\left(x^2+y^2+z^2-1\right)+\mu(x+y+z)$$

令

$$\begin{cases}\dfrac{\partial L}{\partial x}=yz+2\lambda x+\mu=0\\ \dfrac{\partial L}{\partial y}=xz+2\lambda y+\mu=0\\ \dfrac{\partial L}{\partial z}=xy+2\lambda z+\mu=0\\ \dfrac{\partial L}{\partial \lambda}=x^2+y^2+z^2-1=0\\ \dfrac{\partial L}{\partial \mu}=x+y+z=0\end{cases}$$

第一与第二个方程相减得 $(y-x)(z-2\lambda)=0$，解得 $y=x$ 或 $2\lambda=z$. 当 $y=x$ 时，与第四、第五个方程联立

$$\begin{cases}y=x\\ x^2+y^2+z^2-1=0\\ x+y+z=0\end{cases}$$

解得 $x=\pm\frac{1}{\sqrt{6}}$，$y=\pm\frac{1}{\sqrt{6}}$，$z=\mp\frac{2}{\sqrt{6}}$，得极值嫌疑点

$$\left(\frac{1}{\sqrt{6}},\frac{1}{\sqrt{6}},-\frac{2}{\sqrt{6}}\right),\quad \left(-\frac{1}{\sqrt{6}},-\frac{1}{\sqrt{6}},\frac{2}{\sqrt{6}}\right)$$

当 $2\lambda = z$ 时，第一与第三个方程相减得 $(z-x)(y-2\lambda)=0$，将 $2\lambda = z$ 代入得 $(z-x)(y-z)=0$，所以 $y-z=0$ 或 $z-x=0$，类似地，可解得极值嫌疑点

$$\left(-\frac{2}{\sqrt{6}},\frac{1}{\sqrt{6}},\frac{1}{\sqrt{6}}\right),\ \left(\frac{2}{\sqrt{6}},-\frac{1}{\sqrt{6}},-\frac{1}{\sqrt{6}}\right)$$

$$\left(\frac{1}{\sqrt{6}},-\frac{2}{\sqrt{6}},\frac{1}{\sqrt{6}}\right),\ \left(-\frac{1}{\sqrt{6}},\frac{2}{\sqrt{6}},-\frac{1}{\sqrt{6}}\right)$$

又

$$u\Big|_{\left(\frac{1}{\sqrt{6}},\frac{1}{\sqrt{6}},-\frac{2}{\sqrt{6}}\right)} = u\Big|_{\left(-\frac{2}{\sqrt{6}},\frac{1}{\sqrt{6}},\frac{1}{\sqrt{6}}\right)} = u\Big|_{\left(\frac{1}{\sqrt{6}},-\frac{2}{\sqrt{6}},\frac{1}{\sqrt{6}}\right)} = -\frac{1}{3\sqrt{6}}$$

$$u\Big|_{\left(-\frac{1}{\sqrt{6}},-\frac{1}{\sqrt{6}},\frac{2}{\sqrt{6}}\right)} = u\Big|_{\left(\frac{2}{\sqrt{6}},-\frac{1}{\sqrt{6}},-\frac{1}{\sqrt{6}}\right)} = u\Big|_{\left(-\frac{1}{\sqrt{6}},\frac{2}{\sqrt{6}},-\frac{1}{\sqrt{6}}\right)} = \frac{1}{3\sqrt{6}}$$

所以函数在约束条件下的极大值为 $\frac{1}{3\sqrt{6}}$（也为最大值），极小值为 $-\frac{1}{3\sqrt{6}}$（也为最大值）.

3. 求函数 $z = x^2 y(4-x-y)$ 在由直线 $x+y=6$ 与 x 轴、y 轴所围成闭区域上的最大值和最小值.

解 求偏导数

$$\frac{\partial z}{\partial x} = xy(8-3x-2y),\ \frac{\partial z}{\partial y} = x^2(4-x-2y)$$

令 $\frac{\partial z}{\partial x}=0, \frac{\partial z}{\partial y}=0$ 得

$$\begin{cases} xy(8-3x-2y)=0 \\ x^2(4-x-2y)=0 \end{cases}$$

注意到闭区域内 $x \neq 0, y \neq 0$，所以

$$\begin{cases} 8-3x-2y=0 \\ 4-x-2y=0 \end{cases}$$

解得 $x=2, y=1$，区域内的驻点为 $(2,1)$，且 $z\big|_{(2,1)} = 4$.

在边界 $x=0, y=0$ 上，均有 $z = 0$.

在边界 $x+y=6\,(0\le x\le 6)$ 上，有

$$z=x^2y(4-x-y)=x^2(6-x)\cdot(-2)=2x^3-12x^2$$

令 $\dfrac{\mathrm{d}z}{\mathrm{d}x}=6x^2-24x=0$ 得 $x=4$，且 $z\big|_{(4,2)}=-64$，又 $z\big|_{(0,6)}=z\big|_{(6,0)}=0$.

因此, 函数在闭区域上的最大值为 $z\big|_{(2,1)}=4$，最小值为 $z\big|_{(4,2)}=-64$.

4. 在曲面 $z=\sqrt{2+x^2+4y^2}$ 上求一点，使它到平面 $x-2y+3z=1$ 的距离最近.

解　设 (x,y,z) 是曲面 $z=\sqrt{2+x^2+4y^2}$ 上任意一点, 该点到平面 $x-2y+3z=1$ 的距离为

$$d=\frac{|x-2y+3z-1|}{\sqrt{1^2+(-2)^2+3^2}}=\frac{1}{\sqrt{14}}|x-2y+3z-1|$$

又函数 $d^2=f(x,y,z)=(x-2y+3z-1)^2$ 与 d 同时取极值，所以问题化为求 $f(x,y,z)$ 在条件 $\varphi(x,y,z)=x^2+4y^2-z^2+2=0\ (z>0)$ 下的极值问题.

设拉格朗日函数

$$L(x,y,z,\lambda)=(x-2y+3z-1)^2+\lambda\left(x^2+4y^2-z^2+2\right)$$

令

$$\begin{cases}\dfrac{\partial L}{\partial x}=2(x-2y+3z-1)+2\lambda x=0\\ \dfrac{\partial L}{\partial y}=-4(x-2y+3z-1)+8\lambda y=0\\ \dfrac{\partial L}{\partial z}=6(x-2y+3z-1)-2\lambda z=0\\ \dfrac{\partial L}{\partial \lambda}=x^2+4y^2-z^2+2=0\end{cases}$$

由前三个方程得 $x=-2y,\ z=6y$，代人第四个方程解得 $x=\dfrac{2}{\sqrt{14}},\ y=\dfrac{1}{\sqrt{14}},\ z=\dfrac{6}{\sqrt{14}}$，根据问题的实际意义, 距离的最小值存在, 因此, 曲面 $z=\sqrt{2+x^2+4y^2}$ 上到平面 $x-2y+3z=1$ 的距离最近的点是 $\left(\dfrac{2}{\sqrt{14}},\dfrac{1}{\sqrt{14}},\dfrac{6}{\sqrt{14}}\right)$.

5. 抛物线 $z=x^2+y^2$ 被平面 $x+y+z=1$ 截成一椭圆，求这个椭圆上的点到原点的距离的最大值与最小值.

解　设 (x,y,z) 是椭圆任意一点, 该点到原点的距离为

$$d=\sqrt{x^2+y^2+z^2}$$

又函数 $d^2=f(x,y,z)=x^2+y^2+z^2$ 与 d 同时取极值，所以问题化为求 $f(x,y,z)$ 在条件 $\varphi(x,y,z)=z-x^2-y^2=0,\psi(x,y,z)=x+y+z-1=0$ 下的极值问题.

设拉格朗日函数

$$L(x,y,z,\lambda,\mu)=x^2+y^2+z^2+\lambda(z-x^2-y^2)+\mu(x+y+z-1)$$

令

$$\begin{cases}\dfrac{\partial L}{\partial x}=2x-2\lambda x+\mu=0\\ \dfrac{\partial L}{\partial y}=2y-2\lambda y+\mu=0\\ \dfrac{\partial L}{\partial z}=2z+\lambda+\mu=0\\ \dfrac{\partial L}{\partial \lambda}=z-x^2-y^2=0\\ \dfrac{\partial L}{\partial \mu}=x+y+z-1=0\end{cases}$$

由前两个方程得 $x=y$, 与最后两个方程联立

$$\begin{cases}x=y\\ z-x^2-y^2=0\\ x+y+z-1=0\end{cases}$$

解得 $x=y=\dfrac{-1\pm\sqrt{3}}{2}$，$z=2\mp\sqrt{3}$，且

$$d\Big|_{\left(\frac{-1-\sqrt{3}}{2},\frac{-1-\sqrt{3}}{2},2+\sqrt{3}\right)}=\sqrt{9+5\sqrt{3}}$$

$$d\Big|_{\left(\frac{-1+\sqrt{3}}{2},\frac{-1+\sqrt{3}}{2},2-\sqrt{3}\right)}=\sqrt{9-5\sqrt{3}}$$

根据问题的实际意义,距离的最大值和最小值存在,因此,椭圆上距离原点最远的点为$\left(\frac{-1+\sqrt{3}}{2},\frac{-1+\sqrt{3}}{2},2-\sqrt{3}\right)$，最近的点为$\left(\frac{-1-\sqrt{3}}{2},\frac{-1-\sqrt{3}}{2},2+\sqrt{3}\right)$，距离的最大值为$\sqrt{9+5\sqrt{3}}$,距离的最小值为$\sqrt{9-5\sqrt{3}}$.

6. 修建一体积为V的长方体水池（无盖），已知底面与侧面单位面积造价之比为 3:2，问如何设计水池的长、宽、高，使总造价最低.

解　设水池的长、宽、高分别为x,y,z，则造价为

$$f(x,y,z)=3xy+4(xz+yz)$$

且满足条件$xyz=V$.

设拉格朗日函数

$$L(x,y,z,\lambda)=3xy+4(xz+yz)+\lambda(xyz-V)$$

令

$$\begin{cases}\dfrac{\partial L}{\partial x}=3y+4z+\lambda yz=0\\ \dfrac{\partial L}{\partial y}=3x+4z+\lambda xz=0\\ \dfrac{\partial L}{\partial z}=4(x+y)+\lambda xy=0\\ \dfrac{\partial L}{\partial \lambda}=xyz-V=0\end{cases}$$

由前三个方程得$x=y=\frac{4}{3}z$,代入第四个方程解得$x=y=\sqrt[3]{\frac{4}{3}V}$，$z=\frac{3}{4}\sqrt[3]{\frac{4V}{3}}$，根据问题的实际意义,总造价的最小值存在,所以当水池的长、宽、高分别为$\sqrt[3]{\frac{4}{3}V},\sqrt[3]{\frac{4}{3}V},\frac{3}{4}\sqrt[3]{\frac{4}{3}V}$时总造价最低.

9.9

1. 求函数$f(x,y)=\mathrm{e}^x\ln(1+y)$在点$(0,0)$的三阶泰勒公式.

解　求偏导数

$$\frac{\partial f}{\partial x}=\frac{\partial^2 f}{\partial x^2}=\frac{\partial^3 f}{\partial x^3}=\frac{\partial^4 f}{\partial x^4}=\mathrm{e}^x\ln(1+y),\ \frac{\partial f}{\partial y}=\frac{\partial^2 f}{\partial x\partial y}=\frac{\partial^3 f}{\partial x^2\partial y}=\frac{\partial^4 f}{\partial x^3\partial y}=\frac{\mathrm{e}^x}{1+y}$$

$$\frac{\partial^2 f}{\partial y^2}=\frac{\partial^3 f}{\partial x\partial y^2}=\frac{\partial^4 f}{\partial x^2\partial y^2}=-\frac{\mathrm{e}^x}{(1+y)^2},\ \frac{\partial^3 f}{\partial y^3}=\frac{\partial^4 f}{\partial x\partial y^3}=\frac{2\mathrm{e}^x}{(1+y)^3},\ \frac{\partial^4 f}{\partial y^4}=-\frac{6\mathrm{e}^x}{(1+y)^4}$$

因为

$f(0,0)=0$

$$\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)f(0,0)=hf'_x(0,0)+kf'_y(0,0)=k$$

$$\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^2 f(0,0)=h^2 f''_{xx}(0,0)+2hkf''_{xy}(0,0)+k^2 f''_{yy}(0,0)=2hk-k^2$$

$$\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^3 f(0,0)=h^3 f'''_{xxx}(0,0)+3h^2kf'''_{xxy}(0,0)+3hk^2 f'''_{xyy}(0,0)+k^3 f'''_{yyy}(0,0)$$
$$=3h^2k-3hk^2+2k^3$$

$$\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^4 f(\theta h,\theta k)=\left[h^4\ln(1+\theta k)+\frac{4h^2k}{1+\theta k}-\frac{6h^2k^2}{(1+\theta k)^2}+\frac{8hk^3}{(1+\theta k)^3}-\frac{6k^4}{(1+\theta k)^4}\right]\mathrm{e}^{\theta h}\ (0<\theta<1)$$

令 $h=x,k=y$，所以泰勒公式为

$$\mathrm{e}^x\ln(1+y)=y+\frac{1}{2!}\left(2xy-y^2\right)+\frac{1}{3!}\left(3x^2y-3xy^2+2y^3\right)+R_3$$

其中

$$R_3=\frac{1}{4!}\left[\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^4 f(\theta h,\theta k)\right]_{\substack{h=x\\k=y}}$$
$$=\frac{\mathrm{e}^{\theta x}}{24}\left[x^4\ln(1+\theta y)+\frac{4x^2y}{1+\theta y}-\frac{6x^2y^2}{(1+\theta y)^2}+\frac{8xy^3}{(1+\theta y)^3}-\frac{6y^4}{(1+\theta y)^4}\right]$$

2. 求函数 $f(x,y)=\sin x\sin y$ 在点 $\left(\frac{\pi}{4},\frac{\pi}{4}\right)$ 的二阶泰勒公式.

解 求偏导数

$$f'_x(x,y)=\cos x\sin y,\quad f'_y(x,y)=\sin x\cos y,\quad f''_{xx}(x,y)=-\sin x\sin y,$$
$$f''_{xy}(x,y)=\cos x\cos y,\quad f''_{yy}(x,y)=-\sin x\sin y,\quad f'''_{xxx}(x,y)=-\cos x\sin y,$$
$$f'''_{xxy}(x,y)=-\sin x\cos y,\quad f'''_{xyy}(x,y)=-\cos x\sin y,\quad f'''_{yyy}(x,y)=-\sin x\cos y$$

因为

$$f\left(\frac{\pi}{4},\frac{\pi}{4}\right)=\frac{1}{2}$$

$$\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)f\left(\frac{\pi}{4},\frac{\pi}{4}\right)=hf'_x\left(\frac{\pi}{4},\frac{\pi}{4}\right)+kf'_y\left(\frac{\pi}{4},\frac{\pi}{4}\right)=\frac{1}{2}h+\frac{1}{2}k$$

$$\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^2 f\left(\frac{\pi}{4},\frac{\pi}{4}\right)=h^2 f''_{xx}\left(\frac{\pi}{4},\frac{\pi}{4}\right)+2hkf''_{xy}\left(\frac{\pi}{4},\frac{\pi}{4}\right)+k^2 f''_{yy}\left(\frac{\pi}{4},\frac{\pi}{4}\right)=-\frac{1}{2}h^2+hk-\frac{1}{2}k^2$$

令 $h=x-\frac{\pi}{4},k=y-\frac{\pi}{4}$，所以泰勒公式为

$$\sin x\sin y=\frac{1}{2}+\frac{1}{2}\left(x-\frac{\pi}{4}\right)+\frac{1}{2}\left(y-\frac{\pi}{4}\right)$$
$$+\frac{1}{2!}\left[-\frac{1}{2}\left(x-\frac{\pi}{4}\right)^2+\left(x-\frac{\pi}{4}\right)\left(y-\frac{\pi}{4}\right)-\frac{1}{2}\left(y-\frac{\pi}{4}\right)^2\right]+R_2$$

其中

$$R_2=\frac{1}{3!}\left[\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^3 f\left(\frac{\pi}{4}+\theta h,\frac{\pi}{4}+\theta k\right)\right]_{h=x-\frac{\pi}{4},\ k=y-\frac{\pi}{4}}$$
$$=-\frac{1}{6}\left[\cos\xi\sin\eta\cdot\left(x-\frac{\pi}{4}\right)^3+3\sin\xi\cos\eta\left(x-\frac{\pi}{4}\right)^2\left(y-\frac{\pi}{4}\right)\right.$$
$$\left.+3\cos\xi\sin\eta\left(x-\frac{\pi}{4}\right)\left(y-\frac{\pi}{4}\right)^2+\sin\xi\cos\eta\cdot\left(y-\frac{\pi}{4}\right)^3\right]$$

这里 $\xi=\frac{\pi}{4}+\theta\left(x-\frac{\pi}{4}\right),\eta=\frac{\pi}{4}+\theta\left(y-\frac{\pi}{4}\right),0<\theta<1$.

总习题九

1. 设函数 $f(x,y)$ 在点 $(0,0)$ 的某邻域内有定义，且 $f'_x(0,0)=3$，$f'_y(0,0)=-1$，则有（　）

(A) $\mathrm{d}x\big|_{(0,0)}=3\mathrm{d}x-\mathrm{d}y$

(B) 曲面 $z=f(x,y)$ 在点 $(0,0,f(0,0))$ 处的切平面方程为 $3x-y-(z-f(0,0))=0$

(C) 曲面 $z=f(x,y)$ 在点 $(0,0,f(0,0))$ 处的法线方程为 $\frac{x}{3}=\frac{y}{-1}=\frac{z-f(0,0)}{-1}$

(D) 曲线 $\begin{cases}z=f(x,y)\\ y=0\end{cases}$ 在点 $(0,0,f(0,0))$ 处的切线方程为 $\frac{x}{1}=\frac{y}{0}=\frac{z-f(0,0)}{3}$

解　偏导数的存在性不能保证可微性，也不能保证切平面与法线存在，所以(A)，(B)，(C)都不对．选(D)．

2. 函数 $f(x,y)$ 在点 $(0,0)$ 处可微的一个充分条件是（　）

(A) $\lim\limits_{(x,y)\to(0,0)}\left[f(x,y)-f(0,0)\right]=0$

(B) $\lim_{x\to0}\frac{f(x,0)-f(0,0)}{x}=0,\lim_{y\to0}\frac{f(0,y)-f(0,0)}{y}=0$

(C) $\lim_{(x,y)\to(0,0)}\frac{f(x,y)-f(0,0)}{\sqrt{x^2+y^2}}=0$

(D) $\lim_{x\to0}\left[f_x'(x,0)-f_x'(0,0)\right]=0$, $\lim_{y\to0}\left[f_y'(0,y)-f_y'(0,0)\right]=0$

解 (A)仅表明 $f(x,y)$ 在点 $(0,0)$ 处连续,(B)仅表明 $f_x'(0,0)=0,f_y'(0,0)=0$,(D)仅表明 $f_x'(x,0)$ 在 $x=0$ 处连续,$f_y'(0,y)$ 在 $y=0$ 处连续,都不能推出可微性.利用全微分定义,(C)可推出 $f(x,y)$ 在点 $(0,0)$ 处可微,故选(C).

3.设 $f(u,v)$ 由关系式 $f(xg(y),y)=x+g(y)$ 所确定，其中 g 可微，求 $\frac{\partial^2 f}{\partial u\partial v}$.

解 令 $u=xg(y)$，$v=y$，则 $x=\frac{u}{g(v)},y=v$，所以

$$f(u,v)=\frac{u}{g(v)}+g(v)$$

求偏导数得

$$\frac{\partial f}{\partial u}=\frac{1}{g(v)}\quad,\quad\frac{\partial^2 f}{\partial u\partial v}=\frac{\partial}{\partial v}\left(\frac{1}{g(v)}\right)=-\frac{g'(v)}{(g(v))^2}$$

4.证明：$f(x,y)=\begin{cases}xy\sin\frac{1}{x^2+y^2},x^2+y^2\neq0\\0,x^2+y^2=0\end{cases}$ 在点 $(0,0)$ 处可微，并讨论其偏导数在点 $(0,0)$ 处是否连续.

证 在点 $(0,0)$ 处,有

$$f_x'(0,0)=\lim_{x\to0}\frac{f(x,0)-f(0,0)}{x}=\lim_{x\to0}\frac{0-0}{x}=0$$

$$f_y'(0,0)=\lim_{y\to0}\frac{f(0,y)-f(0,0)}{y}=\lim_{y\to0}\frac{0-0}{y}=0$$

因为

$$\lim_{(x,y)\to(0,0)}\frac{f(x,y)-f(0,0)-f_x'(0,0)x-f_y'(0,0)y}{\sqrt{x^2+y^2}}=\lim_{(x,y)\to(0,0)}\frac{xy\sin\frac{1}{x^2+y^2}-0}{\sqrt{x^2+y^2}}$$

$$=\lim_{r\to0}\frac{(r\cos\theta)(r\sin\theta)\sin\frac{1}{r^2}}{r}=\lim_{r\to0}r\cdot\left(\cos\theta\sin\theta\sin\frac{1}{r^2}\right)=0$$

所以

$$f(x,y)-f(0,0)=f_x'(0,0)x+f_y'(0,0)y+o\left(\sqrt{x^2+y^2}\right)$$

即 $f(x,y)$在点$(0,0)$处可微.

当$(x,y)\neq(0,0)$时，有

$$f_x'(x,y)=y\sin\frac{1}{x^2+y^2}-\frac{2x^2y}{(x^2+y^2)^2}\cos\frac{1}{x^2+y^2}$$

$$f_y'(x,y)=x\sin\frac{1}{x^2+y^2}-\frac{2xy^2}{(x^2+y^2)^2}\cos\frac{1}{x^2+y^2}$$

因为

$$\lim_{\substack{(x,y)\to(0,0)\\y=x}}f_x'(x,y)=\lim_{x\to0}f_x'(x,x)=\lim_{x\to0}\left[x\sin\frac{1}{2x^2}-\frac{1}{2x}\cos\frac{1}{2x^2}\right]$$

不存在，所以 $f_x'(x,y)$在$(0,0)$处不连续. 同理， $f_y'(x,y)$在$(0,0)$处不连续.

5. 求函数$u=x^2+y^2+z^2$在椭球面$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$上点$M_0(x_0,y_0,z_0)$处沿外法线方向的方向导数.

解 设$F(x,y,z)=\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1$，则椭球面在点$M_0$处的外法向量为

$$\vec{n}=\{F_x',F_y',F_z'\}\big|_{M_0}=\left\{\frac{2x}{a^2},\frac{2y}{b^2},\frac{2z}{c^2}\right\}\bigg|_{M_0}=\left\{\frac{2x_0}{a^2},\frac{2y_0}{b^2},\frac{2z_0}{c^2}\right\},$$

其单位向量为

$$\vec{n}^0=\frac{\vec{n}}{|\vec{n}|}=\frac{1}{\sqrt{\frac{x_0^2}{a^4}+\frac{y_0^2}{b^4}+\frac{z_0^2}{c^4}}}\left\{\frac{x_0}{a^2},\frac{y_0}{b^2},\frac{z_0}{c^2}\right\}=\{\cos\alpha,\cos\beta,\cos\gamma\}$$

又函数的偏导数连续，且

$$\frac{\partial u}{\partial x}\Big|_{M_0}=2x\Big|_{M_0}=2x_0,\frac{\partial u}{\partial y}\Big|_{M_0}=2y\Big|_{M_0}=2y_0,\frac{\partial u}{\partial z}\Big|_{M_0}=2z\Big|_{M_0}=2z_0$$

所以方向导数为

$$\frac{\partial u}{\partial \vec{n}}\Big|_{M_0}=\frac{\partial u}{\partial x}\Big|_{M_0}\cos\alpha+\frac{\partial u}{\partial y}\Big|_{M_0}\cos\beta+\frac{\partial u}{\partial z}\Big|_{M_0}\cos\gamma$$

$$=2x_0\frac{\frac{x_0}{a^2}}{\sqrt{\frac{x_0^2}{a^4}+\frac{y_0^2}{b^4}+\frac{z_0^2}{c^4}}}+2y_0\frac{\frac{y_0}{b^2}}{\sqrt{\frac{x_0^2}{a^4}+\frac{y_0^2}{b^4}+\frac{z_0^2}{c^4}}}+2z_0\frac{\frac{z_0}{c^2}}{\sqrt{\frac{x_0^2}{a^4}+\frac{y_0^2}{b^4}+\frac{z_0^2}{c^4}}}$$

$$=\frac{2}{\sqrt{\frac{x_0^2}{a^4}+\frac{y_0^2}{b^4}+\frac{z_0^2}{c^4}}}$$

6. 有一圆板占有平面闭区域 $\{(x,y)|x^2+y^2\le 1\}$. 设圆板被加热，以致在点 (x,y) 的温度是 $T=x^2+2y^2-x$. 求该圆板的最热点和最冷点.

解 求偏导数

$$\frac{\partial T}{\partial x}=2x-1,\quad \frac{\partial T}{\partial y}=4y$$

令 $\frac{\partial T}{\partial x}=0,\frac{\partial T}{\partial y}=0$ 得

$$\begin{cases}2x-1=0\\4y=0\end{cases}\Rightarrow\begin{cases}x=\frac{1}{2}\\y=0\end{cases}$$

驻点为 $\left(\frac{1}{2},0\right)$，且 $T\Big|_{\left(\frac{1}{2},0\right)}=-\frac{1}{4}$.

在闭区域的边界 $x^2+y^2=1$ 上，有

$$T=x^2+2y^2-x=x^2+2(1-x^2)-x=\frac{9}{4}-\left(x+\frac{1}{2}\right)^2\quad(-1\le x\le 1)$$

当 $x=-\frac{1}{2}$ 时，T 取最大值，最大值 $T\Big|_{\left(-\frac{1}{2},\pm\frac{\sqrt{3}}{2}\right)}=\frac{9}{4}$，当 $x=1$ 时，T 取最大值，最小值为 $T\Big|_{(1,0)}=0$.

综上，最热点在$\left(-\frac{1}{2},\pm\frac{\sqrt{3}}{2}\right)$，温度为$T\Big|_{\left(-\frac{1}{2},\pm\frac{\sqrt{3}}{2}\right)}=\frac{9}{4}$，最冷点在$\left(\frac{1}{2},0\right)$，温度为$T\Big|_{\left(\frac{1}{2},0\right)}=-\frac{1}{4}$.

7. 设在第一卦限内作椭球面$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$的切平面，使该切平面与三个坐标面所围成的四面体的体积最小，求这个切平面的切点，并求此最小体积.

解　设切点坐标为(x, y, z)，则切平面方程为

$$\frac{2x}{a^2}(X-x)+\frac{2y}{b^2}(Y-y)+\frac{2z}{c^2}(Z-z)=0$$

即

$$\frac{X}{\frac{a^2}{x}}+\frac{Y}{\frac{b^2}{y}}+\frac{Z}{\frac{c^2}{z}}=1$$

所以切平面在x,y,z轴上的截距分别为$\frac{a^2}{x},\frac{b^2}{y},\frac{c^2}{z}$，于是四面体的体积为

$$V=\frac{1}{6}\frac{a^2b^2c^2}{xyz}(x>0,y>0,z>0)$$

且满足条件$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$.

设拉格朗日函数

$$L(x,y,z,\lambda)=xyz+\lambda\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1\right)$$

令

$$\begin{cases}\frac{\partial L}{\partial x}=yz+\frac{2\lambda x}{a^2}=0\\ \frac{\partial L}{\partial y}=xz+\frac{2\lambda y}{b^2}=0\\ \frac{\partial L}{\partial z}=xy+\frac{2\lambda z}{c^2}=0\\ \frac{\partial L}{\partial \lambda}=\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1=0\end{cases}$$

由前三个方程得 $y=\frac{b}{a}x, z=\frac{c}{a}x$，代入最后一个方程解得 $x=\frac{\sqrt{3}}{3}a, y=\frac{\sqrt{3}}{3}b, z=\frac{\sqrt{3}}{3}c$.

由于体积的最小值存在，所以必在点 $\left(\frac{\sqrt{3}}{3}a, \frac{\sqrt{3}}{3}b, \frac{\sqrt{3}}{3}c\right)$ 取得，故所求切点为 $\left(\frac{\sqrt{3}}{3}a, \frac{\sqrt{3}}{3}b, \frac{\sqrt{3}}{3}c\right)$，最小体积为 $V=\frac{\sqrt{3}}{2}abc$.

8. 函数 $u=F(x,y,z)$ 在条件 $\varphi(x,y,z)=0$ 和 $\psi(x,y,z)=0$ 下，在点 (x_0,y_0,z_0) 处取极值 m. 试证：三个曲面 $F(x,y,z)=m, \varphi(x,y,z)=0, \psi(x,y,z)=0$ 在点 (x_0,y_0,z_0) 处的三条法线共面，这里 F,φ,ψ 都具有一阶连续偏导数，且每个函数的三个偏导数不同时为零.

证 设拉格朗日函数

$$G(x,y,z,\lambda,\mu)=F(x,y,z)+\lambda\varphi(x,y,z)+\mu\psi(x,y,z)$$

则在点 (x_0,y_0,z_0) 处存在 λ_0,μ_0 使得

$$\begin{cases} \frac{\partial G}{\partial x}\Big|_{(x_0,y_0,z_0,\lambda_0,\mu_0)}=\left(\frac{\partial F}{\partial x}+\lambda\frac{\partial \varphi}{\partial x}+\mu\frac{\partial \psi}{\partial x}\right)\Big|_{(x_0,y_0,z_0,\lambda_0,\mu_0)}=0 \\ \frac{\partial G}{\partial y}\Big|_{(x_0,y_0,z_0,\lambda_0,\mu_0)}=\left(\frac{\partial F}{\partial y}+\lambda\frac{\partial \varphi}{\partial y}+\mu\frac{\partial \psi}{\partial y}\right)\Big|_{(x_0,y_0,z_0,\lambda_0,\mu_0)}=0 \\ \frac{\partial G}{\partial z}\Big|_{(x_0,y_0,z_0,\lambda_0,\mu_0)}=\left(\frac{\partial F}{\partial z}+\lambda\frac{\partial \varphi}{\partial z}+\mu\frac{\partial \psi}{\partial z}\right)\Big|_{(x_0,y_0,z_0,\lambda_0,\mu_0)}=0 \\ \frac{\partial G}{\partial \lambda}\Big|_{(x_0,y_0,z_0,\lambda_0,\mu_0)}=\varphi(x_0,y_0,z_0)=0 \\ \frac{\partial G}{\partial \mu}\Big|_{(x_0,y_0,z_0,\lambda_0,\mu_0)}=\psi(x_0,y_0,z_0)=0 \end{cases}$$

曲面 $F(x,y,z)=m, \varphi(x,y,z)=0, \psi(x,y,z)=0$ 在点 (x_0,y_0,z_0) 处的法向量分别为

$$\vec{n}_1=\left\{\frac{\partial F}{\partial x},\frac{\partial F}{\partial y},\frac{\partial F}{\partial z}\right\}\Big|_{(x_0,y_0,z_0)}, \vec{n}_2=\left\{\frac{\partial \varphi}{\partial x},\frac{\partial \varphi}{\partial y},\frac{\partial \varphi}{\partial z}\right\}\Big|_{(x_0,y_0,z_0)}, \vec{n}_3=\left\{\frac{\partial \psi}{\partial x},\frac{\partial \psi}{\partial y},\frac{\partial \psi}{\partial z}\right\}\Big|_{(x_0,y_0,z_0)}$$

则

$$\vec{n}_1 \cdot (\vec{n}_2 \times \vec{n}_3) = \begin{vmatrix} \frac{\partial F}{\partial x} & \frac{\partial F}{\partial y} & \frac{\partial F}{\partial z} \\ \frac{\partial \varphi}{\partial x} & \frac{\partial \varphi}{\partial y} & \frac{\partial \varphi}{\partial z} \\ \frac{\partial \psi}{\partial x} & \frac{\partial \psi}{\partial y} & \frac{\partial \psi}{\partial z} \end{vmatrix}_{(x_0, y_0, z_0)}$$

$$= \begin{vmatrix} \frac{\partial F}{\partial x} + \lambda_1^0 \frac{\partial \varphi}{\partial x} + \mu_0 \frac{\partial \psi}{\partial x} & \frac{\partial F}{\partial y} + \lambda_1^0 \frac{\partial \varphi}{\partial y} + \mu_0 \frac{\partial \psi}{\partial y} & \frac{\partial F}{\partial z} + \lambda_1^0 \frac{\partial \varphi}{\partial z} + \mu_0 \frac{\partial \psi}{\partial z} \\ \frac{\partial \varphi}{\partial x} & \frac{\partial \varphi}{\partial y} & \frac{\partial \varphi}{\partial z} \\ \frac{\partial \psi}{\partial x} & \frac{\partial \psi}{\partial y} & \frac{\partial \psi}{\partial z} \end{vmatrix}_{(x_0, y_0, z_0)}$$

$$= \begin{vmatrix} 0 & 0 & 0 \\ \frac{\partial \varphi}{\partial x} & \frac{\partial \varphi}{\partial y} & \frac{\partial \varphi}{\partial z} \\ \frac{\partial \psi}{\partial x} & \frac{\partial \psi}{\partial y} & \frac{\partial \psi}{\partial z} \end{vmatrix}_{(x_0, y_0, z_0)} = 0$$

即向量$\vec{n}_1, \vec{n}_2, \vec{n}_3$共面，从而三个曲面在点$(x_0, y_0, z_0)$处的三条法线共面.